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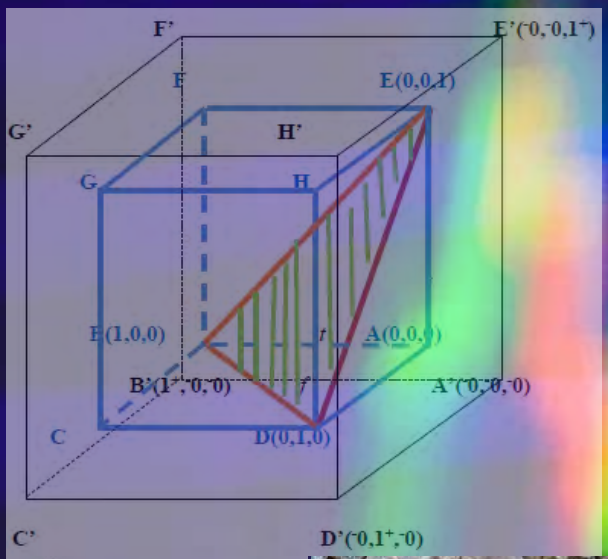

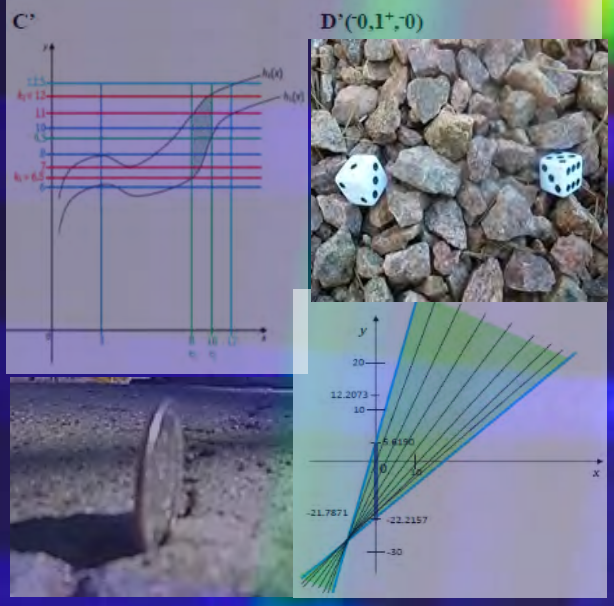

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<A> <neutA> <antiA>

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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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FOLDOC Dictionary of Computing (**1, 2**), Webster
Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free
Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Some Fixed Point Theorems in Complete Neutrosophic Metric Spaces for Neutrosophic ψ-Quasi Contractions

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Abstract. Fixed point theory is a fundamental tool in mathematics and numerous scientific fields. This study employs the concept of neutrosophic metric spaces as defined by Kirişci and Şimşek to introduce fixed point theorems related to neutrosophic ψ-quasi contractions. We demonstrate that each ψ-quasi contractions possesses a unique fixed point, provided the fuzzy sets are continuous. Additionally, we provide some examples to illustrate our primary theorem. Moreover, we derive several fixed point results based on the main theorem.

Keywords: Fixed point; ψ-Quasi-contractions; Banach contraction principle; Chatterjea contractions; Kannan contractions; Neutrosophic metric; Quasi-contraction; Non-linear contractions.

1. **Introduction**

In 1965, Zadeh [1] introduced fuzzy sets, enabling elements to have varying degrees of membership. Despite initial skepticism, this concept became influential across many fields. However, fuzzy sets faced limitations in solving certain problems. In 1986, Atanassov [2] proposed Intuitionistic Fuzzy Sets, incorporating both membership and non-membership. Later, Smarandache [3] advanced the theory further with Neutrosophic Sets, adding indeterminacy

to membership and non-membership, providing a more robust framework for handling uncertainty. These developments have significantly expanded the scope of applications in decision-making and optimization.

In recent years, the study of fuzzy sets and their generalizations has garnered significant attention due to their wide-ranging applications in decision-making and optimization. For instance, [4] explored bipolar complex fuzzy soft sets and their practical applications, while [5] investigated the normality and cosets of (γ, ∂)-fuzzy HX-subgroups. The concept of time fuzzy soft sets and their role in decision-making was introduced by [6], and the influence of time on fuzzy soft expert sets was further examined by [7]. By integrating time elements and neutrosophic logic, [8] introduced the idea of time-effective fuzzy soft sets and examined some of their applications, both with and without neutrosophic components, adding a new dimension to the theory of fuzzy soft sets. In a different study, [9] introduced the idea of time-shadow soft sets and examined their applications, further broadening the field of soft set theory and its application in dynamic systems and decision-making processes. A thorough investigation of complicated hesitant fuzzy graphs was presented by [10], which broadened the theory and offered fresh perspectives on its uses in graph theory and decision-making. A numerical solution to the fuzzy heat equation with complicated Dirichlet boundary conditions was described in a later study by [11], providing a useful technique for dealing with fuzzy parameters in heat conduction issues. Last but not least, [12] presented the idea of homomorphism of tripolar fuzzy soft Γ-semirings, a noteworthy advancement in the algebraic structure of fuzzy systems that facilitates comprehension of their characteristics and their applications in a variety of domains.

Neutrosophic sets, as a powerful extension of fuzzy sets, have demonstrated remarkable utility across diverse domains. For example, [13] provided a detailed characterization of effective and optimal solutions for scalar optimization problems, including the derivation of Kuhn-Tucker conditions for efficiency and proper efficiency. [14] initiated a new approach known as possibility interval-valued neutrosophic soft sets (PIVNSSs) as a new development in a fuzzy soft computing environment. Further, [15] explored applications of interval-valued Q-neutrosophic soft matrices, while [16] introduced the concept of a mapping on classes of complex multi-fuzzy soft expert sets, and studied the images and inverse images of complex multi-fuzzy soft expert sets. [17] Introduced a more flexible and extended approach to the previously established concept of Interval-Valued Q-Neutrosophic Soft Matrix (IV-Q-NSM), offering a broader generalization of existing mathematical frameworks.

Additionally, neutrosophic logic has been applied to address real-world challenges. For instance, [18] utilized neutrosophic logic to navigate uncertainty in security events in Mexico, and in [19], combined Neutrosophic Set Theory (NST) with machine learning (ML) to develop

a novel hybrid methodology (NS-ML) aimed at enhancing breast cancer diagnosis. In the agricultural domain, [20] introduced a multi-attribute neutrosophic optimization technique for optimal crop selection in the Ariyalur District. Meanwhile, [21] presented an innovative telemedical framework by integrating vague Type-2 Neutrosophic Sets (T2NSs) with the Ordered Weighted Continuous Multiplicative Ratio Assessment Method (OWCM-RAM) for intelligent Medical 4.0 evaluations. These studies underscore the versatility and applicability of neutrosophic sets in addressing complex, real-world problems.

Fixed point theory plays a crucial role in both applied and theoretical mathematics. At its core is the Banach contraction principle [22], introduced by Banach in 1922. Since then, numerous researchers have extended and generalized this principle in various ways. Notably, common fixed point theorems have been established in [23–25]. In [26,27], the authors explored fixed point results in neutrosophic metric spaces. Additionally, [28,29] investigated fixed points by introducing new types of distance spaces. Additional research on fixed point theory in generalized metric spaces can be found in the following studies: [30] introduced some results on b-metric spaces and related it with results in metric and G-metric spaces, [31] established fixed point results and investigated (α, β)-triangular admissibility within the framework of complete extended b-metric spaces, and [32] introduced new type of contractions and proved fixed point results based on extended quasi b-metric spaces, broadening the field of fixed point theory. [33] contributed to the study of distance functions in nonlinear spaces by presenting fixed point findings for nonlinear contractions with generalized Ω-distance mappings. Further insights into G-metric spaces and their associated fixed point theorems were provided by [34]. Finally, [35] advanced the subject of fixed point theory by introducing common fixed point theorems in G-metric spaces with Ω-distance.

The following studies are cited for further research on fixed points in various distance spaces and their uses in order to better grasp fixed point theory in partly ordered metric spaces, [36] introduced tripled coincidence point theorems for weak Φ-contractions. Important findings in ordered b-metric spaces and graphic b-metric spaces were obtained by [37], who proposed coincidence and common fixed point theorems for four mappings fulfilling $(\alpha(s), F)$-contractions. In a different study, [38] addressed asymptotic regularity and (α, p)-convex contractions, providing a fresh perspective on regularity and convergence in fixed point theory. Fixed point results for Geraghty-type contractions with comparable distance were introduced by [39] through a new type of distance spaces. [40] contributed to the study of approximating fixed point theory by presenting a numerical scheme to approximate common fixed points for a wide class of operators.

2. Preliminary

In this document, we define the sets as follows: $\mathbb{R}^+ = [0, \infty)$ and $I = [0, 1]$.

The following section revisits the definitions of triangular norms and t-norms, concepts initially proposed by Menger [41] (see also [42]). These definitions are essential for the characterization of neutrosophic metric spaces.

Definition 2.1. An operation $\bullet : I \times I \to I$ is called as continuous t-norm (CTN) if for each $\kappa, \kappa', \tau, \tau' \in I$ it satisfies the following:

(1) $\kappa \bullet 1 = \kappa$,
(2) If $\kappa \leq \kappa'$ and $\tau \leq \tau'$, than $\kappa \bullet \tau \leq \kappa' \bullet \tau'$,
(3) $\bullet$ is continuous,
(4) $\bullet$ is commutative and associate.

Definition 2.2. An operation $\diamond : I \times I \to I$ is called as continuous t-conorm (CTCN) if for each $\kappa, \kappa', \tau, \tau' \in I$ it satisfies the following:

(1) $\kappa \diamond 0 = \kappa$,
(2) If $\kappa \leq \kappa'$ and $\tau \leq \tau'$, than $\kappa \diamond \tau \leq \kappa' \diamond \tau'$,
(3) $\diamond$ is continuous,
(4) $\diamond$ is commutative and associate.

The concept of neutrosophic metric spaces was proposed by Kirişci and Şimşek in the following manner.

Definition 2.3. [42] A 6-tuple $(\mathcal{Z}, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond)$ is defined as a Neutrophic Metric Space (NMS) when the set $\mathcal{Z}$ constitutes a non-empty arbitrary collection. In this context, $\bullet$ represents a continuous t-norm, while $\diamond$ denotes a continuous t-conorm. Additionally, the elements $\mathcal{T}, \mathcal{F}$, and $\mathcal{I}$ are three fuzzy sets defined on the Cartesian product $\mathcal{Z}^2 \times (0, \infty)$. These components are required to fulfill specific conditions applicable to all elements $\zeta, \varrho, c \in \mathcal{Z}$ and for all positive real numbers γ, ρ.

(1) $0 \leq \mathcal{T}(\zeta, \varrho, \gamma) \leq 1$, $0 \leq \mathcal{F}(\zeta, \varrho, \gamma) \leq 1$, $0 \leq \mathcal{I}(\zeta, \varrho, \gamma) \leq 1$,
(2) $0 \leq \mathcal{T}(\zeta, \varrho, \gamma) + \mathcal{F}(\zeta, \varrho, \gamma) + \mathcal{I}(\zeta, \varrho, \gamma) \leq 3$,
(3) $\mathcal{T}(\zeta, \varrho, \gamma) = 1$, for $\gamma > 0$ iff $\zeta = \varrho$
(4) $\mathcal{T}(\zeta, \varrho, \gamma) = H(\varrho, \zeta, \gamma)$, for $\gamma > 0$
(5) $\mathcal{T}(\zeta, \varrho, \gamma) \bullet \mathcal{T}(\varrho, c, \rho) \leq \mathcal{T}(\zeta, c, \gamma + \rho)$
(6) $\mathcal{T}(\zeta, \varrho, \cdot) : \mathbb{R}^+ \to I$ is continuous
(7) $\lim_{\gamma \to \infty} \mathcal{T}(\zeta, \varrho, \gamma) = 1$
(8) $\mathcal{F}(\zeta, \varrho, \gamma) = 0$ iff $\zeta = \varrho$
(9) $\mathcal{F}(\zeta, \varrho, \gamma) = \mathcal{F}(\varrho, \zeta, \gamma)$,

(10) $\mathcal{F}(\zeta,\varrho,\gamma)\diamond\mathcal{F}(\varrho,c,\rho)\geq\mathcal{F}(\zeta,c,\gamma+\rho)$,

(11) $\mathcal{F}(\zeta,\varrho,\cdot):\mathbb{R}^{+}\to I$ is continuous

(12) $\lim\limits_{\gamma\to\infty}\mathcal{F}(\zeta,\varrho,\gamma)=0$

(13) $\mathcal{I}(\zeta,\varrho,\gamma)=0$, for $\gamma>0$ iff $\zeta=\varrho$

(14) $\mathcal{I}(\zeta,\varrho,\gamma)=\mathcal{I}(\varrho,\zeta,\gamma)$,

(15) $\mathcal{I}(\zeta,\varrho,\gamma)\diamond\mathcal{I}(\varrho,c,\rho)\geq S(\zeta,c,\gamma+\rho)$,

(16) $\mathcal{I}(\zeta,\varrho,\cdot):\mathbb{R}\to I$ is continuous

(17) $\lim\limits_{\gamma\to\infty}\mathcal{I}(\zeta,\varrho,\gamma)=0$

(18) If $\gamma\leq 0$, then $\mathcal{T}(\zeta,\varrho,\gamma)=0$, $\mathcal{F}(\zeta,\varrho,\gamma)=\mathcal{I}(\zeta,\varrho,\gamma)=1$

The functions $\mathcal{T}(\zeta,\varrho,\gamma)$, $\mathcal{F}(\zeta,\varrho,\gamma)$, and $\mathcal{I}(\zeta,\varrho,\gamma)$ represent the degrees of nearness, neutralness, and non-nearness between the elements ζ and ϱ in relation to the parameter γ, respectively.

The topologies concepts including convergence, Cauchy sequences, and completeness in NMS, are outlined as follows.

Definition 2.4. [42] Let (ζ_n) be a sequence in a NMS $(\mathcal{Z},\mathcal{T},\mathcal{F},\mathcal{I},\bullet,\diamond)$. Then

(1) (ζ_n) converges to $\zeta\in\mathcal{Z}$ iff for a given $\epsilon\in(0,1),\ \gamma>0$ there is $n_0\in\mathbb{N}$ such that for each $n\geq n_0$

$$\mathcal{T}(\zeta_n,\zeta,\gamma)>1-\epsilon,\ \mathcal{F}(\zeta_n,\zeta,\gamma)<\epsilon,\ \mathcal{I}(\zeta_n,\zeta,\gamma)<\epsilon$$

i.e.,

$$\lim_{n\to\infty}\mathcal{T}(\zeta_n,\zeta,\gamma)=1,\ \lim_{n\to\infty}\mathcal{F}(\zeta_n,\zeta,\gamma)=0,\ \lim_{n\to\infty}\mathcal{I}(\zeta_n,\zeta,\gamma)=0$$

(2) (ζ_n) is called Cauchy iff for a given $\epsilon\in(0,1),\ \gamma>0$ there is $n_0\in\mathbb{N}$ such that for each $n,m\geq n_0$

$$\mathcal{T}(\zeta_n,\zeta_m,\gamma)>1-\epsilon,\ \mathcal{F}(\zeta_n,\zeta_m,\gamma)<\epsilon,\ \mathcal{I}(\zeta_n,\zeta_m,\gamma)<\epsilon$$

i.e.,

$$\lim_{n.m\to\infty}\mathcal{T}(\zeta_n,\zeta_m,\gamma)=1,\ \lim_{n,m\to\infty}\mathcal{F}(\zeta_n,\zeta_m,\gamma)=0,\ \lim_{n,m\to\infty}\mathcal{I}(\zeta_n,\zeta_m,\gamma)=0$$

(3) $(\mathcal{Z},\mathcal{T},\mathcal{F},\mathcal{I},\bullet,\diamond)$ is called complete if each Cauchy sequence is convergent to an element in $\mathcal{Z}$.

In the research conducted by Simsek and Kirisci [43], the concept of NC-contractions was presented in the setting of neutrosophic metric spaces, establishing that every NC-contraction in a complete neutrosophic metric space has a distinct fixed point when certain conditions are met.

The following useful lemma presented in [44].

Lemma 2.5. *[44] Let $(\mathcal{Z}, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond)$ be an NMS. Then*

(1) $\mathcal{T}(\zeta, \varrho, \cdot) : \mathbb{R} \to \mathbb{R}$ *is non-decreasing*

(2) $\mathcal{F}(\zeta, \varrho, \cdot) : \mathbb{R} \to \mathbb{R}$ *is non-increasing*

(3) $\mathcal{I}(\zeta, \varrho, \cdot) : \mathbb{R} \to \mathbb{R}$ *is non-increasing*

Bataihah and Hazaymeh [44] established fixed point results for quasi-contractions within the framework of neutrosophic fuzzy metric spaces. Building upon this foundation, we introduce the concept of neutrosophic ψ-quasi contractions, a broader class of mappings that generalizes and unifies various contraction conditions. We prove that such contractions, under suitable assumptions, guarantee the existence of a unique fixed point. Our results not only extend several well-known types of contractions but also contribute to the ongoing development of fixed point theory in neutrosophic settings, offering potential applications in diverse mathematical and applied sciences.

3. Main Result

In this section, we introduce the concept of neutrosophic ψ-quasi contractions within the framework of complete neutrosophic metric spaces (NMS). We then establish a fixed point theorem, proving the existence and uniqueness of fixed points for such mappings.

To facilitate our analysis, we first present the following necessary notations and definitions: If $\mathcal{G} : \mathcal{Z}^2 \times \mathbb{R} \to \mathbb{R}$, and f is a self-map on $\mathcal{Z}$, then

(1) $\mathscr{M}_{\mathcal{G},f}(\zeta, \varrho, \gamma) =$
$$\max\left\{\frac{1}{\mathcal{G}(\zeta, \varrho, \gamma)} - 1, \frac{1}{\mathcal{G}(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\mathcal{G}(\varrho, f\varrho, \gamma)} - 1, \frac{1}{\mathcal{G}(\zeta, f\varrho, \gamma)} - 1, \frac{1}{\mathcal{G}(f\zeta, \varrho, \gamma)} - 1\right\},$$

(2) $\mathscr{N}_{\mathcal{G},f}(\zeta, \varrho, \gamma) = \max\left\{\mathcal{G}(\zeta, \varrho, \gamma), \mathcal{G}(\zeta, f\zeta, \gamma), \mathcal{G}(\varrho, f\varrho, \gamma), \mathcal{G}(\zeta, f\varrho, \gamma), \mathcal{G}(f\zeta, \varrho, \gamma)\right\}.$

And

(1) $\delta_{\mathcal{G}}(\zeta, f, \gamma) = \max\limits_{i,j\in\mathbb{N}}\left\{\frac{1}{\mathcal{G}(f^i\zeta, f^j\zeta, \gamma)} - 1\right\},$

(2) $\sigma_{\mathcal{G}}(\zeta, f, \gamma) = \max\limits_{i,j\in\mathbb{N}}\left\{\mathcal{G}(f^i\zeta, f^j\zeta, \gamma)\right\}.$

Definition 3.1. [45] A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called comparison function if it fullfiled the following:

(1) Where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is monotone increasing

(2) $\lim_{n\to\infty} \psi^n(a) = 0$ for each $a > 0$.

Remark 3.2. [45] If ψ is comparison function, then

(1) $\psi(0) = 0$

(2) $\psi(a) < a$ for each $a > 0$

We will now present the concept of neutrosophic ψ-quasi contractions.

Definition 3.3. Let $(\mathcal{Z}, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond)$ be a NMS. A mapping $f : \mathcal{Z} \to \mathcal{Z}$ is called neutrosophic ψ-quasi-contraction if for each $\zeta, \varrho \in \mathcal{Z}$ and each $\gamma > 0$, we have

$$\frac{1}{\mathcal{T}(f\zeta, f\varrho, \gamma)} - 1 \leq \psi(\mathscr{M}_{\mathcal{T},f}(\zeta, \varrho, \gamma)),$$

$$\mathcal{F}(f\zeta, f\varrho, \gamma) \leq \psi(\mathscr{N}_{\mathcal{F},f}(\zeta, \varrho, \gamma)),$$

and

$$\mathcal{I}(f\zeta, f\varrho, \gamma) \leq \psi(\mathscr{N}_{\mathcal{I},f}(\zeta, \varrho, \gamma)).$$

Theorem 3.4. *Let $(\mathcal{Z}, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond)$ be a complete NMS, Suppose that $f : \mathcal{Z} \to \mathcal{Z}$ is neutrosophic ψ-quasi-contraction. Also, assume that one of the following holds true:*

(a) *f is sequential continuous i.e if $\zeta_n \to \zeta$, then $f\zeta_n \to f\zeta$,*

(b) *the fuzzy sets $\mathcal{T}, \mathcal{F}$ and $\mathcal{I}$ are continuous in the first two coordinates.*

Consequently, the function f possesses a unique fixed point.

Proof. Let $\zeta_0 \in \mathcal{Z}$ represent an arbitrary point. We examine the Picard sequence (ζ_n) characterized by the relation $\zeta_{n+1} = f^n\zeta_0$ for all $n \geq 0$. By Definition 3.3 we have for all $i, j \in \mathbb{N}$

$$\begin{aligned}\frac{1}{\mathcal{T}(f^{n+i}\zeta_0, f^{n+j}\zeta_0, \gamma)} - 1 \leq \psi \ \max\Big\{ & \frac{1}{\mathcal{T}(f^{n+i-1}\zeta_0, f^{n+j-1}\zeta_0, \gamma)} - 1, \frac{1}{\mathcal{T}(f^{n+i-1}\zeta_0, f^{n+i}\zeta_0, \gamma)} - 1, \\ & \frac{1}{\mathcal{T}(f^{n+j-1}\zeta_0, f^{n+j}\zeta_0, \gamma)} - 1, \frac{1}{\mathcal{T}(f^{n+i-1}\zeta_0, f^{n+j}\zeta_0, \gamma)} - 1, \\ & \frac{1}{\mathcal{T}(f^{n+j-1}\zeta_0, f^{n+i}\zeta_0, \gamma)} - 1 \Big\},\end{aligned}$$

$$\begin{aligned}\mathcal{F}(f^{n+i}\zeta_0, f^{n+j}\zeta_0, \gamma) \leq \psi \ \max\big\{ & \mathcal{F}(f^{n+i-1}\zeta_0, f^{n+j-1}\zeta_0, \gamma), \mathcal{F}(f^{n+i-1}\zeta_0, f^{n+i}\zeta_0, \gamma), \\ & \mathcal{F}(f^{n+j-1}\zeta_0, f^{n+j}\zeta_0, \gamma), \mathcal{F}(f^{n+i-1}\zeta_0, f^{n+j}\zeta_0, \gamma), \\ & \mathcal{F}(f^{n+j-1}\zeta_0, f^{n+i}\zeta_0, \gamma) \big\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}(f^{n+i}\zeta_0, f^{n+j}\zeta_0, \gamma) \leq \psi \ \max\big\{ & \mathcal{I}(f^{n+i-1}\zeta_0, f^{n+j-1}\zeta_0, \gamma), \mathcal{I}(f^{n+i-1}\zeta_0, f^{n+i}\zeta_0, \gamma), \\ & \mathcal{I}(f^{n+j-1}\zeta_0, f^{n+j}\zeta_0, \gamma), \mathcal{I}(f^{n+i-1}\zeta_0, f^{n+j}\zeta_0, \gamma), \\ & \mathcal{I}(f^{n+j-1}\zeta_0, f^{n+i}\zeta_0, \gamma) \big\}.\end{aligned}$$

Thus, we conclude

$$\delta_{\mathcal{T}}(f^n\zeta_0, f, \gamma) \leq \psi \ \delta_{\mathcal{T}}(f^{n-1}\zeta_0, f, \gamma),$$

$$\sigma_{\mathcal{F}}(f^n\zeta_0, f, \gamma) \leq \psi \ \sigma_{\mathcal{F}}(f^{n-1}\zeta_0, f, \gamma),$$

and

$$\sigma_{\mathcal{I}}(f^n\zeta_0, f, \gamma) \leq \psi\ \sigma_{\mathcal{I}}(f^{n-1}\zeta_0, f, \gamma).$$

Hence, we deduce that for all $n \geq 1$

$$\delta_{\mathcal{T}}(f^n\zeta_0, f, \gamma) \leq \psi^n\ \delta_{\mathcal{T}}(\zeta_0, f, \gamma),$$
$$\sigma_{\mathcal{F}}(f^n\zeta_0, f, \gamma) \leq \psi^n\ \sigma_{\mathcal{F}}(\zeta_0, f, \gamma),$$

and

$$\sigma_{\mathcal{I}}(f^n\zeta_0, f, \gamma) \leq \psi^n\ \sigma_{\mathcal{I}}(\zeta_0, f, \gamma).$$

From the above inequalities, we have

$$\frac{1}{\mathcal{T}(f^n\zeta_0, f^{n+m}\zeta_0, \gamma)} - 1 \leq \delta_{\mathcal{T}}(f^n\zeta_0, f, \gamma) \leq \psi^n\ \delta_{\mathcal{T}}(\zeta_0, f, \gamma),$$
$$\mathcal{F}(f^n\zeta_0, f^{n+m}\zeta_0, \gamma) \leq \sigma_{\mathcal{F}}(f^n\zeta_0, f, \gamma) \leq \psi^n\ \sigma_{\mathcal{F}}(\zeta_0, f, \gamma),$$

and

$$\mathcal{I}(f^n\zeta_0, f^{n+m}\zeta_0, \gamma) \leq \sigma_{\mathcal{I}}(f^n\zeta_0, f, \gamma) \leq \psi^n\ \sigma_{\mathcal{I}}(\zeta_0, f, \gamma).$$

By taking the limit when both n, m approach ∞, we get

$$\lim_{n,m\to\infty} \mathcal{T}(f^n\zeta_0, f^{n+m}\zeta_0, \gamma) = 1,$$
$$\lim_{n,m\to\infty} \mathcal{F}(f^n\zeta_0, f^{n+m}\zeta_0, \gamma) = 0,$$
$$\lim_{n,m\to\infty} \mathcal{I}(f^n\zeta_0, f^{n+m}\zeta_0, \gamma) = 0.$$

Hence $(f^n\zeta_0)$ is a Cauchy sequence, thus, there is $u \in \mathcal{Z}$ such that $f^n\zeta_0 \to u$.
mathcalIf (a) holds, then $f^{n+1}\zeta_0 = ff^n\zeta_0 \to fu$ and so, $u = fu$.
mathcalIf (b) holds, then Definition 3.3 implies that

$$\begin{aligned}\tfrac{1}{\mathcal{T}(fu, f^{n+1}\zeta_0, \gamma)} - 1 \quad &\leq \psi\ \max\Big\{ \tfrac{1}{\mathcal{T}(u, f^n\zeta_0, \gamma)} - 1, \tfrac{1}{\mathcal{T}(u, fu, \gamma)} - 1, \tfrac{1}{\mathcal{T}(f^n\zeta_0, f^{n+1}\zeta_0, \gamma)} - 1, \\ &\qquad \tfrac{1}{\mathcal{T}(u, f^{n+1}\zeta_0, \gamma)} - 1, \tfrac{1}{\mathcal{T}(f^n\zeta_0, fu, \gamma)} - 1 \Big\},\end{aligned}$$

$$\begin{aligned}\mathcal{F}(fu, f^{n+1}\zeta_0, \gamma) \quad &\leq \psi\ \max\big\{\mathcal{F}(u, f^n\zeta_0, \gamma), \mathcal{F}(u, fu, \gamma), \mathcal{F}(f^n\zeta_0, f^{n+1}\zeta_0, \gamma), \\ &\qquad \mathcal{F}(u, f^{n+1}\zeta_0, \gamma), \mathcal{F}(f^n\zeta_0, fu, \gamma) \big\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}(fu, f^{n+1}\zeta_0, \gamma) \quad &\leq \psi \ \max\{\mathcal{I}(u, f^n\zeta_0, \gamma), \mathcal{I}(u, fu, \gamma), \mathcal{I}(f^n\zeta_0, f^{n+1}\zeta_0, \gamma),\\ &\qquad \mathcal{I}(u, f^{n+1}\zeta_0, \gamma), \mathcal{I}(f^n\zeta_0, fu, \gamma)\}.\end{aligned}$$

By taking the limit, we get

$$\frac{1}{\mathcal{T}(fu, u, \gamma)} - 1 \leq \psi(\frac{1}{\mathcal{T}(fu, u, \gamma)} - 1),$$

$$\mathcal{F}(fu, u, \gamma) \leq \psi(\mathcal{F}(fu, u, \gamma)),$$

$$\mathcal{I}(fu, u, \gamma) \leq \psi(\mathcal{I}(fu, u, \gamma)).$$

If one of $\frac{1}{\mathcal{T}(fu,u,\gamma)} - 1, \mathcal{F}(fu, u, \gamma), \mathcal{I}(fu, u, \gamma)$ not equal 0, then

$$\frac{1}{\mathcal{T}(fu, u, \gamma)} - 1 < \frac{1}{\mathcal{T}(fu, u, \gamma)} - 1,$$

or

$$\mathcal{F}(fu, u, \gamma) < \mathcal{F}(fu, u, \gamma),$$

or

$$\mathcal{I}(fu, u, \gamma) < \mathcal{I}(fu, u, \gamma).$$

Which leads to a contradiction in each case. Hence $u = fu$.

Now, let $w \in \mathcal{Z}$ with $w = fw$. If $u \neq w$, then from Definition 3.3, it follows that

$$\begin{aligned}\tfrac{1}{\mathcal{T}(u,w,\gamma)} - 1 = \tfrac{1}{\mathcal{T}(fu,fw,\gamma)} - 1 \quad &\leq \psi \ \max\Big\{\tfrac{1}{\mathcal{T}(u,w,\gamma)} - 1, \tfrac{1}{\mathcal{T}(u,u,\gamma)} - 1, \tfrac{1}{\mathcal{T}(w,w,\gamma)} - 1,\\ &\qquad \tfrac{1}{\mathcal{T}(u,w,\gamma)} - 1, \tfrac{1}{\mathcal{T}(w,u,\gamma)} - 1\Big\}\\ &= \psi \ \left(\tfrac{1}{\mathcal{T}(u,w,\gamma)} - 1\right),\end{aligned}$$

$$\begin{aligned}\mathcal{F}(u, w, \gamma) = \mathcal{F}(fu, fw, \gamma) \quad &\leq \psi \ \max\{\mathcal{F}(u, w, \gamma), \mathcal{F}(u, u, \gamma), \mathcal{F}(w, w, \gamma),\\ &\qquad \mathcal{F}(u, w, \gamma), \mathcal{F}(w, u, \gamma)\}\\ &= \psi \ \mathcal{F}(w, u, \gamma),\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}(u, w, \gamma) = \mathcal{I}(fu, fw, \gamma) \quad &\leq \psi \ \max\{\mathcal{I}(u, w, \gamma), \mathcal{I}(u, u, \gamma), \mathcal{I}(w, w, \gamma),\\ &\qquad \mathcal{I}(u, w, \gamma), \mathcal{I}(w, u, \gamma)\}\\ &= \psi \ \mathcal{I}(w, u, \gamma).\end{aligned}$$

Therefore, by the same argument as in the previous step, we conclude that $u = w$. □

4. Examples and Consequences

In this section, we provide some of illustrative examples to demonstrate the applicability of our primary theorem in various contexts. These examples will highlight the significance of the conditions imposed in the theorem and showcase different scenarios where the results hold. Following this, we derive several interesting consequences and related findings, further emphasizing the theoretical and practical implications of our main result.

Example 4.1. Let $\mathcal{Z} = [0, 1]$ and define the neutrosophic metric $\mathcal{T}, \mathcal{F}, \mathcal{I}$ as follows:

$$\mathcal{T}(\zeta, \varrho, \gamma) = \frac{\gamma}{\gamma + |\zeta - \varrho|},$$
$$\mathcal{F}(\zeta, \varrho, \gamma) = \frac{|\zeta - \varrho|}{\gamma + |\zeta - \varrho|},$$
$$\mathcal{I}(\zeta, \varrho, \gamma) = \frac{|\zeta - \varrho|}{\gamma + |\zeta - \varrho|}.$$

Then, $(\mathcal{Z}, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond)$ is a complete NMS where $\bullet$ and $\diamond$ are as follows: $\bullet(a, b) = \min(a, b)$ and $\diamond(a, b) = \max(a, b)$.

Define the mapping $f : \mathcal{Z} \to \mathcal{Z}$ as: $f(\zeta) = \frac{\zeta}{2}$.

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $\psi(a) = \frac{a}{2}$. Then, ψ is a comparison function.

Proof. Now, we verify the conditions for f being a neutrosophic ψ-quasi-contraction.

For any $\zeta, \varrho \in \mathcal{Z}$ and any $\gamma > 0$, we have

$$\frac{1}{\mathcal{T}(f\zeta, f\varrho, \gamma)} - 1 = \frac{|\zeta/2 - \varrho/2|}{\gamma} = \frac{|\zeta - \varrho|}{2\gamma}.$$

On the other hand:

$$\mathcal{M}_{\mathcal{T},f}(\zeta, \varrho, \gamma) = \max\left\{ \frac{1}{\mathcal{T}(\zeta, \varrho, \gamma)} - 1, \frac{1}{\mathcal{T}(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\mathcal{T}(\varrho, f\varrho, \gamma)} - 1, \frac{1}{\mathcal{T}(\zeta, f\varrho, \gamma)} - 1, \frac{1}{\mathcal{T}(f\zeta, \varrho, \gamma)} - 1 \right\}.$$

Since f is a contraction, it can be shown that:

$$\frac{1}{\mathcal{T}(f\zeta, f\varrho, \gamma)} - 1 \leq \psi\left(\mathcal{M}_{\mathcal{T},f}(\zeta, \varrho, \gamma)\right).$$

Similarly, we can verify:

$$\mathcal{F}(f\zeta, f\varrho, \gamma) \leq \psi\left(\mathcal{N}_{\mathcal{F},f}(\zeta, \varrho, \gamma)\right),$$

$$\mathcal{I}(f\zeta, f\varrho, \gamma) \leq \psi\left(\mathcal{N}_{\mathcal{I},f}(\zeta, \varrho, \gamma)\right).$$

Moreover, the mapping f is sequential continuous because if $\zeta_n \to \zeta$, then $f\zeta_n = \frac{\zeta_n}{2} \to \frac{\zeta}{2} = f\zeta$. Additionally, the fuzzy sets $\mathcal{T}, \mathcal{F}, \mathcal{I}$ are continuous in the first two coordinates.

Hence, by Theorem 3.4, the mapping f has a unique fixed point. In this case, the fixed point is $\zeta = 0$, since $f(0) = 0$. □

Example 4.2. Let $\mathcal{Z} = \left\{\frac{1}{3^n} : n \in \mathbb{N}\right\} \cup \{0\}$ with the standard metric by $d(\zeta, \rho) = |\zeta - \rho|$. Also, Let the t-norm and t-conorm be defined as follows $\zeta \diamond \rho = \min\{\zeta, \rho\}$, $\zeta \bullet \rho = \max\{\zeta, \rho\}$. Additionally, let the fuzzy sets be defined as follows:

$$\mathcal{T}(\zeta, \rho, \gamma) = \frac{\gamma}{\gamma + d(\zeta, \rho)},\ \mathcal{F}(\zeta, \rho, \gamma) = \frac{d(\zeta, \rho)}{\gamma + d(\zeta, \rho)},\ \mathcal{I}(\zeta, \rho, \gamma) = \frac{d(\zeta, \rho)}{d(\zeta, \rho) + \gamma}.$$

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $\psi(a) = \frac{1}{2}a$.
Define $f : \mho \to \mho$ by

$$f(\zeta) = \begin{cases} 0 & ,\ \zeta = 0 \\ \frac{1}{3^{n+1}} & ,\ \zeta = \frac{1}{3^n} \end{cases}$$

Then, f, has a unique fixed point.

Proof. From [42], the six-tuple $(\mathcal{Z}, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond)$ represents a complete NMS. Now, for any elements ζ and ρ in $\mathcal{Z}$ and for any $\gamma > 0$, we have.

$$\begin{aligned} \mathscr{M}_{\mathcal{T},f}(\zeta, \varrho, \gamma) &= \max\left\{ \frac{1}{\mathcal{T}(\zeta, \varrho, \gamma)} - 1, \frac{1}{\mathcal{T}(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\mathcal{T}(\varrho, f\varrho, \gamma)} - 1, \right. \\ &\qquad \left. \frac{1}{\mathcal{T}(\zeta, f\varrho, \gamma)} - 1, \frac{1}{\mathcal{T}(f\zeta, \varrho, \gamma)} - 1 \right\} \\ &= \max\left\{ \frac{|\zeta - \varrho|}{\gamma}, \frac{|\zeta - f\zeta|}{\gamma}, \frac{|\varrho - f\varrho|}{\gamma}, \frac{|\zeta - f\varrho|}{\gamma}, \frac{|f\zeta - \varrho|}{\gamma} \right\}. \end{aligned}$$

If $\zeta = \varrho = 0$, then $0 = \left(\frac{1}{\mathcal{T}(f\zeta, f\varrho, \gamma)} - 1\right) \leq \psi(\mathscr{M}_{\mathcal{T},f}(\zeta, \varrho, \gamma)) = 0$.

If $\zeta = 0,\ \varrho = \frac{1}{3^m}$, then

$$\frac{|\frac{1}{3^{m+1}}|}{\gamma} = \left(\frac{1}{\mathcal{T}(f\zeta, f\varrho, \gamma)} - 1\right) \leq \psi(\mathscr{M}_{\mathcal{T},f}(\zeta, \varrho, \gamma)) = \frac{1}{2}\max\left\{ \frac{|\frac{1}{3^m}|}{\gamma}, \frac{|\frac{2}{3^{m+1}}|}{\gamma}, \frac{|\frac{1}{3^{m+1}}|}{\gamma} \right\}.$$

If $\zeta = \frac{1}{3^n},\ \varrho = \frac{1}{3^m}$, then

$$\left(\frac{1}{\mathcal{T}(f\zeta, f\varrho, \gamma)} - 1\right) = \frac{1}{3}\frac{|\frac{1}{3^m} - \frac{1}{3^n}|}{\gamma}.$$

Also,

$$\psi(\mathscr{M}_{\mathcal{T},f}(\zeta, \varrho, \gamma)) = \frac{1}{2}\max\left\{ \frac{|\frac{1}{3^m} - \frac{1}{3^n}|}{\gamma}, \frac{|\frac{2}{3^{n+1}}|}{\gamma}, \frac{|\frac{2}{3^{m+1}}|}{\gamma}, \frac{|\frac{1}{3^n} - \frac{1}{3^{m+1}}|}{\gamma}, \frac{|\frac{1}{3^m} - \frac{1}{3^{n+1}}|}{\gamma} \right\}.$$

So,

$$\left(\frac{1}{\mathcal{T}(f\zeta, f\varrho, \gamma)} - 1\right) \leq \psi(\mathscr{M}_{\mathcal{T},f}(\zeta, \varrho, \gamma)).$$

Also, we have

$$\psi(\mathscr{N}_{\mathcal{F},f}(\zeta,\varrho,\gamma)) = \frac{1}{2}\max\{\mathcal{F}(\zeta,\varrho,\gamma),\mathcal{F}(\zeta,f\zeta,\gamma),\mathcal{F}(\varrho,f\varrho,\gamma),\mathcal{F}(\zeta,f\varrho,\gamma),\mathcal{F}(f\zeta,\varrho,\gamma)\}$$

$$= \frac{1}{2}\max\left\{\frac{|\zeta-\varrho|}{\gamma+|\zeta-\varrho|},\frac{|\zeta-f\zeta|}{\gamma+|\zeta-f\zeta|},\frac{|\varrho-f\varrho|}{\gamma+|\varrho-f\varrho|},\frac{|\zeta-f\varrho|}{\gamma+|\zeta-f\varrho|},\frac{|f\zeta-\varrho|}{\gamma+|f\zeta-\varrho|}\right\}.$$

If $\zeta=\varrho=0$, then $0=\mathcal{F}(f\zeta,f\varrho,\gamma)\leq\psi(\mathscr{N}_{\mathcal{F},f}(\zeta,\varrho,\gamma))=0$.

If $\zeta=0,\ \varrho=\frac{1}{3^m}$, then

$$\frac{|\frac{1}{3^{m+1}}|}{\gamma+|\frac{1}{3^{m+1}}|}=\mathcal{F}(f\zeta,f\varrho,\gamma)\leq\psi(\mathscr{N}_{\mathcal{F},f}(\zeta,\varrho,\gamma))=\frac{1}{2}\max\left\{\frac{|\frac{1}{3^m}|}{\gamma+|\frac{1}{3^m}|},\frac{|\frac{2}{3^{m+1}}|}{\gamma+|\frac{2}{3^{m+1}}|},\frac{|\frac{1}{3^{m+1}}|}{\gamma+|\frac{1}{3^{m+1}}|}\right\}.$$

If $\zeta=\frac{1}{3^n},\ \varrho=\frac{1}{3^m}$, then

$$\mathcal{F}(f\zeta,f\varrho,\gamma)=\frac{|\frac{1}{3^{m+1}}-\frac{1}{3^{n+1}}|}{\gamma+|\frac{1}{3^{m+1}}-\frac{1}{3^{n+1}}|}.$$

Also,

$$\psi(\mathscr{N}_{\mathcal{F},f}(\zeta,\varrho,\gamma))=\frac{1}{2}\max\left\{\frac{|\frac{1}{3^m}-\frac{1}{3^n}|}{\gamma+|\frac{1}{3^m}-\frac{1}{3^n}|},\frac{|\frac{2}{3^{n+1}}|}{\gamma+|\frac{2}{3^{n+1}}|},\frac{|\frac{2}{3^{m+1}}|}{\gamma+|\frac{2}{3^{m+1}}|},\frac{|\frac{1}{3^n}-\frac{1}{3^{m+1}}|}{\gamma+|\frac{1}{3^n}-\frac{1}{3^{m+1}}|},\frac{|\frac{1}{3^m}-\frac{1}{3^{n+1}}|}{\gamma+|\frac{1}{3^m}-\frac{1}{3^{n+1}}|}\right\}.$$

So,

$$\mathcal{F}(f\zeta,f\varrho,\gamma)\leq\psi(\mathscr{N}_{\mathcal{F},f}(\zeta,\varrho,\gamma)).$$

The remaining portion can be illustrated in a similar fashion. □

The following corollaries are direct consequences of the main theorem, extending its applicability to specific cases and providing further insights into its implications.

Corollary 4.3 (Fixed Point for Quasi-Contractions)**.** *Let $(\mathcal{Z},\mathcal{T},\mathcal{F},\mathcal{I},\bullet,\diamond)$ be a complete NMS, Suppose that there is $k\in[0,1)$ such that for each $\zeta,\varrho\in\mathcal{Z}$ and each $\gamma>0$ the self map $f:\mathcal{Z}\to\mathcal{Z}$ satisfies the following:*

$$\frac{1}{\mathcal{T}(f\zeta,f\varrho,\gamma)}-1\leq k\ \mathscr{M}_{\mathcal{T},f}(\zeta,\varrho,\gamma),$$

$$\mathcal{F}(f\zeta,f\varrho,\gamma)\leq k\ \mathscr{N}_{\mathcal{F},f}(\zeta,\varrho,\gamma),$$

and

$$\mathcal{I}(f\zeta,f\varrho,\gamma)\leq k\ \mathscr{N}_{\mathcal{I},f}(\zeta,\varrho,\gamma).$$

Also, assume that one of the following holds true:

(1) *f is sequential continuous,*

(2) *the fuzzy sets $\mathcal{T},\mathcal{F}$ and $\mathcal{I}$ are continuous in the first two coordinates.*

Therefore, the function f has a unique fixed point.

Corollary 4.4 (Fixed Point for Contraction Mappings). *Let $(\mathcal{Z}, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond)$ be a complete NMS, Suppose that there is $k \in [0,1)$ such that for each $\zeta, \varrho \in \mathcal{Z}$ and each $\gamma > 0$ the self map $f : \mathcal{Z} \to \mathcal{Z}$ satisfies the following:*

$$\frac{1}{\mathcal{T}(f\zeta, f\varrho, \gamma)} - 1 \leq k \left(\frac{1}{\mathcal{T}(\zeta, \varrho, \gamma)} - 1 \right),$$
$$\mathcal{F}(f\zeta, f\varrho, \gamma) \leq k\, \mathcal{F}(\zeta, \varrho, \gamma),$$

and

$$\mathcal{I}(f\zeta, f\varrho, \gamma) \leq k\, \mathcal{I}(\zeta, \varrho, \gamma).$$

Also, assume that one of the following holds true:

(1) *f is sequential continuous,*

(2) *the fuzzy sets $\mathcal{T}, \mathcal{F}$ and $\mathcal{I}$ are continuous in the first two coordinates.*

Therefore, the function f has a unique fixed point.

Corollary 4.5 (Fixed Point for Kannan-Type Contractions). *Let $(\mathcal{Z}, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond)$ be a complete NMS, Suppose that there is $k \in [0,1)$ such that for each $\zeta, \varrho \in \mathcal{Z}$ and each $\gamma > 0$ the self map $f : \mathcal{Z} \to \mathcal{Z}$ satisfies the following:*

$$\frac{1}{\mathcal{T}(f\zeta, f\varrho, \gamma)} - 1 \leq \frac{k}{2} \left(\frac{1}{\mathcal{T}(\zeta, f\zeta, \gamma)} - 1 + \frac{1}{\mathcal{T}(\varrho, f\varrho, \gamma)} - 1 \right),$$
$$\mathcal{F}(f\zeta, f\varrho, \gamma) \leq \frac{k}{2} \left(\mathcal{F}(\zeta, f\zeta, \gamma) + \mathcal{F}(\varrho, f\varrho, \gamma) \right),$$

and

$$\mathcal{I}(f\zeta, f\varrho, \gamma) \leq \frac{k}{2} \left(\mathcal{I}(\zeta, f\zeta, \gamma) + \mathcal{I}(\varrho, f\varrho, \gamma) \right).$$

Also, assume that one of the following holds true:

(1) *f is sequential continuous,*

(2) *the fuzzy sets $\mathcal{T}, \mathcal{F}$ and $\mathcal{I}$ are continuous in the first two coordinates.*

Therefore, the function f has a unique fixed point.

Corollary 4.6 (Fixed Point for Chatterjea type contraction). *Let $(\mathcal{Z}, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond)$ be a complete NMS, Suppose that there is $k \in [0,1)$ such that for each $\zeta, \varrho \in \mathcal{Z}$ and each $\gamma > 0$ the self map $f : \mathcal{Z} \to \mathcal{Z}$ satisfies the following:*

$$\frac{1}{\mathcal{T}(f\zeta, f\varrho, \gamma)} - 1 \leq \frac{k}{2} \left(\frac{1}{\mathcal{T}(\zeta, f\varrho, \gamma)} - 1 + \frac{1}{\mathcal{T}(\zeta, f\varrho, \gamma)} - 1 \right),$$
$$\mathcal{F}(f\zeta, f\varrho, \gamma) \leq \frac{k}{2} \left(\mathcal{F}(\zeta, f\varrho, \gamma) + \mathcal{F}(\varrho, f\zeta, \gamma) \right),$$

and

$$\mathcal{I}(f\zeta, f\varrho, \gamma) \leq \frac{k}{2} \left(\mathcal{I}(\zeta, f\varrho, \gamma) + \mathcal{I}(\varrho, f\zeta, \gamma) \right).$$

Also, assume that one of the following holds true:

(1) *f is sequential continuous,*

(2) *the fuzzy sets $\mathcal{T}, \mathcal{F}$ and $\mathcal{I}$ are continuous in the first two coordinates.*

Therefore, the function f has a unique fixed point.

5. Conclusions

The fixed point theory is a crucial tool in mathematics and various scientific fields. This paper discusses fixed point results in the context of complete neutrosophic metric spaces, focusing on the concept of neutrosophic ψ-quasi contractions, with an example provided to demonstrate the main theorem. Additionally, we build upon previous findings documented in the literature. Future research may aim to generalize the primary results to encompass broader neutrosophic metric spaces and other abstract spaces. Another avenue for future investigation could involve exploring applications to fractional differential equations through the lens of neutrosophic sets, as referenced in [46–48].

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Neutrosophic Z-Number and HyperSoft for Integrated Energy Supply System Performance Inference of Solar Energy and Cogenerations

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Abstract: The increasing global demand for sustainable energy solutions has led to the development of integrated energy supply systems that combine solar energy and cogeneration technologies. So, this study proposes a decision-making approach to evaluate The Energy Supply System Performance of Solar Energy and Cogenerations. Two methods are applied in this study, such as SIWEC to compute the criteria weights and EDAS method to rank the alternatives. These methods are used under the neutrosophic Z-number (NZN). We used the HyperSoft set to deal with various criteria and sub criteria. This study applied the NZN model with nine criteria and eight alternatives. Three experts can evaluate these criteria and alternatives. We applied the sensitivity analysis under nine different cases. The results showed that the ranks of the alternatives are stable in different cases.

Keywords: Neutrosophic Z-Number; HyperSoft; Solar Energy and Cogenerations; Energy Supply System.

1. Introduction and Literature Review

The increasing global demand for sustainable energy solutions has led to the development of integrated energy supply systems that combine solar energy and cogeneration technologies. These hybrid systems aim to improve energy efficiency, reduce carbon emissions, and ensure a reliable and continuous power supply. Solar energy, as a clean and renewable resource, offers a sustainable alternative to fossil fuels, while cogeneration (or combined heat and power, CHP) maximizes energy utilization by simultaneously generating electricity and useful thermal energy. However, the performance of such integrated systems must be systematically evaluated to ensure optimal energy efficiency, economic feasibility, and environmental benefits[1], [2].

To assess the effectiveness of an integrated solar-cogeneration system, a comprehensive evaluation framework is required. Various factors, including energy efficiency, renewable energy utilization, carbon emissions reduction, system reliability, operational costs, scalability, and smart grid compatibility, must be considered. These criteria help determine the suitability and viability of different system configurations for various applications, such as residential, industrial, and commercial energy supply. Since different combinations of solar PV, solar thermal, and cogeneration technologies offer varying levels of performance, a structured multi-criteria decision-making (MCDM) approach is essential for effective evaluation[3], [4].

Several MCDM methods provide a systematic way to compare and rank alternative energy supply models based on multiple performance indicators. These methods facilitate data-driven decision-making by balancing technical, economic, and environmental aspects to identify the most efficient and sustainable system configuration. For instance, systems with high renewable energy utilization may have longer payback periods, while systems with lower operational costs might exhibit lower carbon emission reduction potential. By integrating MCDM techniques, policymakers and energy planners can optimize energy system designs that align with sustainability goals and economic constraints.

The future of integrated energy supply systems lies in continuous innovation and technological advancement. The incorporation of smart grid technologies, advanced energy storage solutions, and AI-driven energy management systems can further enhance the performance of solar-cogeneration systems. By conducting a comprehensive performance evaluation, stakeholders can ensure that integrated energy supply systems not only improve energy efficiency but also contribute to a more sustainable and resilient energy infrastructure. As the world transitions towards low-carbon energy solutions, the evaluation of such hybrid systems will play a crucial role in shaping the future of sustainable energy production[5], [6].

To represent a fuzzy value and a dependability metric in an uncertain and imprecise situation[7], Zadeh originally proposed a Z-number in 2011 as an extension of the fuzzy set notion[8]. An order pair of fuzzy numbers (N, M) with respect to an uncertain variable U make up the Z-number; component N is a fuzzy value of U, and component M is a reliability metric that is strongly associated with N[9]. The Z-number demonstrates the superiority of improving the fuzzy value's dependability in ambiguous and uncertain settings as compared to the conventional fuzzy set[10]. Consequently, Z-numbers have found utility in a variety of applications, including the study of stable strategies for evolutionary games, sensor data fusion, and decision making (DM)[11].

The Z-number loses the indeterminacy and falsity fuzzy values and reliability measures in the inconsistent, incomplete, and indeterminate environment, while being characterized by an order pair of fuzzy numbers (a fuzzy value and a reliability measure). While SNS (SVNS and INS) only include the truth, indeterminacy, and falsity membership degrees without considering their reliability measures, the Z-number is unable to express the hybrid information of the truth, indeterminacy, and falsity Z-numbers [12]. The truth, indeterminacy, and falsity Z-numbers may

be used to convey the inconsistent and ambiguous knowledge and judgments of human cognitions if the Z-number is added to the neutrosophic idea. This is known as the neutrosophic Z-number (NZN) concept [13], [14].

In an uncertain context, Zadeh initially proposed a Z-number using a reliability measure and a fuzzy value. The fuzzy value's tight relationship to the dependability measure is its main feature. Nevertheless, the Z-number is devoid of reliability metrics and fuzzy values for indeterminacy and falsehood. The truth, indeterminacy, and falsity degrees define a single-valued neutrosophic set, however it does not have the reliability measures of these membership degrees. A neutrosophic Z-number (NZN) can be presented to convey the inconsistent, incomplete, and indeterminate knowledge and judgments of human cognitions in the actual world by using hybrid information of the truth, indeterminacy, and falsity Z-numbers.

There is no similarity measure of any NZN set in the body of extant research, even though it is a crucial mathematical tool for pattern identification and DM. Inspired by this notion, Jun Ye [15] offered cosine and cotangent similarity measures based on the weighted generalized distance of NZN sets after first proposing NZN sets and generalized distance and similarity measures between NZN sets. The suggested similarity metrics are then used to create an MCDM approach in the context of NZN sets.

The viability and logic of the suggested approach are illustrated with an example, and the comparative analysis then highlights the two main benefits: In addition to improving the reliability of DM problems, the established DM method based on the suggested similarity measures also demonstrates its superiority over current neutrosophic DM methods. (a) NZN enriches the neutrosophic information expression through the truth, indeterminacy, and falsity Z-numbers.

This paper's structure is built as follows: A NZN set, and its operational links with HyperSoft Sets are introduced in Sect. 2. In the context of NZN sets, an MCDM approach utilizing the suggested NZN is constructed in Sect. 3. The feasibility and effectiveness of the established DM method in the context of NZN sets are demonstrated in Sect. 4 using an illustrative example on the evaluation problem of MCDM. Conclusion is shown in Sect. 5

2. Neutrosophic Z-Number Sets (NZN)

With respect to an uncertain variable U, a Z-number is represented by an order pair of fuzzy numbers (N, M), where M is a reliability measure with respect to N and N is a fuzzy value of U. Consequently, it is devoid of both the falsehood and indeterminacy Z-number information. A NZN set is the extension of the Z-number that may be used to convey hybrid information of the truth, indeterminacy, and falsity Z-numbers [15].

Definition 1

The NZN can be defined as:

$$S_Z = \{(T(N,M)(u), I(N,M)(u), F(N,M)(u)) | u \in U\} \quad (1)$$

$$0 \leq T(N)(u) + I(N)(u) + F(N)(u) \leq 3 \quad (2)$$

$$0 \leq T(M)(u) + I(M)(u) + F(M)(u) \leq 3 \quad (3)$$

Definition 2

The component of NZN is presented as:

$$S_Z = \{(T(N,M), I(N,M), F(N,M))\} \quad (4)$$

$$S_Z = \{(TN, TM), (IN, IM), (FN, FM)\} \quad (5)$$

Definition 3

Let $S_{Z_1} = \begin{pmatrix} T_1(N,M), I_1(N,M), F_1(N,M) \\ ((TN_1, TM_1), (IN_1, IM_1), (FN_1, FM_1)) \end{pmatrix}$ and $S_{Z_2} = \begin{pmatrix} T_2(N,M), I_2(N,M), F_2(N,M) \\ ((TN_2, TM_2), (IN_2, IM_2), (FN_2, FM_2)) \end{pmatrix}$ be two NZNs and their operations can be defined as:

$$S_{Z_1} \sqsupseteq S_{Z_2} \Leftrightarrow TN_1 \geq TN_2, TM_1 \geq TM_2, IN_1 \leq IN_2, IM_1 \leq IM_2, FN_1 \leq FN_2, FM_1 \leq FM_2 \quad (6)$$

$$S_{Z_1} = S_{Z_2} \Leftrightarrow S_{Z_1} \sqsupseteq S_{Z_2} \; and \; S_{Z_2} \sqsupseteq S_{Z_1} \quad (7)$$

$$S_{Z_1} \cup S_{Z_2} = \begin{pmatrix} (TN_1 \vee TN_2, TM_1 \vee TM_2), \\ (IN_1 \wedge IN_2, IM_1 \wedge IM_2), \\ (FN_1 \wedge FN_2, FM_1 \wedge FM_2) \end{pmatrix} \quad (8)$$

$$S_{Z_1} \cap S_{Z_2} = \begin{pmatrix} (TN_1 \wedge TN_2, TM_1 \wedge TM_2), \\ (IN_1 \vee IN_2, IM_1 \vee IM_2), \\ (FN_1 \vee FN_2, FM_1 \vee FM_2) \end{pmatrix} \quad (9)$$

$$(S_{Z_1})^c = \begin{pmatrix} (FN_1, FM_1), \\ (1 - IN_1, 1 - IM_1), \\ (TN_1, TM_1) \end{pmatrix} \quad (10)$$

Definition 4

We can define the HyperSoft (HS) set as

Let X be the universal set and P(X) is the power set of X. Let $G^1, G^2, G^3, \ldots G^n$ be a set of criteria for $n \geq 1$and the corresponding values are:

$S^1, S^2, S^3, \ldots S^n$, $S^i \cap S^j = \emptyset \; for \; i \neq j$ and the relation $S^1 \times S^2 \times S^3 \ldots . S^n = A$

A pair (F, A) is a NHS over X where $(F, S^1 \times S^2 \times S^3 \ldots . S^n) = S^1 \times S^2 \times S^3 \ldots . S^n \rightarrow P(X)$ with $F(S^1 \times S^2 \times S^3 \ldots . S^n) = \{T(F(A)), I(F(A)), F(F(A))\}$ where T,I, and F refer to truth, indeterminacy, and falsity functions[16], [17].

3. NZN- SIWEC-EDAS Approach

This section shows the steps of the proposed approach. We used two methods, such as SIWEC method to compute the criteria weights and EDAS method to rank the alternatives. These methods are used under the NZN to deal with vague and uncertainty information.

Determine the importance of the criteria.

The rate of experts and decision makers are evaluated the criteria to determine their importance.

Build the decision matrix.

Experts build the decision matrix based on their opinions.

Normalize the decision matrix

$$u_{ij} = \frac{x_{ij}}{\max x_{ij}} \tag{11}$$

Computing the standard deviation

Multiply the standard deviation by the normalization values.

$$d_{ij} = u_{ij} \times \text{standard deviation} \tag{12}$$

Compute the sum of each row

$$r_j = \sum_{j=1}^{n} d_{ij} \tag{13}$$

Compute the criteria weights.

$$W_j = \frac{r_j}{\sum_{j=1}^{n} r_{ij}} \tag{14}$$

NZN-EDAS Method

We apply the steps of the EDAS method to rank the alternatives.

Determine the average solution of each criterion

$$V_j = \frac{\sum_{i=1}^{m} x_{ij}}{m} \tag{15}$$

Compute the positive distance from average and negative distance from average

$$PV_{ij} = \frac{\max\left(0,(x_{ij} - V_j)\right)}{V_j} \tag{16}$$

$$CV_{ij} = \frac{\max\left(0,(V_j - x_{ij})\right)}{V_j} \tag{17}$$

And for cost criteria

$$PV_{ij} = \frac{\max\left(0,(V_j - x_{ij})\right)}{V_j} \quad (18)$$

$$CV_{ij} = \frac{\max\left(0,(x_{ij} - V_j)\right)}{V_j} \quad (19)$$

Compute the weighted $PV_{ij}\ and\ CV_{ij}$

$$SPV_{ij} = \sum_{j=1}^{n} PV_{ij}\, w_j \quad (20)$$

$$SCV_{ij} = \sum_{j=1}^{n} CV_{ij}\, w_j \quad (21)$$

Compute the weighted normalized $PV_{ij}\ and\ CV_{ij}$

$$NSPV_{ij} = \frac{SPV_{ij}}{\max(SPV_{ij})} \quad (22)$$

$$NSCV_{ij} = \frac{SCV_{ij}}{\max(SCV_{ij})} \quad (23)$$

Compute the appraisal score

$$F_i = \frac{NSCV_{ij} + NSPV_{ij}}{2} \quad (24)$$

4. An Illustrative Example

This section shows the results of the proposed approach to compute the criteria weights and ranking the alternatives. Three experts evaluated the criteria and alternatives. We gathered nine criteria and eight alternatives to be evaluated. The criteria of this study presented as: Smart Grid Compatibility (Limited, Moderate, High), Operational Cost (High, Moderate, Low), Payback Period (Long, Moderate, Short), Energy Efficiency (Low, Moderate, High), Scalability and Adaptability (Limited, Moderate, Flexible), System Reliability (Low, Moderate, High), Renewable Energy Utilization (Low, Moderate, High), Energy Storage Capability (Weak, Moderate, Strong), Carbon Emissions Reduction (Ineffective, Moderate, Effective). The alternatives of this study are presented as: Solar PV with Wind, Standalone Solar PV System, Solar PV with Biomass Cogeneration, Solar Thermal with Steam Turbine Cogeneration, Hybrid Solar PV, Solar PV with Gas-Based Cogeneration, Fully Integrated Multi-Source System, Solar PV with Geothermal Cogeneration

We build the decision matrix by using the NZN as shown in Table 1.

Eq. (11) is used to normalize the decision matrix as shown in Table 2.

We compute the standard deviation

We multiply the standard deviation by the normalization values using Eq. (12).

Then we compute the sum of each row using Eq. (13).

Then we compute the criteria weights using Eq. (14) as $C_1 = 0.088$, $C_2 = 0.111$, $C_3 = 0.091$, $C_4 = 0.144$, $C_5 = 0.103$, $C_6 = 0.115$, $C_7 = 0.138$, $C_8 = 0.092$, $C9_1 = 0.118$.

Table 1. The values of NZNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.76,0.75),(0.2,0.7),(0.1,0.8))
A_2	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))
A_3	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))
A_4	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))
A_5	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.8,0.8),(0.1,0.8),(0.2,0.8))
A_6	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.76,0.75),(0.2,0.7),(0.1,0.8))
A_7	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.8,0.8),(0.1,0.8),(0.2,0.8))
A_8	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.88,0.75),(0.2,0.8),(0.2,0.8))
A_2	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))
A_3	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))
A_4	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))
A_5	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.8,0.8),(0.1,0.8),(0.2,0.8))
A_6	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))
A_7	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.8,0.8),(0.1,0.8),(0.2,0.8))
A_8	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.76,0.75),(0.2,0.7),(0.1,0.8))
A_2	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))
A_3	((0.8,0.8),(0.1,0.8),(0.2,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.65,0.9),(0.2,0.7),(0.2,0.8))
A_4	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))
A_5	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))

A_6	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.85,0.75),(0.1,0.7),(0.1,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.7,0.8),(0.2,0.7),(0.2,0.85))
A_7	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.65,0.9),(0.2,0.7),(0.2,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.76,0.75),(0.2,0.7),(0.1,0.8))	((0.7,0.8),(0.2,0.7),(0.2,0.85))
A_8	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.88,0.75),(0.2,0.8),(0.2,0.8))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.6,0.85),(0.1,0.8),(0.2,0.9))	((0.7,0.8),(0.2,0.7),(0.2,0.85))	((0.88,0.75),(0.2,0.8),(0.2,0.8))

Table 2. The normalization values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.976584	0.922761	0.944215	1	0.973528	0.943384	0.904523	0.950413	0.943384
A_2	0.96832	0.975735	0.971074	0.904523	0.955068	1	1	0.977961	0.904523
A_3	0.950413	0.948052	0.944215	1	0.940439	0.929313	0.945729	0.983127	0.918593
A_4	1	0.950103	0.962466	0.943384	0.996169	0.929983	0.904523	1	1
A_5	0.972107	0.955571	1	0.956449	1	0.917923	0.942044	0.939394	0.944724
A_6	0.929752	1	0.955234	0.936013	0.957854	0.954439	0.904523	0.943526	0.931323
A_7	0.96832	0.932331	0.990014	0.924623	0.945315	0.941039	0.929983	0.976584	0.944724
A_8	0.971074	0.927546	0.950413	0.953099	0.990944	0.913903	0.909213	0.964187	0.943384

NZN-EDAS Method

Based on the neutrosophic HyperSoft set, three experts are using the values of the criteria. The values are used as: {High, Moderate, Long, High, Flexible, High, Moderate, Strong, Effective}

Eq. (15) is used to determine the average solution of each criterion

Eqs. (16-19) are used to compute the positive distance from average and negative distance from average as shown in Fig 1 and Fig 2.

Eqs. (20 and 21) are used to compute the weighted PV_{ij} and CV_{ij} as shown in Fig 3 and Fig 4.

Eqs. (22 and 23) are used to compute the weighted normalized PV_{ij} and CV_{ij} as shown in Fig 5.

Eq. (24) sis used to compute the appraisal score as shown in Fig 6. Then we ranked the alternatives as shown in Fig 7.

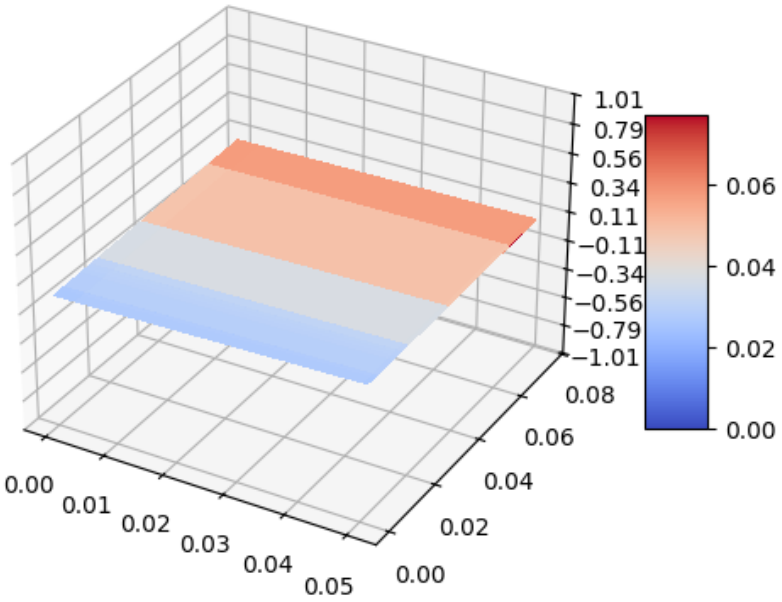

Fig 1. The values of PV_{ij}.

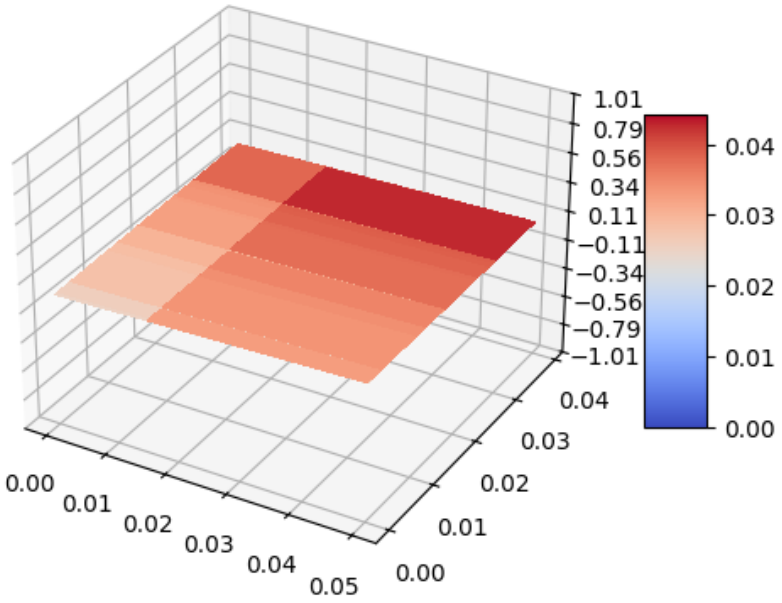

Fig 2. The values of CV_{ij}.

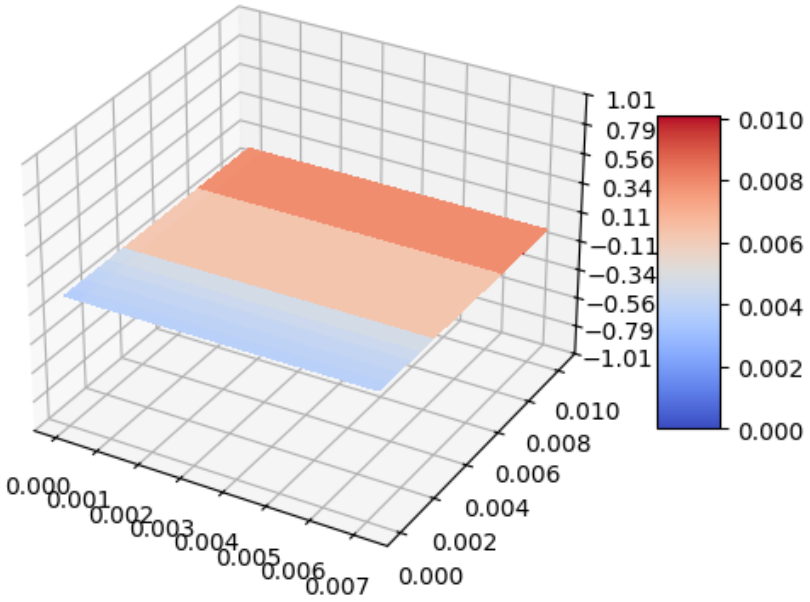

Fig 3. The weighted values of PV_{ij}.

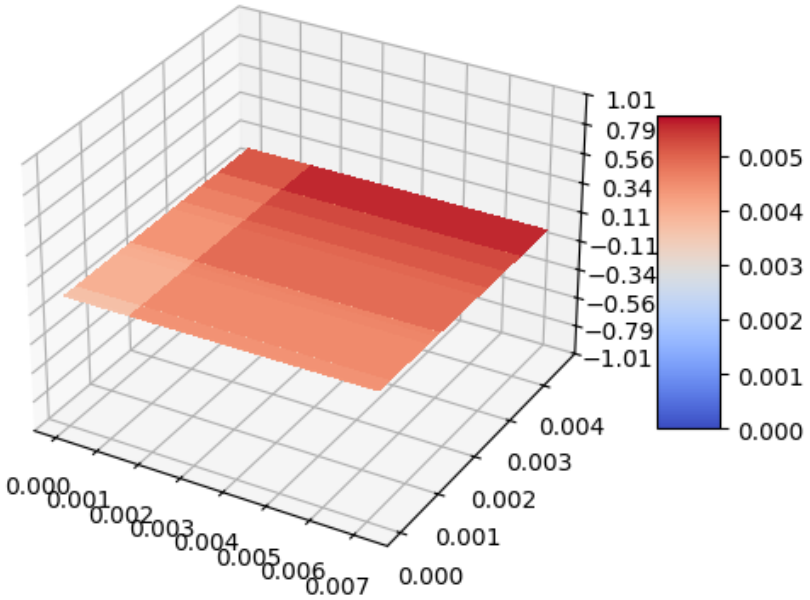

Fig 4. The weighted values of CV_{ij}.

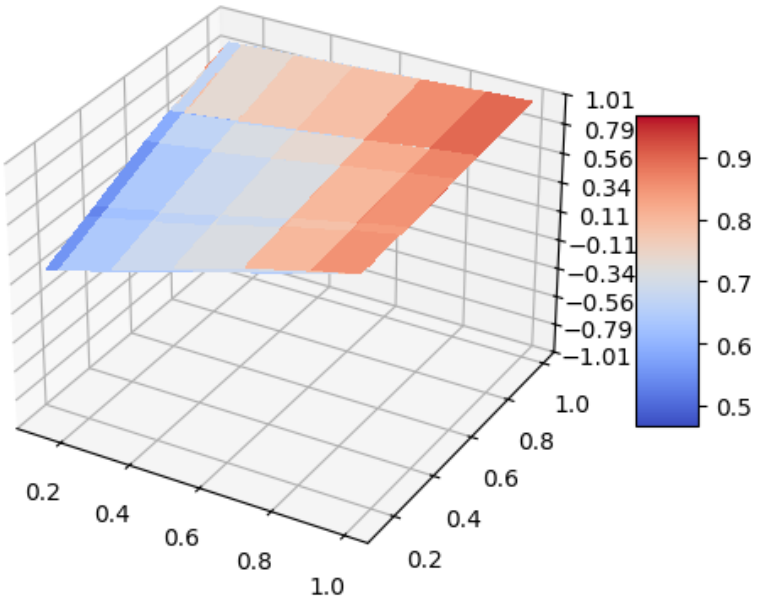

Fig 5. The weighted normalized PV_{ij} and CV_{ij}.

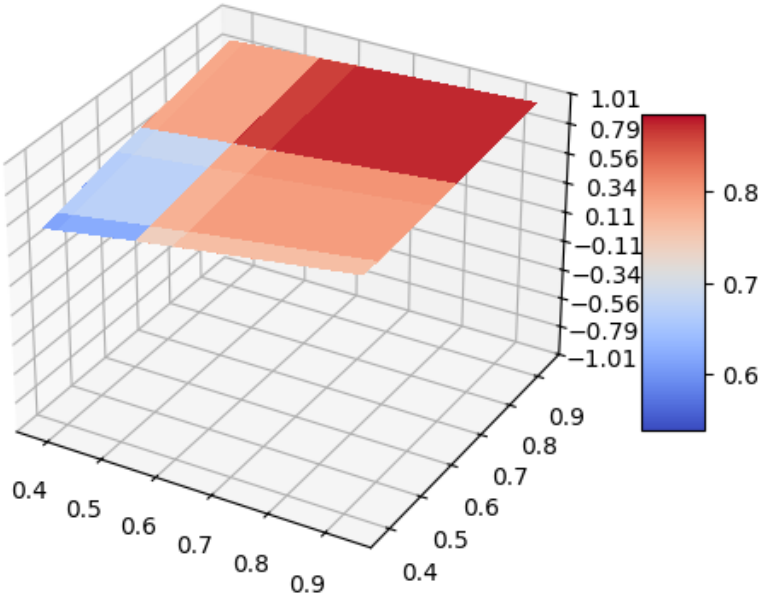

Fig 6. The appraisal values.

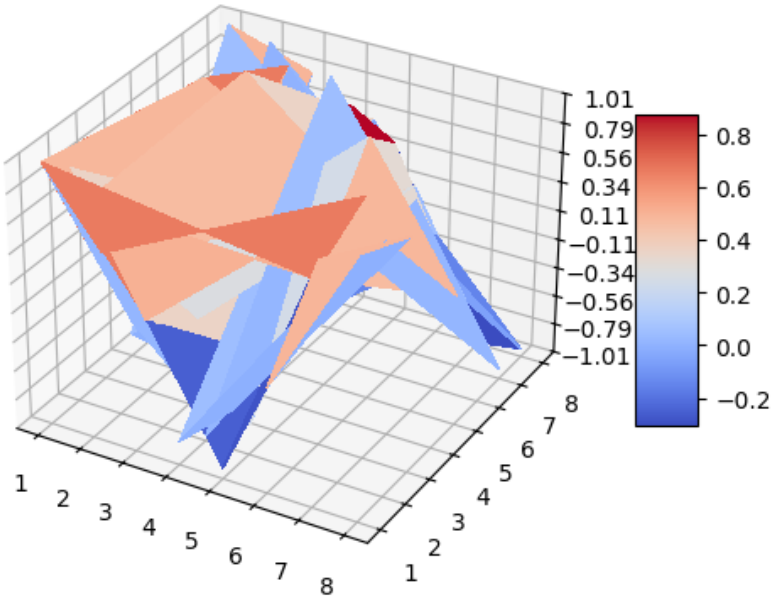

Fig 7. The rank of alternatives.

Sensitivity analysis

This part shows the aims of the sensitivity analysis. We applied the sensitivity analysis to show the stability of the ranks under different values. This study proposed nine cases in different criteria weights. We increase the criteria weights by 15%, then we decrease other criteria. These cases can influence the ranks of the alternatives. Fig 8 shows the nine cases in the criteria weights to show the ranks of alternatives.

Then we applied the EDAS method under these cases. The results show alternative 2 is the best and alternative 7 is the worst. The results of the sensitivity analysis show the proposed approach ranks are stable under different cases. Fig 9 shows the different ranks of the alternatives under different cases.

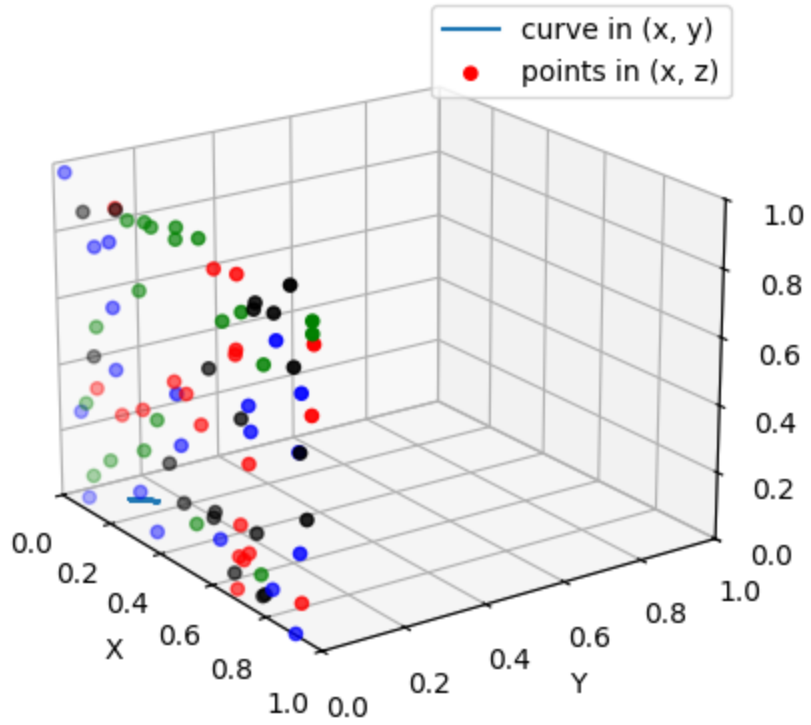

Fig 8. The different criteria weights.

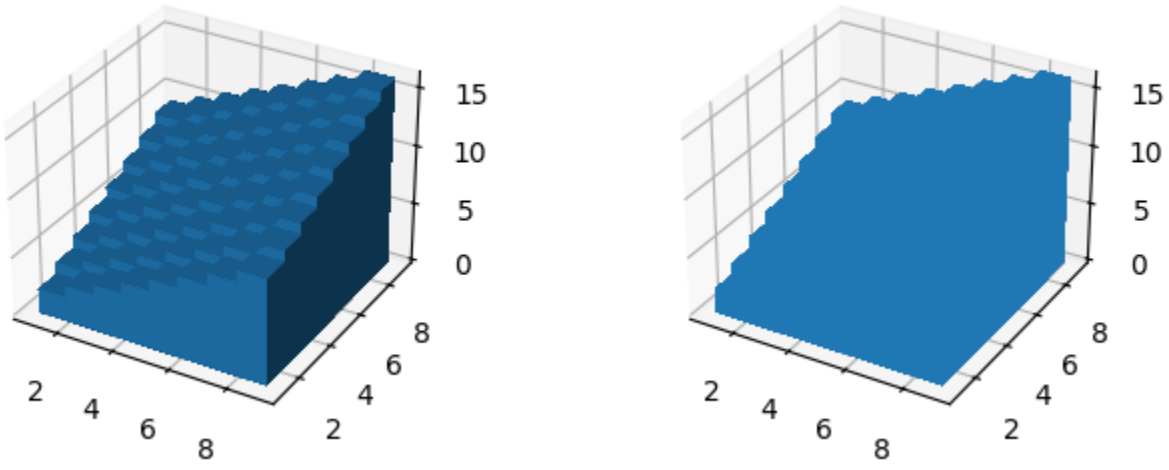

Fig 9. The different ranks of the alternatives.

5. Conclusions and Future Works

The suggested approaches were used to build an MCDM approach in the context of NZN sets. Lastly, to demonstrate the usefulness and rationale of the developed MCDM technique in the

context of NZN sets, it was applied to an example evaluation MCDM problem. Two MCDM methods are used in this study, such as SIWEC to compute the criteria weights and the EDAS method to rank the alternatives. This study illustrated the primary benefits of the MCDM approach developed within the context of NZN sets, as well as the information expression and similarity metrics of NZN sets, through comparison analysis. MCDM approach and NZN are used with the HyperSoft set to deal with different criteria and sub criteria. Nonetheless, the developed MCDM approach based on the suggested NZN not only improves the DM issues' logic and dependability, but it also demonstrates its superiority over the current neutrosophic MCDM approaches. The sensitivity analysis results show that the proposed approach ranks are stable under nine cases.

In terms of future research, we will apply the NZN sets to further applications, including clustering analysis, pattern recognition, and medical diagnostics in the context of NZN sets. The MCDM problem can be solved by different MCDM methods to compute the criteria weights and rank the alternatives.

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The integral of 2- refined irrational neutrosophic functions

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Abstract: In mathematics, we often come across several ways to solve the same problem, so we should find direct rules that make it easier for us to find the direct solution to these problems. This is what prompted us to present this paper, which dealt with the integral of 2- refined irrational neutrosophic functions. Where we presented seven rules to find the integral of 2- refined irrational neutrosophic functions directly, in addition to proving each case.

Keywords: irrational neutrosophic functions; 2- refined irrational; neutrosophic integrals; standard irrational.

1. Introduction and Preliminaries

As an alternative to the existing logics, Smarandache proposed the neutrosophic logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R \text{ or } C$ [1]. Agboola introduced the concept of refined neutrosophic algebraic structures [2]. In addition, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. Abobala presented the papers on some special substructures of refined neutrosophic rings and a study of ah-substructures in n-refined neutrosophic vector spaces [6-7].

Alhasan.Y and Abdulfatah. R also presented a study on the division of refined neutrosophic number [8].

There are papers presented in n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Neutrosophic Rings I [4-5] and studying the integral calculus according to the logic of neutrosophic by presenting a set of papers on that [9-10].

In addition, the AH-Isometry was extended to n-Refined AH-Isometry by Smarandache & Abobala in 2024 [11] https://fs.unm.edu/NSS/RefinedLiteral21.pdf.

2. Integral of standard 2- refined irrational neutrosophic functions

- **Standard 2- refined irrational neutrosophic integral I:**

$$\int \frac{a_1 + b_1 I_1 + c_1 I_2}{\sqrt{(a_2 + b_2 I_1 + c_2 I_2)^2 - x^2}} dx$$

$$= (a_1 + b_1 I_1 + c_1 I_2) sin^{-1}\left(\left(\frac{1}{a_2} + \left[\frac{-b_2}{(a_2 + c_2)(a_2 + b_2 + c_2)}\right] I_1 - \left[\frac{c_2}{a_2(a_2 + c_2)}\right] I_2\right) x\right) + C$$

Whereas $a_2 \neq 0,\ a_2 \neq -c_2,\ a_2 \neq -b_2 - c_2, C = a_0 + b_0 I_1 + c_0 I_2\ and\ a_0,\ b_0,\ c_0$ are real numbers.

Proof:
Let's put: $x = (a_2 + b_2 I_1 + c_2 I_2)\ sin\,\vartheta \implies dx = (a_2 + b_2 I_1 + c_2 I_2)\ cos\,\vartheta\ d\vartheta$

Then:

$$(a_2 + b_2 I_1 + c_2 I_2)^2 - x^2 = (a_2 + b_2 I_1 + c_2 I_2)^2 - (a_2 + b_2 I_1 + c_2 I_2)^2 sin^2\vartheta$$

$$= (a_2 + b_2 I_1 + c_2 I_2)^2 (1 - sin^2\vartheta)$$

$$= (a_2 + b_2 I_1 + c_2 I_2)^2 cos^2\vartheta$$

$$\implies \int \frac{a_1 + b_1 I_1 + c_1 I_2}{\sqrt{(a_2 + b_2 I_1 + c_2 I_2)^2 - x^2}} dx = \int \frac{a_1 + b_1 I_1 + c_1 I_2}{\sqrt{(a_2 + b_2 I_1 + c_2 I_2)^2 cos^2\vartheta}} (a_2 + b_2 I_1 + c_2 I_2)\ cos\,\vartheta\ d\vartheta$$

$$= \int (a_1 + b_1 I_1 + c_1 I_2) d\vartheta = (a_1 + b_1 I_1 + c_1 I_2) \int d\vartheta$$

$$= (a_1 + b_1 I_1 + c_1 I_2)\vartheta = (a_1 + b_1 I_1 + c_1 I_2) sin^{-1}\left(\frac{x}{a_2 + b_2 I_1 + c_2 I_2}\right) + C$$

where

$$\vartheta = sin^{-1}\left(\frac{x}{a_2 + b_2 I_1 + c_2 I_2}\right)$$

hence:

$$\int \frac{a_1 + b_1 I_1 + c_1 I_2}{\sqrt{(a_2 + b_2 I_1 + c_2 I_2)^2 - x^2}} dx$$

$$= (a_1 + b_1 I_1 + c_1 I_2) sin^{-1}\left(\left(\frac{1}{a_2} + \left[\frac{-b_2}{(a_2 + c_2)(a_2 + b_2 + c_2)}\right] I_1 - \left[\frac{c_2}{a_2(a_2 + c_2)}\right] I_2\right) x\right) + C$$

Whereas $a_2 \neq 0,\ a_2 \neq -c_2,\ a_2 \neq -b_2 - c_2$

Example 1

$$\int \frac{1 + I_1 + I_2}{\sqrt{100 + 75 I_1 + 21 I_2 - x^2}} dx$$

Solution:

$$100 + 75 I_1 + 21 I_2 - x^2 = \left(\sqrt{100 + 75 I_1 + 21 I_2}\right)^2 - x^2$$

Whereas: $\sqrt{100 + 75 I_1 + 21 I_2} = \sqrt{100} + \left[\sqrt{100 + 75 + 21} - \sqrt{100 + 21}\right] I_1 + \left[\sqrt{100 + 21} - \sqrt{100}\right] I_2$

$$\Longrightarrow \sqrt{100+60I_1+36I_2}=10+3I_1+I_2$$

$$\int \frac{1+I_1+I_2}{\sqrt{100+75I_1+21I_2-x^2}}dx=\int \frac{1+I_1+I_2}{\sqrt{(10+3I_1+I_2)^2-x^2}}dx$$

$$=(1+I_1+I_2)sin^{-1}\left(\left(\frac{1}{10}-\frac{3}{154}I_1-\frac{1}{110}I_2\right)x\right)+C$$

Let's check the answer:

$$\frac{d}{dx}\left[(1+I_1+I_2)sin^{-1}\left(\left(\frac{1}{10}-\frac{3}{154}I_1-\frac{1}{110}I_2\right)x\right)+C\right]$$

$$=(1+I_1+I_2)\frac{\frac{1}{10}-\frac{3}{154}I_1-\frac{1}{110}I_2}{\sqrt{1-\left(\frac{1}{10}-\frac{3}{154}I_1-\frac{1}{110}I_2\right)^2x^2}}$$

$$=(1+I_1+I_2)\frac{\frac{1}{10}-\frac{3}{154}I_1-\frac{1}{110}I_2}{\left(\frac{1}{10}-\frac{3}{154}I_1-\frac{1}{110}I_2\right)\left(\sqrt{\frac{1}{\left(\frac{1}{10}-\frac{3}{154}I_1-\frac{1}{110}I_2\right)^2}-x^2}\right)}$$

$$=(1+I_1+I_2)\frac{1}{\left(\sqrt{\frac{1}{\frac{1}{100}-\frac{75}{23716}I_1-\frac{21}{12100}I_2}-x^2}\right)}$$

$$=\frac{1+I_1+I_2}{\sqrt{100+75I_1+21I_2-x^2}}\quad (The\ same\ integral\ function)$$

- **Standard 2- refined irrational neutrosophic integral II:**

$$\int \frac{a_1+b_1I_1+c_1I_2}{x\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}}dx$$

$$=\left(\frac{a_1}{a_2}+\left[\frac{a_2b_1+b_1c_2-a_1b_2-b_2c_1}{(a_2+c_2)(a_2+b_2+c_2)}\right]I_1+\left[\frac{a_2c_1-a_1c_2}{a_2(a_2+c_2)}\right]I_2\right)sec^{-1}\left(\left(\frac{1}{a_2}+\left[\frac{-b_2}{(a_2+c_2)(a_2+b_2+c_2)}\right]I_1-\left[\frac{c_2}{a_2(a_2+c_2)}\right]I_2\right)x\right)+C$$

Whereas $a_2\neq 0,\ a_2\neq -c_2,\ a_2\neq -b_2-c_2, C=a_0+b_0I_1+c_0I_2\ and\ a_0,\ b_0,\ c_0$ are real numbers.

Proof:
Let's put: $x=(a_2+b_2I_1+c_2I_2)\,sec\,\vartheta \Longrightarrow dx=(a_2+b_2I_1+c_2I_2)\,sec\,\vartheta\,tan\,\vartheta\ d\vartheta$

Then:
$$x\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}=(a_2+b_2I_1+c_2I_2)\,sec\,\vartheta\sqrt{(a_2+b_2I_1+c_2I_2)^2sec^2\vartheta-(a_2+b_2I_1+c_2I_2)^2}$$

$$= (a_2 + b_2I_1 + c_2I_2)\, sec\,\vartheta \sqrt{(a_2 + b_2I_1 + c_2I_2)^2(sec^2\vartheta - 1)}$$

$$= (a_2 + b_2I_1 + c_2I_2)^2\, sec\,\vartheta\, tan\,\vartheta$$

$$\Longrightarrow \int \frac{a_1 + b_1I_1 + c_1I_2}{x\sqrt{x^2 - (a_2 + b_2I_1 + c_2I_2)^2}}dx$$

$$= \int \frac{a_1 + b_1I_1 + c_1I_2}{(a_2 + b_2I_1 + c_2I_2)^2\, sec\,\vartheta\, tan\,\vartheta}(a_2 + b_2I_1 + c_2I_2)\, sec\,\vartheta\, tan\,\vartheta\, d\vartheta$$

$$= \int \frac{a_1 + b_1I_1 + c_1I_2}{a_2 + b_2I_1 + c_2I_2}d\vartheta = \frac{a_1 + b_1I_1 + c_1I_2}{a_2 + b_2I_1 + c_2I_2}\int d\vartheta$$

$$= \frac{a_1 + b_1I_1 + c_1I_2}{a_2 + b_2I_1 + c_2I_2}\vartheta = \frac{a_1 + b_1I_1 + c_1I_2}{a_2 + b_2I_1 + c_2I_2}sec^{-1}\left(\frac{x}{a_2 + b_2I_1 + c_2I_2}\right) + C$$

Whereas:

$$\vartheta = sec^{-1}\left(\frac{x}{a_2 + b_2I_1 + c_2I_2}\right)$$

Hence:

$$\int \frac{a_1 + b_1I_1 + c_1I_2}{x\sqrt{x^2 - (a_2 + b_2I_1 + c_2I_2)^2}}dx$$

$$= \left(\frac{a_1}{a_2} + \left[\frac{a_2b_1 + b_1c_2 - a_1b_2 - b_2c_1}{(a_2 + c_2)(a_2 + b_2 + c_2)}\right]I_1 + \left[\frac{a_2c_1 - a_1c_2}{a_2(a_2 + c_2)}\right]I_2\right) sec^{-1}\left(\left(\frac{1}{a_2} + \left[\frac{-b_2}{(a_2 + c_2)(a_2 + b_2 + c_2)}\right]I_1 - \left[\frac{c_2}{a_2(a_2 + c_2)}\right]I_2\right)x\right) + C$$

Whereas $a_2 \neq 0,\ a_2 \neq -c_2,\ a_2 \neq -b_2 - c_2$

Example 2

Evaluate:

$$\int \frac{3 + 7I_2}{x\sqrt{x^2 - 144 - 56I_1 - 25I_2}}dx$$

Solution:

$$x^2 - 144 - 56I_1 - 25I_2 = x^2 - \left(\sqrt{144 + 56I_1 + 25I_2}\right)^2$$

$$x^2 - 144 - 56I_1 - 25I_2 = x^2 - (12 + 2I_1 + I_2)^2$$

Whereas: $\sqrt{144 + 56I_1 + 25I_2} = \sqrt{144} + \left[\sqrt{144 + 56 + 25} - \sqrt{144 + 25}\right]I_1 + \left[\sqrt{144 + 25} - \sqrt{144}\right]I_2$

$$\Longrightarrow \sqrt{1 + 72I_1 + 8I_2} = 12 + 2I_1 + I_2$$

$$\int \frac{3 + 7I_2}{x\sqrt{x^2 - 144 - 56I_1 - 25I_2}}dx = \int \frac{3 + 7I_2}{x\sqrt{x^2 - (12 + 2I_1 + I_2)^2}}dx$$

$$= \left(\frac{3 + 7I_2}{12 + 2I_1 + I_2}\right) sec^{-1}\left(\left(\frac{1}{12 + 2I_1 + I_2}\right)x\right)$$

$$= \left(\frac{1}{4} - \frac{4}{39}I_1 + \frac{27}{52}I_2\right) sec^{-1}\left(\left(\frac{1}{12} - \frac{2}{195}I_1 - \frac{1}{156}I_2\right)x\right) + C$$

Let's check the answer:

$$\frac{d}{dx}\left[\left(\frac{1}{4} - \frac{4}{39}I_1 + \frac{27}{52}I_2\right) sec^{-1}\left(\left(\frac{1}{12} - \frac{2}{195}I_1 - \frac{1}{156}I_2\right)x\right) + C\right]$$

$$= \left(\frac{1}{4} - \frac{4}{39}I_1 + \frac{27}{52}I_2\right) \frac{\frac{1}{12} - \frac{2}{195}I_1 - \frac{1}{156}I_2}{\left(\frac{1}{12} - \frac{2}{195}I_1 - \frac{1}{156}I_2\right)x\sqrt{\left(\frac{1}{12} - \frac{2}{195}I_1 - \frac{1}{156}I_2\right)^2 x^2 - 1}}$$

$$= \left(\frac{1}{4} - \frac{4}{39}I_1 + \frac{27}{52}I_2\right) \frac{\frac{1}{12} - \frac{2}{195}I_1 - \frac{1}{156}I_2}{\left(\frac{1}{12} - \frac{2}{195}I_1 - \frac{1}{156}I_2\right)^2 \left(x\sqrt{x^2 - \frac{1}{\left(\frac{1}{12} - \frac{2}{195}I_1 - \frac{1}{156}I_2\right)^2}}\right)}$$

$$= \left(\frac{\frac{1}{4} - \frac{4}{39}I_1 + \frac{27}{52}I_2}{\frac{1}{12} - \frac{2}{195}I_1 - \frac{1}{156}I_2}\right) \frac{1}{\left(x\sqrt{x^2 - \frac{1}{\left(\frac{1}{12} - \frac{2}{195}I_1 - \frac{1}{156}I_2\right)^2}}\right)}$$

$$= (3 + 7I_2) \frac{1}{\left(x\sqrt{x^2 - \frac{1}{\frac{1}{144} - \frac{56}{38025}I_1 - \frac{25}{24336}I_2}}\right)}$$

$$= \frac{3 + 7I_2}{x\sqrt{x^2 - 144 - 56I_1 - 25I_2}} \quad (The\ same\ integral\ function)$$

- **Standard 2- refined irrational neutrosophic integral III:**

$$\int \frac{a_1 + b_1I_1 + c_1I_2}{\sqrt{x^2 - (a_2 + b_2I_1 + c_2I_2)^2}} dx = (a_1 + b_1I_1 + c_1I_2) ln\left|x + \sqrt{x^2 - (a_2 + b_2I_1 + c_2I_2)^2}\right| + C$$

Proof:
Let's put: $x = (a_2 + b_2I_1 + c_2I_2)\, sec\,\vartheta \implies dx = (a_2 + b_2I_1 + c_2I_2)\, sec\,\vartheta\ tan\,\vartheta\, d\vartheta$

Then:

$$\sqrt{x^2 - (a_2 + b_2I_1 + c_2I_2)^2} = \sqrt{(a_2 + b_2I_1 + c_2I_2)^2 sec^2\vartheta - (a_2 + b_2I_1 + c_2I_2)^2}$$

$$= \sqrt{(a_2 + b_2I_1 + c_2I_2)^2 (sec^2\vartheta - 1)}$$

$$= (a_2 + b_2I_1 + c_2I_2)\, tan\,\vartheta$$

$$\implies \int \frac{a_1 + b_1I_1 + c_1I_2}{\sqrt{x^2 - (a_2 + b_2I_1 + c_2I_2)^2}} dx = \int \frac{a_1 + b_1I_1 + c_1I_2}{(a_2 + b_2I_1 + c_2I_2)\, tan\,\vartheta} (a_2 + b_2I_1 + c_2I_2)\, sec\,\vartheta\, tan\,\vartheta\ d\vartheta$$

$$= \int (a_1 + b_1 I_1 + c_1 I_2)\, sec\, \vartheta \;\; d\vartheta = (a_1 + b_1 I_1 + c_1 I_2) \int sec\, \vartheta \;\; d\vartheta$$

$$= (a_1 + b_1 I_1 + c_1 I_2) ln|sec\, \vartheta + tan\, \vartheta| + C_1 = (a_1 + b_1 I_1 + c_1 I_2) ln \left| sec\, \vartheta + \sqrt{sec^2 \vartheta - 1} \right| + C_1$$

$$= (a_1 + b_1 I_1 + c_1 I_2) ln \left| \frac{x}{a_2 + b_2 I_1 + c_2 I_2} + \sqrt{\left(\frac{x}{a_2 + b_2 I_1 + c_2 I_2} \right)^2 - 1} \right| + C_1$$

$$= (a_1 + b_1 I_1 + c_1 I_2) ln \left| \frac{x}{a_2 + b_2 I_1 + c_2 I_2} + \frac{1}{a_2 + b_2 I_1 + c_2 I_2} \sqrt{x^2 - (a_2 + b_2 I_1 + c_2 I_2)^2} \right| + C_1$$

$$= (a_1 + b_1 I_1 + c_1 I_2) ln \left| \frac{1}{a_2 + b_2 I_1 + c_2 I_2} \left(x + \sqrt{x^2 - (a_2 + b_2 I_1 + c_2 I_2)^2} \right) \right| + C_1$$

$$= (a_1 + b_1 I_1 + c_1 I_2) \left[ln \left| \left(x + \sqrt{x^2 - (a_2 + b_2 I_1 + c_2 I_2)^2} \right) \right| - ln|a_2 + b_2 I_1 + c_2 I_2| + C_1 \right]$$

hence:

$$\int \frac{a_1 + b_1 I_1 + c_1 I_2}{\sqrt{x^2 - (a_2 + b_2 I_1 + c_2 I_2)^2}} dx = (a_1 + b_1 I_1 + c_1 I_2) ln \left| x + \sqrt{x^2 - (a_2 + b_2 I_1 + c_2 I_2)^2} \right| + C$$

Whereas:

$$sec\, \vartheta = \frac{x}{a_2 + b_2 I_1 + c_2 I_2} \;\; and \;\; C = -ln|a_2 + b_2 I_1 + c_2 I_2| + C_1$$

Example 3

Evaluate:

$$\int \frac{1 - I_1 + 2I_2}{\sqrt{x^2 - 1 - 32I_1 - 3I_2}} dx$$

Solution:

$$x^2 - 1 - 32I_1 - 3I_2 = x^2 - \left(\sqrt{1 + 32I_1 + 3I_2} \right)^2$$

$$x^2 - 1 - 32I_1 - 3I_2 = x^2 - (1 + 4I_1 + 3I_2)^2$$

Whereas: $\sqrt{1 + 32I_1 + 3I_2} = 1 + \left[\sqrt{1 + 32 + 3} - \sqrt{4} \right] I_1 + \left[\sqrt{4} - 1 \right] I_2$

$$\Longrightarrow \;\; \sqrt{1 + 32I_1 + 3I_2} = 1 + 4I_1 + 3I_2$$

$$\int \frac{1 - I_1 + 2I_2}{\sqrt{x^2 - 1 - 32I_1 - 3I_2}} dx = \int \frac{1 - I_1 + 2I_2}{\sqrt{x^2 - (1 + 4I_1 + 3I_2)^2}} dx$$

$$= (1 - I_1 + 2I_2) ln \left| x + \sqrt{x^2 - (1 + 4I_1 + 3I_2)^2} \right| + C$$

Let's check the answer:

$$\frac{d}{dx} \left[(1 - I_1 + 2I_2) ln \left| x + \sqrt{x^2 - (1 + 4I_1 + 3I_2)^2} \right| + C \right]$$

$$= (1 - I_1 + 2I_2)\frac{1 + \frac{2x}{2\sqrt{x^2 - (1 + 4I_1 + 3I_2)^2}}}{x + \sqrt{x^2 - (1 + 4I_1 + 3I_2)^2}} = (1 - I_1 + 2I_2)\frac{\frac{\sqrt{x^2 - (1 + 4I_1 + 3I_2)^2} + x}{\sqrt{x^2 - (1 + 4I_1 + 3I_2)^2}}}{x + \sqrt{x^2 - (1 + 4I_1 + 3I_2)^2}}$$

$$= (1 - I_1 + 2I_2)\frac{1}{\sqrt{x^2 - (1 + 4I_1 + 3I_2)^2}} = \frac{1 - I_1 + 2I_2}{\sqrt{x^2 - 1 - 32I_1 - 3I_2}} \ (The\ same\ integral\ function)$$

- **Standard 2- refined irrational neutrosophic integral IV:**

$$\int \frac{a_1 + b_1I_1 + c_1I_2}{\sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}}dx = (a_1 + b_1I_1 + c_1I_2)ln\left|x + \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}\right| + C$$

Proof:
Let's put: $x = (a_2 + b_2I_1 + c_2I_2)\tan\vartheta \implies dx = (a_2 + b_2I_1 + c_2I_2)sec^2\vartheta\, d\vartheta$

Then:

$$\sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2} = \sqrt{(a_2 + b_2I_1 + c_2I_2)^2tan^2\vartheta + (a_2 + b_2I_1 + c_2I_2)^2}$$

$$= \sqrt{(a_2 + b_2I_1 + c_2I_2)^2(tan^2\vartheta + 1)}$$

$$= (a_2 + b_2I_1 + c_2I_2)\sec\vartheta$$

$$\implies \int \frac{a_1 + b_1I_1 + c_1I_2}{\sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}}dx = \int \frac{a_1 + b_1I_1 + c_1I_2}{(a_2 + b_2I_1 + c_2I_2)\sec\vartheta}(a_2 + b_2I_1 + c_2I_2)sec^2\theta\, d\theta$$

$$= \int (a_1 + b_1I_1 + c_1I_2)\sec\vartheta\, d\vartheta = (a_1 + b_1I_1 + c_1I_2)\int \sec\vartheta\, d\vartheta$$

$$= (a_1 + b_1I_1 + c_1I_2)ln|\sec\vartheta + \tan\vartheta| = (a_1 + b_1I_1 + c_1I_2)ln\left|\sqrt{tan^2\vartheta + 1} + \tan\vartheta\right|$$

$$= (a_1 + b_1I_1 + c_1I_2)ln\left|\frac{x}{a_2 + b_2I_1 + c_2I_2} + \sqrt{\left(\frac{x}{a_2 + b_2I_1 + c_2I_2}\right)^2 + 1}\right|$$

$$= (a_1 + b_1I_1 + c_1I_2)ln\left|\frac{x}{a_2 + b_2I_1 + c_2I_2} + \frac{1}{a_2 + b_2I_1 + c_2I_2}\sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}\right|$$

$$= (a_1 + b_1I_1 + c_1I_2)ln\left|\frac{1}{a_2 + b_2I_1 + c_2I_2}\left(x + \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}\right)\right|$$

$$= (a_1 + b_1I_1 + c_1I_2)\left[ln\left|\left(x + \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}\right)\right| - ln|a_2 + b_2I_1 + c_2I_2| + C_1\right]$$

Hence:

$$\int \frac{a_1 + b_1I_1 + c_1I_2}{\sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}}dx = (a_1 + b_1I_1 + c_1I_2)ln\left|x + \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}\right| + C$$

Whereas:

$$\sec\vartheta = \frac{x}{a_2 + b_2I_1 + c_2I_2} \ and \ C = -ln|a_2 + b_2I_1 + c_2I_2| + C_1$$

Example 4

Evaluate:

$$\int \frac{12-14I_1-92I_2}{\sqrt{x^2+25+13I_1+11I_2}}dx$$

Solution:

$$x^2+25+13I_1+11I_2 = x^2+\left(\sqrt{25+13I_1+11I_2}\right)^2$$

$$x^2+25+13I_1+11I_2 = x^2-(5+I_1+I_2)^2$$

Whereas: $\sqrt{25+13I_1+11I_2} = 5+\left[\sqrt{25+13+11}-\sqrt{36}\right]I_1+\left[\sqrt{36}-5\right]I_2$

$$\Longrightarrow \quad \sqrt{25+13I_1+11I_2} = 5+I_1+I_2$$

$$\int \frac{12-14I_1-92I_2}{\sqrt{x^2+25+13I_1+11I_2}}dx = \int \frac{12-14I_1-92I_2}{\sqrt{x^2+(5+I_1+I_2)^2}}dx$$

$$= (12-14I_1-92I_2)ln\left|x+\sqrt{x^2+(5+I_1+I_2)^2}\right|+C$$

Let's check the answer:

$$\frac{d}{dx}\left[(12-14I_1-92I_2)ln\left|x+\sqrt{x^2+(5+I_1+I_2)^2}\right|+C\right]$$

$$= (12-14I_1-92I_2)\frac{1+\frac{2x}{2\sqrt{x^2+(5+I_1+I_2)^2}}}{x+\sqrt{x^2+(5+I_1+I_2)^2}} = (12-14I_1-92I_2)\frac{\frac{\sqrt{x^2+(5+I_1+I_2)^2}+x}{\sqrt{x^2+(5+I_1+I_2)^2}}}{x+\sqrt{x^2+(5+I_1+I_2)^2}}$$

$$= (12-14I_1-92I_2)\frac{1}{\sqrt{x^2+(5+I_1+I_2)^2}}$$

$$= \frac{12-14I_1-92I_2}{\sqrt{x^2+25+13I_1+11I_2}} \quad (The\ same\ integral\ function)$$

- **Standard 2- refined irrational neutrosophic integral V:**

$$\int \sqrt{(a_2+b_2I_1+c_2I_2)^2-x^2}\,dx$$
$$= \frac{x}{2}\sqrt{(a_2+b_2I_1+c_2I_2)^2-x^2}$$
$$+\frac{(a_2+b_2I_1+c_2I_2)^2}{2}sin^{-1}\left(\left(\frac{1}{a_2}+\left[\frac{-b_2}{(a_2+c_2)(a_2+b_2+c_2)}\right]I_1-\left[\frac{c_2}{a_2(a_2+c_2)}\right]I_2\right)x\right)$$
$$+C$$

Whereas $a_2 \neq 0,\ a_2 \neq -c_2,\ a_2 \neq -b_2-c_2$

Proof:

Let's put: $x = (a_2+b_2I_1+c_2I_2)\,sin\,\vartheta \quad \Longrightarrow dx = (a_2+b_2I_1+c_2I_2)\cos\vartheta\ d\vartheta$

Then:

$$(a_2+b_2I_1+c_2I_2)^2-x^2 = (a_2+b_2I_1+c_2I_2)^2-(a_2+b_2I_1+c_2I_2)^2sin^2\vartheta$$

$$= (a_2 + b_2I_1 + c_2I_2)^2(1 - sin^2\vartheta)$$

$$= (a_2 + b_2I_1 + c_2I_2)^2cos^2\vartheta$$

$$\Longrightarrow \quad \int \sqrt{(a_2 + b_2I_1 + c_2I_2)^2 - x^2}\, dx = \int \sqrt{(a_2 + b_2I_1 + c_2I_2)^2cos^2\vartheta}\, (a_2 + b_2I_1 + c_2I_2)\cos\vartheta \; d\vartheta$$

$$= \int (a_2 + b_2I_1 + c_2I_2)^2cos^2\vartheta \, d\vartheta = \frac{(a_2 + b_2I_1 + c_2I_2)^2}{2}\int (cos\, 2\vartheta + 1)d\vartheta$$

$$= \frac{(a_2 + b_2I_1 + c_2I_2)^2}{2}\int (cos\, 2\vartheta + 1)d\vartheta = \frac{(a_2 + b_2I_1 + c_2I_2)^2}{2}\left(\frac{1}{2}\sin 2\vartheta + \vartheta\right)$$

$$= \frac{(a_2 + b_2I_1 + c_2I_2)^2}{2}(sin\,\vartheta\, cos\,\vartheta + \vartheta)$$

$$= \frac{(a_2 + b_2I_1 + c_2I_2)^2}{2}\left(\frac{x}{a_2 + b_2I_1 + c_2I_2}\,\frac{\sqrt{(a_2 + b_2 + c_2)^2 - x^2}}{(a_2 + b_2 + c_2)}\right.$$
$$\left. + sin^{-1}\left(\left(\frac{1}{a_2} + \left[\frac{-b_2}{(a_2 + c_2)(a_2 + b_2 + c_2)}\right]I_1 - \left[\frac{c_2}{a_2(a_2 + c_2)}\right]I_2\right)x\right)\right)$$

Whereas:

$$\vartheta = sin^{-1}\left(\frac{x}{a_2 + b_2I_1 + c_2I_2}\right)$$

Hence:

$$\int \sqrt{(a_2 + b_2I_1 + c_2I_2)^2 - x^2}\, dx$$

$$= \frac{x}{2}\sqrt{(a_2 + b_2I_1 + c_2I_2)^2 - x^2}$$
$$+ \frac{(a_2 + b_2I_1 + c_2I_2)^2}{2}sin^{-1}\left(\left(\frac{1}{a_2} + \left[\frac{-b_2}{(a_2 + c_2)(a_2 + b_2 + c_2)}\right]I_1 - \left[\frac{c_2}{a_2(a_2 + c_2)}\right]I_2\right)x\right)$$
$$+ C$$

Example 5

$$\int \sqrt{25 + 80I_1 + 39I_2 - x^2}\, dx$$

Solution:

$$25 + 80I_1 + 39I_2 - x^2 = \left(\sqrt{25 + 80I_1 + 39I_2}\right)^2 - x^2$$

$$25 + 80I_1 + 39I_2 - x^2 = (5 + 4I_1 + 3I_2)^2 - x^2$$

Whereas: $\sqrt{25 + 80I_1 + 39I_2} = \sqrt{25} + \left[\sqrt{25 + 80 + 39} - \sqrt{25 + 39}\right]I_1 + \left[\sqrt{25 + 39} - \sqrt{25}\right]I_2$

$$\Longrightarrow \quad \sqrt{1 + 72I_1 + 8I_2} = 5 + 4I_1 + 3I_2$$

$$\int \sqrt{25+80I_1+39I_2-x^2}\,dx = \int \sqrt{(5+4I_1+3I_2)^2-x^2}\,dx$$

$$= \frac{x}{2}\sqrt{(5+4I_1+3I_2)^2-x^2} + \frac{(5+4I_1+3I_2)^2}{2} sin^{-1}\left(\left(\frac{1}{5+4I_1+3I_2}\right)x\right) + C$$

$$= \frac{x}{2}\sqrt{25+80I_1+39I_2-x^2} + \left(\frac{25}{2}+40I_1+\frac{39}{2}I_2\right) sin^{-1}\left(\left(\frac{1}{5}-\frac{1}{24}I_1+\frac{3}{40}I_2\right)x\right) + C$$

Let's check the answer:

$$\frac{d}{dx}\left[\frac{x}{2}\sqrt{25+80I_1+39I_2-x^2} + \left(\frac{25}{2}+40I_1+\frac{39}{2}I_2\right) sin^{-1}\left(\left(\frac{1}{5}-\frac{1}{24}I_1+\frac{3}{40}I_2\right)x\right) + C\right]$$

$$= \frac{1}{2}\sqrt{25+80I_1+39I_2-x^2} + \frac{x}{2}\frac{-2x}{2\sqrt{25+80I_1+39I_2-x^2}}$$
$$+ \left(\frac{25}{2}+40I_1+\frac{39}{2}I_2\right)\frac{\frac{1}{5}-\frac{1}{24}I_1+\frac{3}{40}I_2}{\left(\frac{1}{5}-\frac{1}{24}I_1+\frac{3}{40}I_2\right)\sqrt{1-\left(\frac{1}{5}-\frac{1}{24}I_1+\frac{3}{40}I_2\right)^2 x^2}}$$

$$= \frac{1}{2}\left(\frac{25+80I_1+39I_2-2x^2}{\sqrt{25+80I_1+39I_2-x^2}}\right) + \frac{\frac{25}{2}+40I_1+\frac{39}{2}I_2}{\sqrt{\frac{1}{\left(\frac{1}{5}-\frac{1}{24}I_1+\frac{3}{40}I_2\right)^2}-x^2}}$$

$$= \frac{1}{2}\left(\frac{25+80I_1+39I_2-2x^2}{\sqrt{25+80I_1+39I_2-x^2}}\right) + \frac{1}{2}\frac{25+80I_1+39I_2}{\sqrt{25+80I_1+39I_2-x^2}} = \frac{25+80I_1+39I_2-x^2}{\sqrt{25+80I_1+39I_2-x^2}}$$

$$= \sqrt{25+80I_1+39I_2-x^2} \quad (The\ same\ integral\ function)$$

- **Standard 2- refined irrational neutrosophic integral VI:**

$$\int \sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}\,dx$$

$$= \frac{x}{2}\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2} + \frac{(a_2+b_2I_1+c_2I_2)^2}{2} ln\left|x+\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}\right| + C$$

Proof:
Put: $x = (a_2+b_2I_1+c_2I_2)\,sec\,\vartheta \implies dx = (a_2+b_2I_1+c_2I_2)\,sec\,\vartheta\,tan\,\vartheta\;d\vartheta$

Then:

$$\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2} = \sqrt{(a_2+b_2I_1+c_2I_2)^2 sec^2\vartheta - (a_2+b_2I_1+c_2I_2)^2}$$

$$= \sqrt{(a_2+b_2I_1+c_2I_2)^2(sec^2\vartheta-1)}$$

$$= (a_2+b_2I_1+c_2I_2)\,tan\,\vartheta$$

$$\Rightarrow \int \sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}dx = \int (a_2+b_2I_1+c_2I_2)\tan\vartheta\,(a_2+b_2I_1+c_2I_2)\sec\vartheta\tan\vartheta\,d\vartheta$$

$$= (a_2+b_2I_1+c_2I_2)^2\int \sec\vartheta\tan^2\vartheta\,d\vartheta = (a_2+b_2I_1+c_2I_2)^2\int \sec\vartheta\,(\sec^2\theta-1)d\vartheta$$

$$= (a_2+b_2I_1+c_2I_2)^2\int (\sec^3\vartheta-\sec\vartheta)d\vartheta$$

$$= (a_2+b_2I_1+c_2I_2)^2\left[\frac{1}{2}\sec\vartheta\tan\vartheta+\frac{1}{2}ln|\sec\vartheta+\tan\vartheta|-ln|\sec\vartheta+\tan\vartheta|\right]+C_1$$

$$= (a_2+b_2I_1+c_2I_2)^2\left[\frac{1}{2}\sec\vartheta\sqrt{\sec^2\vartheta-1}-\frac{1}{2}ln\left|\sec\vartheta+\sqrt{\sec^2\theta-1}\right|\right]+C_1$$

$$= (a_2+b_2I_1+c_2I_2)^2\left[\frac{1}{2}\sec\vartheta\sqrt{\sec^2\vartheta-1}-\frac{1}{2}ln|\sec\vartheta+\tan\vartheta|\right]+C_1 \quad (1)$$

But:

$$\sec\vartheta = \frac{x}{a_2+b_2I_1+c_2I_2}$$

$$\sqrt{\sec^2\vartheta-1} = \sqrt{\left(\frac{x}{a_2+b_2I_1+c_2I_2}\right)^2-1} = \frac{1}{a_2+b_2I_1+c_2I_2}\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}$$

By substitution in (1), we get:

$$\int \sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}dx$$

$$= (a_2+b_2I_1+c_2I_2)^2\left[\frac{1}{2}\frac{x}{(a_2+b_2I_1+c_2I_2)^2}\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}\right.$$
$$\left.-\frac{1}{2}ln\left|\frac{x}{a_2+b_2I_1+c_2I_2}+\frac{1}{a_2+b_2I_1+c_2I_2}\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}\right|\right]+C_1$$

$$= \frac{x}{2}\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}$$
$$-\frac{(a_2+b_2I_1+c_2I_2)^2}{2}ln\left|\frac{1}{a_2+b_2I_1+c_2I_2}\left(x+\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}\right)\right|$$

$$= \frac{x}{2}\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}-\frac{(a_2+b_2I_1+c_2I_2)^2}{2}ln\left|x+\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}\right|$$
$$-ln|a_2+b_2I_1+c_2I_2|+C_1$$

Hence:

$$\int \sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}dx$$
$$= \frac{x}{2}\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}-\frac{(a_2+b_2I_1+c_2I_2)^2}{2}ln\left|x+\sqrt{x^2-(a_2+b_2I_1+c_2I_2)^2}\right|$$
$$+C$$

Whereas:

$$C = -ln|a_2+b_2I_1+c_2I_2|+C_1$$

Example 6

$$\int \sqrt{x^2-(1-4I_1-3I_2)^2}dx$$

$$=\frac{x}{2}\sqrt{x^2-(1-4I_1-3I_2)^2}-\frac{(1-4I_1-3I_2)^2}{2}ln\left|\left(x+\sqrt{x^2-(1-4I_1-3I_2)^2}\right)\right|+C$$

$$=\frac{x}{2}\sqrt{x^2-(1-4I_1-3I_2)^2}-(1+32I_1+3I_2)ln\left|\left(x+\sqrt{x^2-(1-4I_1-3I_2)^2}\right)\right|+C$$

Let's check the answer:

$$\frac{d}{dx}\left[\frac{x}{2}\sqrt{x^2-(1-4I_1-3I_2)^2}-(1+32I_1+3I_2)ln\left|\left(x+\sqrt{x^2-(1-4I_1-3I_2)^2}\right)\right|+C\right]$$

$$=\frac{1}{2}\sqrt{x^2-(1-4I_1-3I_2)^2}+\frac{x}{2}.\frac{2x}{2\sqrt{x^2-(1-4I_1-3I_2)^2}}$$
$$-\frac{(1-4I_1-3I_2)^2}{2}\frac{1+\frac{2x}{2\sqrt{x^2-(1-4I_1-3I_2)^2}}}{x+\sqrt{x^2-(1-4I_1-3I_2)^2}}$$

$$=\frac{1}{2}\left(\sqrt{x^2-(1-4I_1-3I_2)^2}+\frac{x^2}{\sqrt{x^2-(1-4I_1-3I_2)^2}}-(1-4I_1-3I_2)^2.\frac{\frac{x+\sqrt{x^2-(1-4I_1-3I_2)^2}}{\sqrt{x^2-(1-4I_1-3I_2)^2}}}{x+\sqrt{x^2-(1-4I_1-3I_2)^2}}\right)$$

$$=\frac{1}{2}\left(\sqrt{x^2-(1-4I_1-3I_2)^2}+\frac{x^2}{\sqrt{x^2-(1-4I_1-3I_2)^2}}-(1-4I_1-3I_2)^2.\frac{1}{\sqrt{x^2-(1-4I_1-3I_2)^2}}\right)$$

$$=\frac{1}{2}\left(\sqrt{x^2-(1-4I_1-3I_2)^2}+\frac{x^2-(1-4I_1-3I_2)^2}{\sqrt{x^2-(1-4I_1-3I_2)^2}}\right)$$

$$=\frac{1}{2}\left(\frac{x^2-(1-4I_1-3I_2)^2+x^2-(1-4I_1-3I_2)^2}{\sqrt{x^2-(1-4I_1-3I_2)^2}}\right)=\frac{1}{2}\left(\frac{2(x^2-(1-4I_1-3I_2)^2)}{\sqrt{x^2-(1-4I_1-3I_2)^2}}\right)$$

$$=\sqrt{x^2-(1-4I_1-3I_2)^2}\ (The\ same\ integral\ function)$$

- **Standard 2- refined irrational neutrosophic integral VII:**

$$\int \sqrt{x^2+(a_2+b_2I_1+c_2I_2)^2}dx$$

$$=\frac{x}{2}\sqrt{x^2+(a_2+b_2I_1+c_2I_2)^2}+\frac{(a_2+b_2I_1+c_2I_2)^2}{2}ln\left|\left(x+\sqrt{x^2+(a_2+b_2I_1+c_2I_2)^2}\right)\right|+C$$

Proof:
Put $x=(a_2+b_2I_1+c_2I_2)\tan\vartheta \implies dx=(a_2+b_2I_1+c_2I_2)sec^2\vartheta\, d\vartheta$

Then:

$$\sqrt{x^2+(a_2+b_2I_1+c_2I_2)^2}=\sqrt{(a_2+b_2I_1+c_2I_2)^2tan^2\vartheta+(a_2+b_2I_1+c_2I_2)^2}$$

$$=\sqrt{(a_2+b_2I_1+c_2I_2)^2(tan^2\vartheta+1)}$$

$$= (a_2 + b_2I_1 + c_2I_2)\, sec\, \vartheta$$

$$\Longrightarrow \quad \int \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}dx = \int (a_2 + b_2I_1 + c_2I_2)\, sec\, \vartheta\, (a_2 + b_2I_1 + c_2I_2) sec^2\vartheta\, d\vartheta$$

$$= (a_2 + b_2I_1 + c_2I_2)^2 \int sec^3\vartheta\, d\vartheta$$

$$= (a_2 + b_2I_1 + c_2I_2)^2 \left[\frac{1}{2} sec\, \vartheta\, tan\, \vartheta + \frac{1}{2} ln|sec\, \vartheta + tan\, \vartheta|\right] + C_1$$

$$= (a_2 + b_2I_1 + c_2I_2)^2 \left[\frac{1}{2} tan\, \vartheta \sqrt{tan^2\vartheta + 1} + \frac{1}{2} ln\left|tan\, \vartheta + \sqrt{tan^2\vartheta + 1}\right|\right] + C_1$$

$$= (a_2 + b_2I_1 + c_2I_2)^2 \left[\frac{1}{2} tan\, \vartheta \sqrt{tan^2\vartheta + 1} + \frac{1}{2} ln\left|tan\, \vartheta + \sqrt{tan^2\vartheta + 1}\right|\right] + C_1 \quad (2)$$

But:

$$tan\, \vartheta = \frac{x}{(a_2 + b_2I_1 + c_2I_2)}$$

$$\sqrt{tan^2\theta + 1} = \sqrt{\left(\frac{x}{(a_2 + b_2I_1 + c_2I_2)}\right)^2 + 1} = \frac{1}{(a_2 + b_2I_1 + c_2I_2)} \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}$$

By substitution in (2), we get:

$$\int \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}dx$$

$$= (a_2 + b_2I_1 + c_2I_2)^2 \left[\frac{1}{2} \frac{x}{(a_2 + b_2I_1 + c_2I_2)^2} \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2} + \frac{1}{2} ln\left|\frac{x}{a_2 + b_2I_1 + c_2I_2} + \frac{1}{a_2 + b_2I_1 + c_2I_2} \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}\right|\right] + C_1$$

$$= \frac{x}{2} \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2} + \frac{(a_2 + b_2I_1 + c_2I_2)^2}{2} ln\left|\frac{1}{a_2 + b_2I_1 + c_2I_2}\left(x + \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}\right)\right| + C_1$$

$$= \frac{x}{2} \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2} + \frac{(a_2 + b_2I_1 + c_2I_2)^2}{2} ln\left|x + \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}\right| - ln|a_2 + b_2I_1 + c_2I_2| + C_1$$

Hence:

$$\int \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}dx$$

$$= \frac{x}{2} \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2} + \frac{(a_2 + b_2I_1 + c_2I_2)^2}{2} ln\left|x + \sqrt{x^2 + (a_2 + b_2I_1 + c_2I_2)^2}\right| + C$$

Whereas:

$$C = -ln|a_2 + b_2I_1 + c_2I_2| + C_1$$

Example 7

$$1)\int \sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}dx$$

$$= \frac{x}{2}\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2} + \frac{(8 + 2I_1 - 5I_2)^2}{2} ln\left|\left(x + \sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}\right)\right| + C$$

$$= \frac{x}{2}\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2} + \left(32 + 8I_1 - \frac{55}{2}I_2\right) ln\left|\left(x + \sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}\right)\right| + C$$

Let's check the answer:

$$\frac{d}{dx}\left[\frac{x}{2}\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2} + \left(32 + 8I_1 - \frac{55}{2}I_2\right) ln\left|\left(x + \sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}\right)\right| + C\right]$$

$$= \frac{1}{2}\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2} + \frac{x}{2}.\frac{2x}{2\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}} + \frac{(8 + 2I_1 - 5I_2)^2}{2}\frac{1 + \frac{2x}{2\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}}}{x + \sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}}$$

$$= \frac{1}{2}\left(\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2} + \frac{x^2}{\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}} + (8 + 2I_1 - 5I_2)^2.\frac{\frac{x + \sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}}{\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}}}{x + \sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}}\right)$$

$$= \frac{1}{2}\left(\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2} + \frac{x^2}{\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}} + (8 + 2I_1 - 5I_2)^2.\frac{1}{\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}}\right)$$

$$= \frac{1}{2}\left(\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2} + \frac{x^2 + (8 + 2I_1 - 5I_2)^2}{\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}}\right)$$

$$= \frac{1}{2}\left(\frac{x^2 + (8 + 2I_1 - 5I_2)^2 + x^2 + (8 + 2I_1 - 5I_2)^2}{\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}}\right) = \frac{1}{2}\left(\frac{2(x^2 + (8 + 2I_1 - 5I_2)^2)}{\sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}}\right)$$

$$= \sqrt{x^2 + (8 + 2I_1 - 5I_2)^2}\ (The\ same\ integral\ function)$$

3. Conclusions

The importance of this paper lies in the fact that it provides rules for calculating the integral of 2- refined irrational neutrosophic functions directly, without the need to use long methods to solve these integrals, in addition to use the division of refined neutrosophic numbers. We reached accurate results in the solution, which were verified. Finally, we discussed a set of examples that illustrate the ideas presented.

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Safety Analysis of Natural Medicines Strategies based on SuperHyperSoft Set: Case Study and Analysis of Results

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Abstract: Natural medicines, derived from plant, animal, and mineral sources, have been widely used in traditional and modern healthcare systems due to their perceived effectiveness and minimal side effects. This study proposes a decision-making approach for safety evaluation of natural medicines. We used two methods, such as Entropy method to compute the criteria weights and the VIKOR method to rank the alternatives. These methods are used under the single valued neutrosophic sets (SVNSs) to deal with uncertainty and vague information. We used the SuperHyperSoft set to deal with the criteria and sub criteria values in this evaluation. We used six criteria and seven alternatives in this study. We conducted sensitivity analysis to show the stability of the ranks.

Keywords: Single Valued Neutrosophic Numbers (SVNNs); SuperHyperSoft; Natural Medicines; Strategies Selection.

1. Introduction and Literature Review

Natural medicines, derived from plant, animal, and mineral sources, have been widely used in traditional and modern healthcare systems due to their perceived effectiveness and minimal side effects. However, despite their therapeutic benefits, concerns about their safety, toxicity, and interactions with conventional drugs have led to an increasing demand for rigorous safety evaluation frameworks. Unlike synthetic pharmaceuticals, many natural medicines lack standardized formulations, making it essential to assess their chemical composition, dosage consistency, and potential risks before widespread clinical use[1], [2]. The safety evaluation of natural medicines is therefore crucial for ensuring their reliability and long-term health benefits.

A comprehensive safety evaluation framework must consider multiple factors, including toxicity levels, side effects, pharmacokinetics, and drug interactions. Some natural medicines contain potent bioactive compounds that, if improperly administered, may lead to adverse reactions or

long-term health risks. Additionally, variations in plant species, growing conditions, and extraction methods can influence the potency and purity of herbal products, further complicating safety assessments[3], [4]. As a result, pharmacovigilance programs and scientific validation studies are necessary to minimize risks and ensure the safe use of these remedies.

To enhance the safety of natural medicines, Multi-Criteria Decision-Making (MCDM) methodologies can be employed to systematically assess and rank different herbal products based on their safety profiles. MCDM techniques such as Entropy and VIKOR allow for the consideration of multiple evaluation criteria, such as toxicity levels, dosage safety, clinical validation, and long-term effects, enabling healthcare professionals and regulatory bodies to make informed decisions[5], [6]. By integrating quantitative and qualitative assessments, MCDM methods offer a structured approach to prioritizing safe and effective natural medicines.

Ultimately, the goal of safety evaluation is to establish a scientific and regulatory framework that ensures the responsible use of natural medicines while maintaining their therapeutic potential. Governments, researchers, and healthcare institutions must collaborate to develop standardized safety protocols, promote clinical trials, and enhance consumer awareness regarding the proper usage of herbal remedies[7]. Through rigorous risk assessment, regulatory compliance, and evidence-based research, natural medicines can be safely integrated into modern healthcare systems, bridging the gap between traditional healing practices and contemporary medical science.

By establishing membership measures in 1965, Zadeh [8] initially presented the fuzzy set to address uncertainty and modeling of practical and scientific situations. Bellman and Zadeh made significant contributions to the field of fuzzy decision-making by employing the max and min operators. In 1986, Atanassov [9] created the intuitionistic fuzzy set (IFS) by including the non-membership measure as a stand-alone component.

The literature has documented both theoretical and practical uses of IFSs in MCDM. In the fuzzy environment, Zadeh proposed the entropy measure. Burillo and Bustince [10] provided an axiomatic description of the entropy measure and suggested a distance measure between IFSs.

Following the release of the neutrosophic set (NS) and single-valued neutrosophic set (SVNS), the study of uncertainty underwent a transformation. Researchers are particularly drawn to SVNS because of its decision-making application[11].

Indeterminacy, inconsistency, and incompleteness are characteristics of uncertainty that may be expressed by a SVNS. The majority of the SVN cross entropy that is now in use exhibits asymmetrical behavior and occasionally results in an ill-defined phenomenon. To address these drawbacks, Pramanik et al. [12] provided a novel cross entropy metric, called NS-cross entropy, in a SVNS setting and demonstrate its fundamental characteristics. Additionally, they defined the weighted NS-cross entropy measure and examined its fundamental characteristics. They created a new MCDM approach that does not suffer from undefined phenomena or unbalanced behavior.

It can handle both an unknown weight of decision-makers and an unknown weight of characteristics. Lastly, a numerical illustration of a group decision-making issue with several attributes is provided.

The following is how the remainder of the paper is presented: A few SVNS ideas are explained in Section 2. We developed a decision-making approach using two methods Entropy and VIKOR to compute the criteria weights and rank the alternatives as in section 2 with the SuperHyperSoft set. To show the usefulness and effectiveness of the created Entropy and VIKOR techniques in the SVNS environment, an example problem is handled in Section 3. We provide sensitivity analysis and discussion in Section 4. Conclusions and the direction of future study are provided in Section 5.

2. SuperHyperSoft Set (SHS)

This section shows the definition of the SHS and the neutrosophic set to deal with uncertainty and vague information.

SHS is used to deal with the criteria and sub criteria with different values. It is defined based on the HyperSoft set[13], [14]. Let the universe set $U = \{Q_1, Q_2, \dots . Q_n\}$. The power set of U is a $P(U)$ and S_1, S_2, S_3 are select as a criteria. $P(S_1) \times P(S_2)$ and $P(S_3)$ are $powersets$ of S_1, S_2, S_3

Let F: $P(S_1) \times P(S_2) \times P(S_3) \to P(S)$ $where$ $\times refers$ to $cartesian$ $product$, and this called SHS over S.

$$P(S_1) \times P(S_2) \times P(S_3) = \left\{ \begin{matrix} \{S_{11}\}, \{S_{12}\}, \{S_{11}, S_{12}\} \times \\ \{S_{21}\}, \{S_{22}\}, \{S_{21}, S_{22}\} \times \\ \{S_{31}\}, \{S_{32}\}, \{S_{33}\}, \{S_{31}, S_{32}\}, \{S_{31}, S_{33}\}, \\ \{S_{32}, S_{33}\}, \{S_{31}, S_{32}, S_{33}\} \end{matrix} \right\} \quad (1)$$

Single Valued Neutrosophic Sets (SVNSs)

We can define some definitions of the SVNSs as[12]:

Definition 1

Neutrosophic set can be defined by truth, indeterminacy, and falsity values as $T_D(B), I_D(B), and\ F_D(B)$ (2)

$$D = \left(T_D(B), I_D(B), F_D(B)\right) \quad (3)$$

$$0 \le T_D(B) + I_D(B) + F_D(B) \le 3 \quad (4)$$

Example 1

We can define the neutrosophic number as:

$D = (0.6,0.3,0.4)$ these numbers present the truth, indeterminacy, and falsity functions.

Definition 2

We can define the inclusion of two single valued neutrosophic numbers (SVNNs) as:

$$D_1 \sqsubseteq D_2, T_{D_1}(B) \leq T_{D_2}(B), I_{D_1}(B) \geq I_{D_2}(B), F_{D_1}(B) \geq F_{D_2}(B) \quad (5)$$

Definition 3

The equality of two SVNNs can be defined as:

$$T_{D_1}(B) = T_{D_2}(B), I_{D_1}(B) = I_{D_2}(B), and\ F_{D_1}(B) = F_{D_2}(B) \quad (6)$$

Definition 4

The complement of the SVNN can be defined as:

$$D_1^c = \left(1 - T_{D_1}(B), 1 - I_{D_1}(B), 1 - F_{D_1}(B)\right) \quad (7)$$

Definition 5

The union of the SVNNs can be defined as:

$$D_1 \cup D_2 = \begin{pmatrix} \max\{T_{D_1}(B), T_{D_2}(B)\}, \\ \min\{I_{D_1}(B), I_{D_2}(B)\}, \\ \min\{F_{D_1}(B), F_{D_2}(B)\} \end{pmatrix} \quad (8)$$

Definition 6

The intersection of the SVNNs can be defined as:

$$D_1 \cap D_2 = \begin{pmatrix} \min\{T_{D_1}(B), T_{D_2}(B)\}, \\ \max\{I_{D_1}(B), I_{D_2}(B)\}, \\ \max\{F_{D_1}(B), F_{D_2}(B)\} \end{pmatrix} \quad (9)$$

SVN-Entropy-VIKOR Methodology

This part shows the steps of the SVN-Entropy methodology to compute the criteria weights and the SVN-VIKOR methodology to rank the alternatives.

SVN-Entropy

We use SVNNs to build the decision matrix between the criteria and alternatives. Then we obtain crisp values by the score function, then we combine these numbers.

Normalize the decision matrix

We can normalize the decision matrix by the Entropy method between the criteria and alternatives such as:

$$q_{ij} = \frac{y_{ij}}{\sum_{i=1}^{m} y_{ij}} \quad (10)$$

Where y_{ij} refers to the values in the combined decision matrix, m refers to the number of alternatives and n refers to the number of criteria.

Compute the entropy

$$r_j = -g \sum_{i=1}^{m} q_{ij} \ln q_{ij} \; i = 1, \dots m; j = 1, \dots, n \tag{11}$$

$$g = \frac{1}{\ln m} \tag{12}$$

Compute the criteria weights.

We can obtain the criteria weights such as:

$$w_j = \frac{(1-r_j)}{\sum_{j=1}^{n}(1-r_j)} \tag{13}$$

SVN-VIKOR method

The VIKOR method is used under the SVNS to rank the alternatives.

We start with the combined decision matrix between the criteria and alternatives.

Obtain the values of max and min values for positive and cost criteria such as:

$$p^* = \max y_{ij} \tag{14}$$

$$p^- = \min y_{ij} \tag{15}$$

And for cost criteria such as

$$p^* = \min y_{ij} \tag{16}$$

$$p^- = \max y_{ij} \tag{17}$$

Compute the S and R indexes

$$S_i = \sum_{j=1}^{n} w_j \left(\frac{(p^* - y_{ij})}{p^* - p^-} \right) \tag{18}$$

$$R_i = \max \left[w_j \left(\frac{(p^* - y_{ij})}{p^* - p^-} \right) \right] \tag{19}$$

Compute the VIKOR index

$$K_i = h \times \left[\frac{S_i - S^*}{S^- - S^*} \right] + (1-h) \times \left[\frac{R_i - R^*}{R^- - R^*} \right] \tag{20}$$

$$\begin{pmatrix} R^* = \min R_i \\ R^- = \max R_i \end{pmatrix} \tag{21}$$

$$\begin{pmatrix} S^* = \min S_i \\ S^- = \max S_i \end{pmatrix} \tag{22}$$

Rank the alternatives.

3. Case Study

In this section, we solve the MCDM issue by obtaining the criteria weights and ranking the alternatives. This study proposed a case study to evaluate a set of strategies for safety of natural medicines. These strategies are evaluated using the decision-making model based on a set of criteria to select the best one. We proposed a set of criteria and alternatives.

I. Integration with Modern Medicine
II. Scientific Research
III. Quality Control
IV. Standardization of Herbal Extracts
V. Public Awareness
VI. Risk Assessment
VII. Pharmacovigilance

The criteria of this study can be evaluated such as:

I. Side Effects {Rare, Mild, Severe}
II. Interaction with Other Drugs {Minimal, Moderate, Significant}
III. Scientific Validation {Strong, Moderate, Weak}
IV. Dosage Safety {Sufficient, Uncertain, Insufficient}
V. Toxicity Level {Low, High}
VI. Long-Term Safety {Safe, Risky}

We build the decision matrix by using the SVNNs between the criteria and alternatives as shown in Tables 1-3. Then we obtain crisp values. Then we combined these values.

Eq. (10) is used to normalize the decision matrix as shown in Table 4.

Then we compute the entropy using Eq. (11).

Then we compute the criteria weights using Eq. (13). The criteria weights of this study are organized as follows: C1= 0.273192556, C2= 0.203378481, C3= 0.08053423, C4= 0.106331396, C5= 0.31510081, C6= 0.021462523.

Table 1. The first SVNNs.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.3,0.6,0.7)	(0.1,0.8,0.9)
A_2	(0.1,0.8,0.9)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)
A_3	(0.3,0.6,0.7)	(0.9,0.1,0.2)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.3,0.6,0.7)	(0.3,0.6,0.7)
A_4	(0.5,0.5,0.5)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.8,0.2,0.3)	(0.9,0.1,0.2)	(0.8,0.2,0.3)
A_5	(0.6,0.4,0.5)	(0.9,0.1,0.2)	(0.5,0.5,0.5)	(0.3,0.6,0.7)	(0.1,0.8,0.9)	(0.3,0.6,0.7)
A_6	(0.7,0.3,0.4)	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.8,0.2,0.3)	(0.1,0.8,0.9)	(0.8,0.2,0.3)
A_7	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.9,0.1,0.2)	(0.7,0.3,0.4)	(0.7,0.3,0.4)

Table 2. The second SVNNs.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	(0.9,0.1,0.2)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.3,0.6,0.7)	(0.8,0.2,0.3)
A_2	(0.8,0.2,0.3)	(0.3,0.6,0.7)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)
A_3	(0.7,0.3,0.4)	(0.3,0.6,0.7)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.3,0.6,0.7)	(0.9,0.1,0.2)
A_4	(0.6,0.4,0.5)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.1,0.8,0.9)	(0.9,0.1,0.2)	(0.1,0.8,0.9)
A_5	(0.5,0.5,0.5)	(0.3,0.6,0.7)	(0.5,0.5,0.5)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)
A_6	(0.7,0.3,0.4)	(0.1,0.8,0.9)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
A_7	(0.1,0.8,0.9)	(0.7,0.3,0.4)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)

Table 3. The third SVNNs.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	(0.1,0.8,0.9)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.8,0.2,0.3)
A_2	(0.1,0.8,0.9)	(0.5,0.5,0.5)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)
A_3	(0.9,0.1,0.2)	(0.6,0.4,0.5)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
A_4	(0.5,0.5,0.5)	(0.7,0.3,0.4)	(0.5,0.5,0.5)	(0.8,0.2,0.3)	(0.8,0.2,0.3)	(0.7,0.3,0.4)
A_5	(0.6,0.4,0.5)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.8,0.2,0.3)
A_6	(0.7,0.3,0.4)	(0.5,0.5,0.5)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.6,0.4,0.5)	(0.5,0.5,0.5)
A_7	(0.8,0.2,0.3)	(0.6,0.4,0.5)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.7,0.3,0.4)	(0.6,0.4,0.5)

Table 4. The normalized SVNNs.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.16208	0.167598	0.128463	0.118421	0.102639	0.140845
A_2	0.094801	0.142458	0.173804	0.157895	0.14956	0.126761
A_3	0.171254	0.148045	0.128463	0.118421	0.117302	0.149296
A_4	0.143731	0.184358	0.138539	0.131579	0.219941	0.132394
A_5	0.149847	0.111732	0.118388	0.134211	0.123167	0.149296
A_6	0.183486	0.094972	0.15869	0.165789	0.120235	0.15493
A_7	0.094801	0.150838	0.153652	0.173684	0.167155	0.146479

SVN-VIKOR method

This method is used to rank the alternatives. in the SuperHyperSoft set, we used the values of this study as:

I. {Severe}
II. {Significant}
III. {Strong}
IV. {Sufficient}
V. {Low, High}
VI. {Safe, Risky}

Then we proposed four portions such as:

Portion 1: Severe, Significant, Strong, sufficient, Low, and Safe.

Portion 2: Severe, Significant, Strong, sufficient, Low, and Risky.

Portion 3: Severe, Significant, Strong, sufficient, High, and Safe.

Portion 4: Severe, Significant, Strong, sufficient, High, and Risky.

We rank the alternative based on these portions such as:

Ranking the alternatives based on the Portion 1

We obtain the values of max and min values for positive and cost criteria using Eqs. (14-17).

We compute the S and R indexes using Eqs. (18 and 19). We show the S and R matrix as shown in Fig 1. Then we obtained the S and R index values as shown in Fig 2.

Then we compute the VIKOR index using Eq. (20). In this study we use the h value with 0.5.

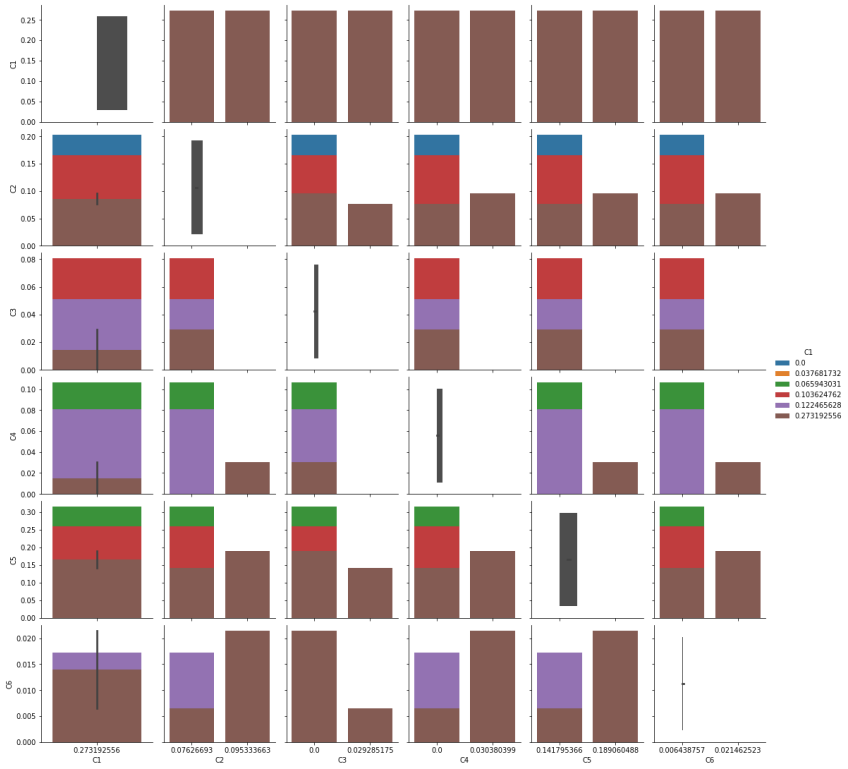

Fig 1. The S and R matrix

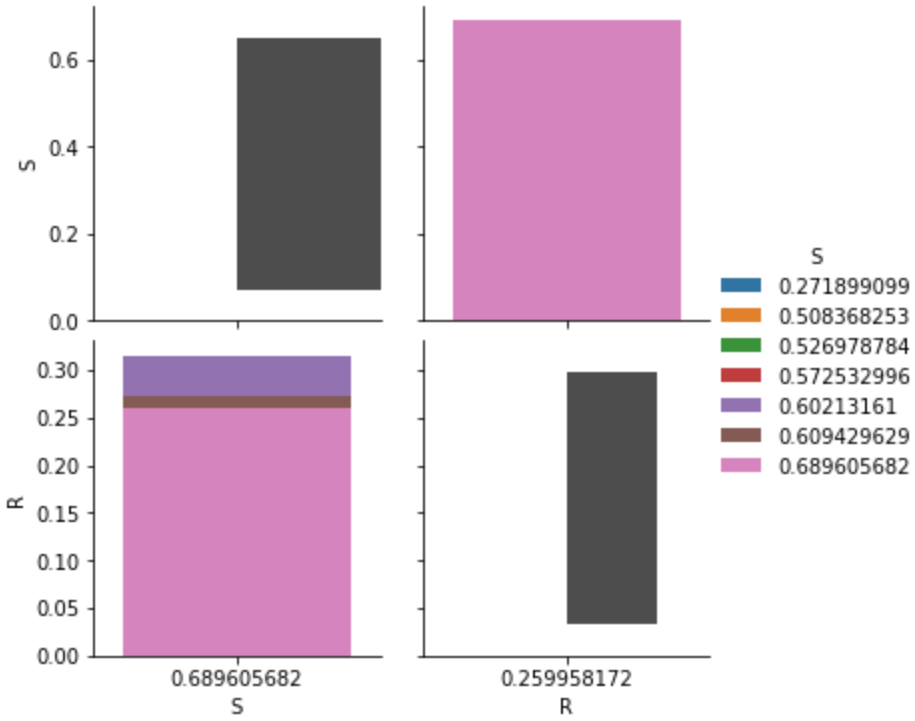

Fig 2. The S and R indexes values.

Ranking the alternatives based on the Portion 2

We compute the S and R indexes, then we show the S and R matrix as shown in Fig 2. Then we obtained the S and R index values as shown in Fig 4.

Then we computed the VIKOR index.

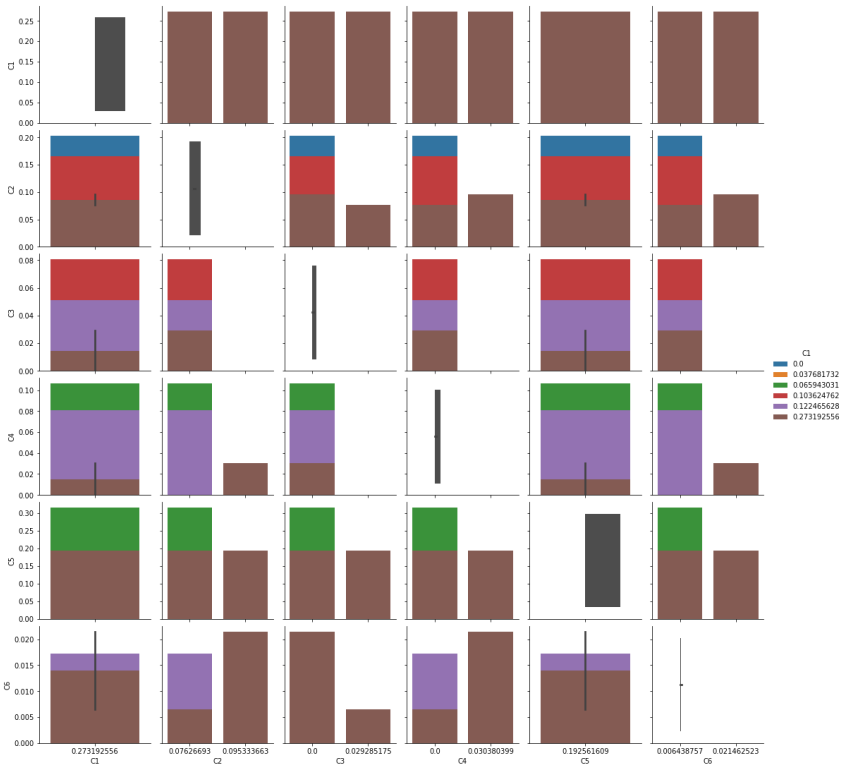

Fig 3. The S and R matrix

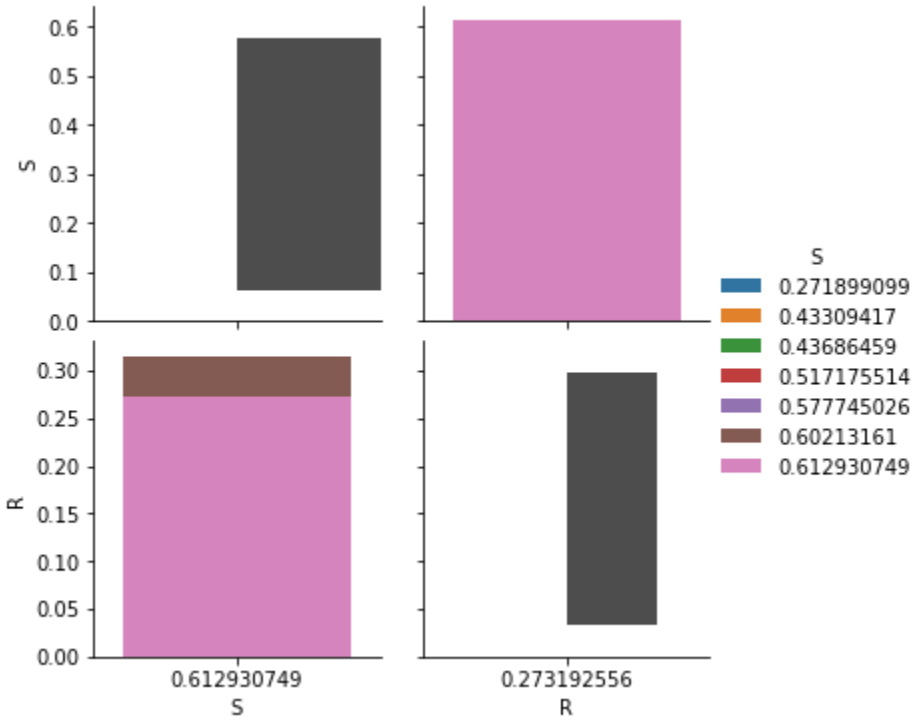

Fig 4. The S and R indexes values.

Ranking the alternatives based on the Portion 3

We compute the S and R indexes, then we show the S and R matrix as shown in Fig 5. Then we obtained the S and R index values as shown in Fig 6.

Then we computed the VIKOR index.

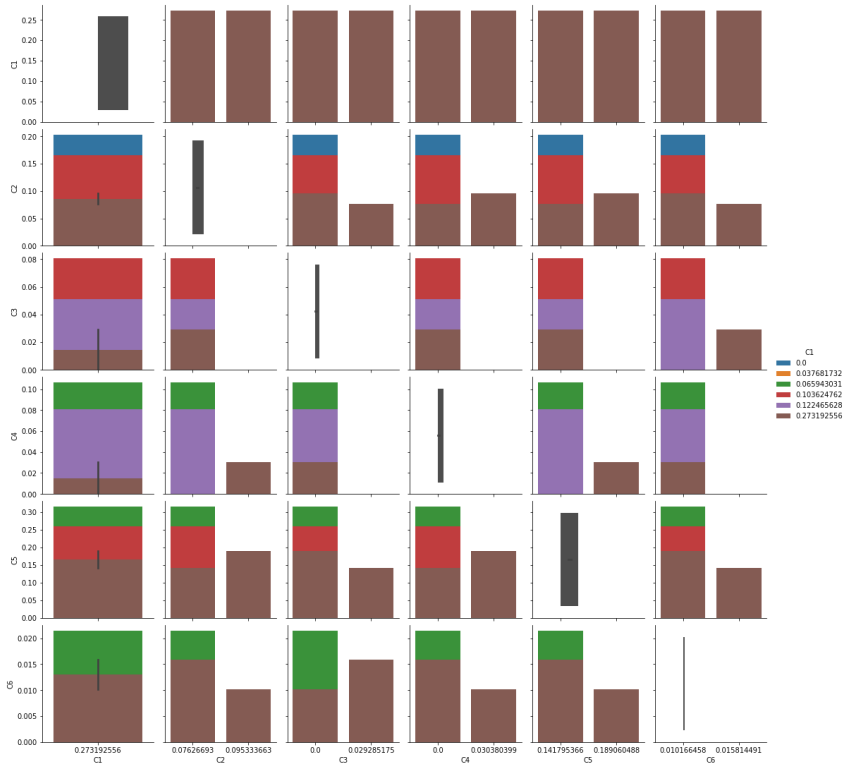

Fig 5. The S and R matrix

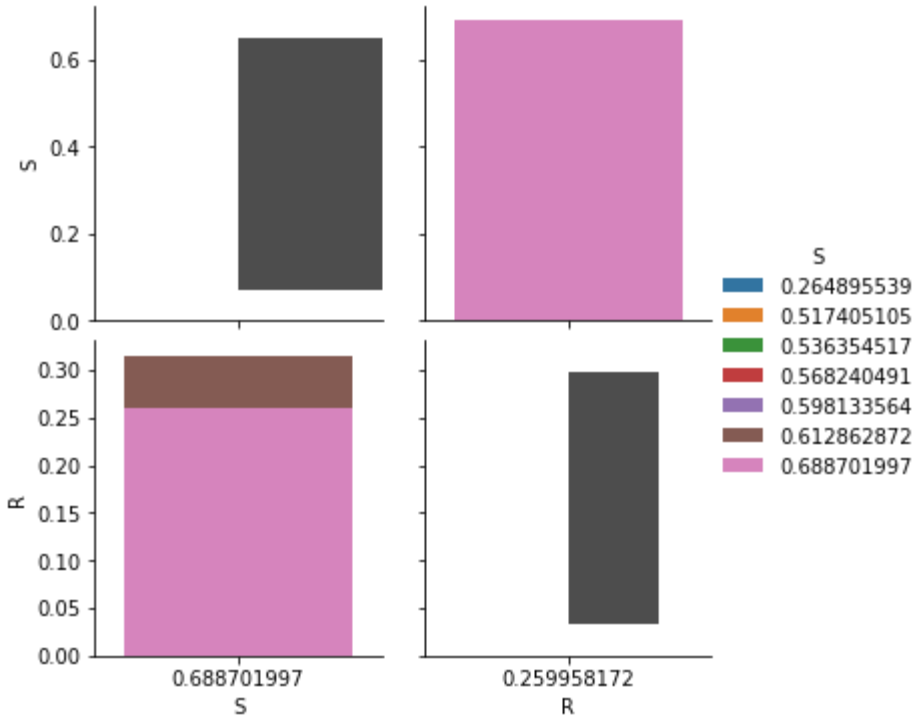

Fig 6. The S and R indexes values.

Ranking the alternatives based on the Portion 4

We compute the S and R indexes, then we show the S and R matrix as shown in Fig 7. Then we obtained the S and R index values as shown in Fig 8.

Then we computed the VIKOR index.

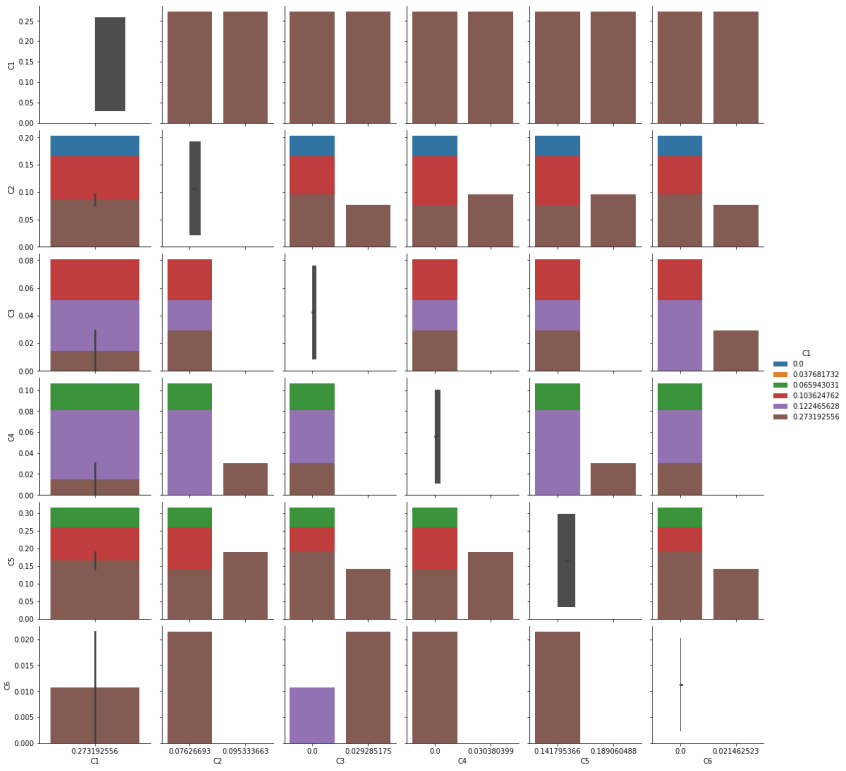

Fig 7. The S and R matrix

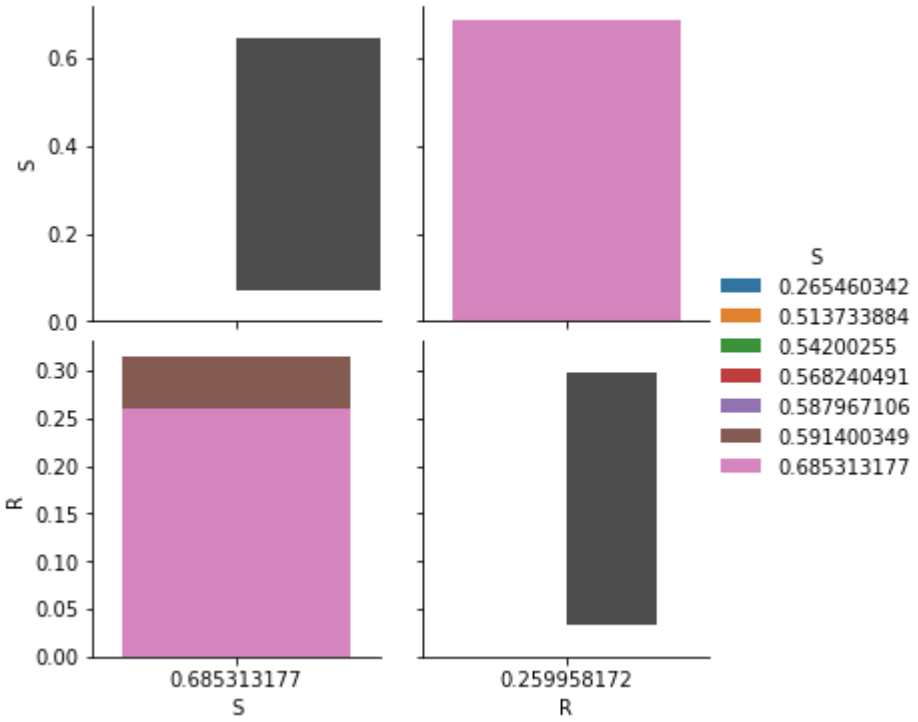

Fig 8. The S and R indexes values.

Then we obtained the VIKOR index values as shown in Fig 9. Then we ranked the alternatives based on each portion as shown in Fig 10.

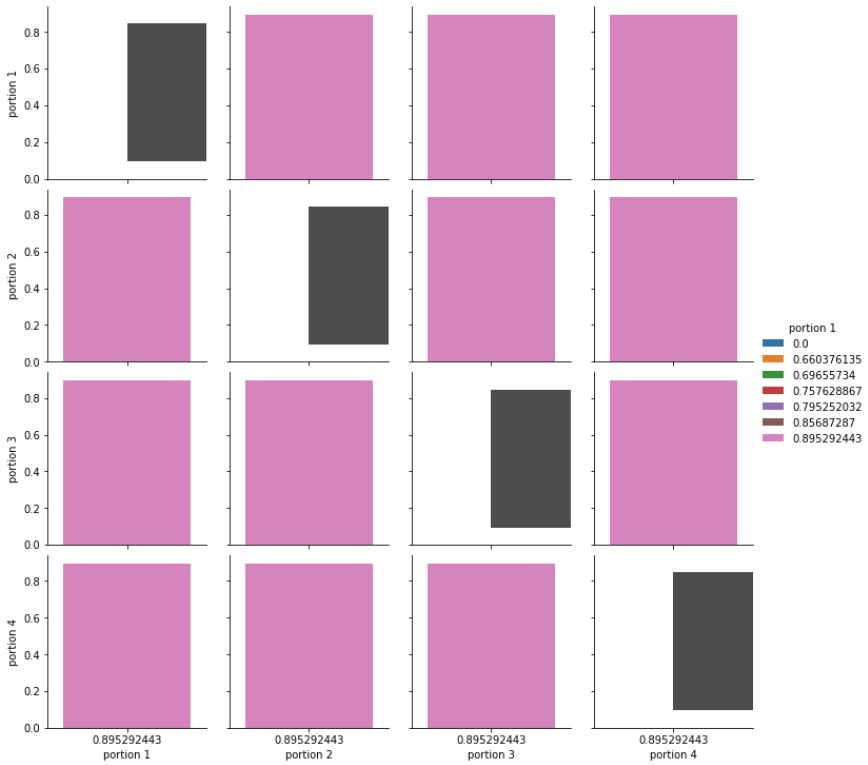

Fig 9. The VIKOR index values.

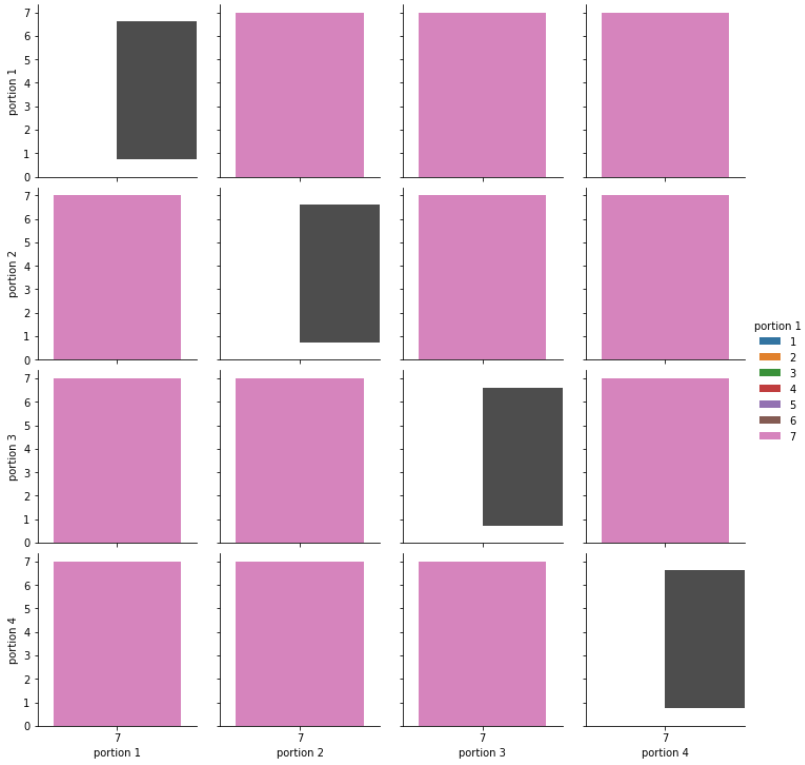

Fig 10. The final ranks.

4. Analysis

This section shows the sensitivity analysis to show the ranks of the alternatives under different cases. In this study, we change the h value between 0 and 1. Then we compute the VIKOR index based on these values as shown in Fig 11. Then we rank the alternatives under different cases as shown on Fig 12. We applied the VIKOR index based on the same weights of the criteria by the Entropy method.

In the first case, we show alternative 4 is the best, followed by alternative 5, alternative 6, and alternative 7. We show alternative 1 is the worst. In the second case, we show alternative 4 is the best, followed by alternative 5, alternative 6, and alternative 7. We show alternative 1 is the worst. In the tenth case, we show alternative 4 is the best, followed by alternative 5, alternative 7, and alternative 3. We show alternative 5 is the worst. In the ninth case, we show alternative 4 is the best, followed by alternative 5, alternative 7, and alternative 3. We show alternative 5 is the worst.

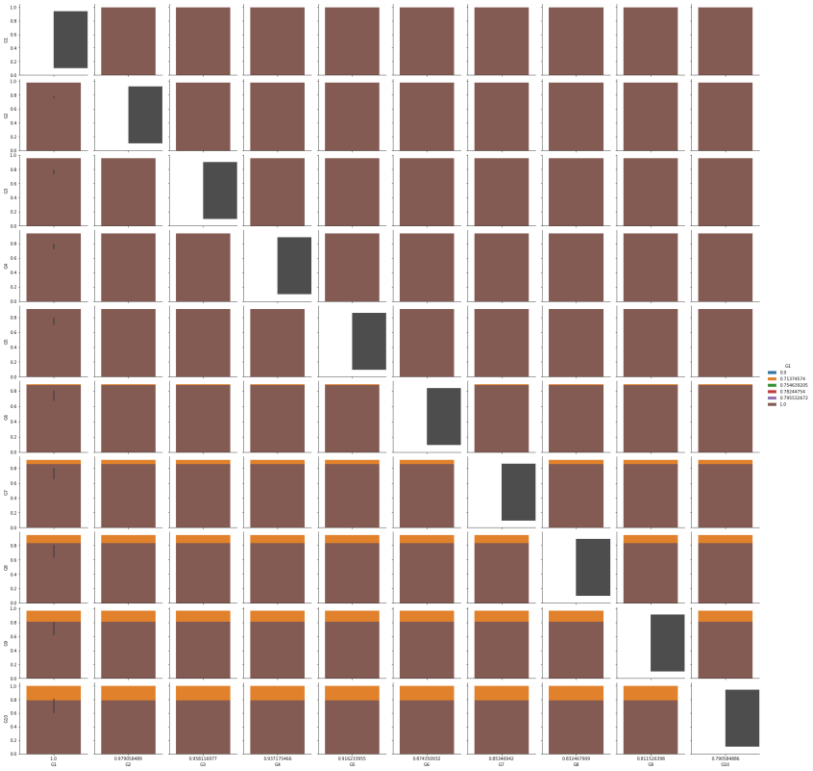

Fig 11. The different VIKOR index values.

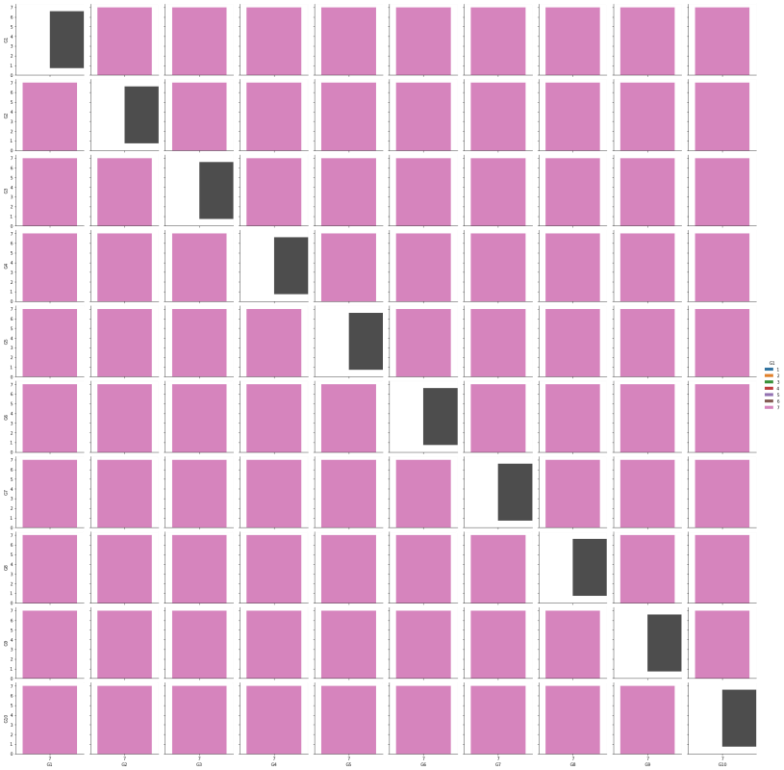

Fig 12. The different ranks.

5. Conclusions

This study evaluated the MCDM approach for safety evaluation of strategies for natural medicines. We used the MCDM approaches to compute the criteria weights and ranks the alternatives. The Entropy method is used to compute the criteria weights. The VIKOR method is used to rank alternatives. We used the single valued neutrosophic sets to deal with uncertainty and vague information. These methods used with the SuperHyperSoft set to treat various criteria and sub values. This study proposed four portions in the SuperHyperSoft set. In each portion we applied the VIKOR method to obtain the ranks of alternatives. Then we show the final ranks of alternatives. The results show the ranks of alternatives is stable. Then we conducted the sensitivity analysis between the ranks of alternatives. We proposed ten cases of ranks. The results show alternative 4 is the best and alternative 6 is the worst.

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New approach towards different ideals using bipolar valued intuitionistic neutrosophic set of an ordered semigroups

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Abstract. This paper introduces the notion of bipolar valued intuitionistic neutrosophic subsemigroup (BIntNS), bipolar valued intuitionistic neutrosophic left ideal (BIntNLI), bipolar valued intuitionistic neutrosophic right ideal (BIntNRI), bipolar valued intuitionistic neutrosophic ideal (NI), and bipolar valued intuitionistic neutrosophic bi-ideal (BIntNBI) of an ordered semigroups, along with some of its properties. A new extension over ordered semigroups of the bipolar valued intuitionistic neutrosophic ideal. The lower level set is a subsemigroup $(BVLI, BVRI, BVBI)$ of an ordered semigroup. Examples are provided to exemplify our results, however the contrary may not be true. Every BIntNS is a BIntNS of an ordered semigroup.

Keywords: Left ideal, right ideal, ideal, bi-ideal.

1. Introduction

Uncertainty has resulted in the development of several uncertain theories, including FS [1], intuitionistic FS (IFS) [2], Pythagorean FS (PFS) [3], and spherical FS (SFS) [4]. An FS consists of sets of membership values, known as MV, ranging from 0 to 1. Although Atanassov's claim that non-membership values (NMV) may only have a value of 1, IFS is classified as MV. The total of MVs and NMVs may occasionally exceed 1 during the decision-making process. Yager [3] developed the generalized MV and NMV logic using PFS logic, with a value of no more than 1 and based on the square of the MVs and NMVs. These notions cannot describe the neutral situation, which is neither positive nor negative. He explored their features in a manner

similar to that of set theory. Rosenfeld [5] defined fuzzy subgroups and their features in 1971. Kuroki [6] introduced fuzzy semigroups as an extension of classical semigroups. Mordeson [7] proposed a fuzzy semigroup characterization. Sen et al. [8, 9] described semigroups and their properties. Kehayopula [10] examined ordered semigroups. Somsak Lekkoksung [11] discusses Q-FIs in ordered semigroups.

Kehayopula et al. [12] began research on fuzzy ordered semigroups. Faiz Muhammad Khan et al. [13] introduced the (τ_1, τ_2)-fuzzy bi-ideal and (τ_1, τ_2)-fuzzy subsemigroup. Jun et al. [14] provided results for ordered semigroups with $(\hbar, \hbar \veebar q)$-fuzzy bi-ideals. Bhakat et al. [15] developed $(\hbar, \hbar \veebar q)$-fuzzy subgroups based on the concept of membership. Kazanci et al. [16] defined FBI characteristics using $(\hbar, \hbar \veebar q)$ and proposed a generalized fuzzy bi-ideal in semigroups. Smarandache [17] invented the neutrosophic set (NS) to handle ambiguous and contradictory data. Huseyin et al. [18] explored the idea of intuitionistic neutrosophic subgroups. Palanikumar et al. [19] introduced an intuitionistic fuzzy normal subbisemiring of bisemiring. Palanikumar et al. [20] introduced bisemiring using bipolar-valued neutrosophic normal sets. The neutrosophic ideals of the ordered semigroup (τ, υ) are explored, and numerous properties are demonstrated using examples. Hila [21] et al. explored the concept of bi-ideals on ordered semigroups. Recently, Udhayakumar et al. discussed the many fuzzy applications and its generalization [22–25].

2. **Preliminaries**

Definition 2.1. An ordered semigroup $\mathscr{O}$ together with an order relation $\leq$ such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in \mathscr{O}$.

Definition 2.2. Let $\beth$ and $\daleth$ be two non empty subsets of $\mathscr{O}$. We denote

(1) $(\beth] = \{t \in \mathscr{O} \mid t \leq h \, for \, some \, h \in \beth\}$,
(2) $\beth\daleth = \{a_1a_2 : a_1 \in \beth, \, a_2 \in \daleth\}$,
(3) $\beth_a = \{(b, c) \in \mathscr{O} \times S \mid a \leq bc\}$.

Definition 2.3. Let $\flat$ be a fuzzy subset of $\mathscr{O}$. A mapping $\flat : \mathscr{O} \to [0, 1]$ is called a fuzzy subsemigroup (FSS) of $\mathscr{O}$ if $\flat(xy) \geq \min\{\flat(x), \flat(y)\}$ for all $x, y \in \mathscr{O}$.

Definition 2.4. A fuzzy $\flat$ subset of $\mathscr{O}$ is called a FBI of $\mathscr{O}$ if

(1) $a_1 \leq a_2 \implies \flat(a_1) \geq \flat(a_2)$ and
(2) $\flat(xyz) \geq \min\{\flat(x), \flat(z)\}$ for all $x, y, z \in \mathscr{O}$.

Definition 2.5. A fuzzy subset υ of an ordered semigroup $\mathscr{O}$ is called a FRI(FLI) of $\mathscr{O}$ if

(1) $x \leq y \Rightarrow \upsilon(x) \geq \upsilon(y)$ for all $x, y \in \mathscr{O}$,
(2) $\upsilon(xy) \geq \upsilon(x)$ (resp. $\upsilon(xy) \geq \upsilon(y)$) for all $x, y \in \mathscr{O}$,

(3) A fuzzy subset υ of an ordered semigroup $\mathscr{O}$ is called a FI of $\mathscr{O}$, if it is both FRI and FLI.

Definition 2.6. Let $\flat$ be a fuzzy subset of $\mathscr{O}$ and $t \in [0,1]$. The set $\flat_t = \{x \in \mathscr{O} | \flat(x) \geq t\}$ is called the level subset of $\flat$. Clearly $\flat_t \subseteq \flat_s$ whenever $t \geq s$.

Definition 2.7. Let A be a fuzzy set,if $\Lambda_{_A}$ is the characteristic function of A, then $(\Lambda_{_A})^{\ell}_{\xi}$ is defined as

$$(\Lambda_{_A})^{\ell}_{\xi}(x) := \begin{cases} \ell \, if \, x \in A, \\ \xi \, if \, x \notin A. \end{cases}$$

Definition 2.8. An intuitionistic neutrosophic set $\tilde{N}$ on the set X is defined as follows:
$\tilde{N} = \{(x, \langle \tilde{\Re}(x), \tilde{\Im}(x), \tilde{\mho}(x) \rangle) : x \in X\}$ where for all $x \in X$,
$\min\{\tilde{\Re}(x), \tilde{\Im}(x)\} \leq 0.5, \min\{\tilde{\Re}(x), \tilde{\mho}(x)\} \leq 0.5, \min\{\tilde{\Im}(x), \tilde{\mho}(x)\} \leq 0.5$ with the condition
$0 = \tilde{\Re}(x) + \tilde{\Im}(x) + \tilde{\mho}(x) \leq 2$.

3. (τ, υ) bipolar valued intuitionistic neutrosophic ideals

The ordered semigroup is indicated in this section by $\mathscr{O}$. If $(\tau, \upsilon) \in [0,1]$ is such that $0 \geq \tau^- > \upsilon^- \geq -1$ and $0 \leq \tau^+ < \upsilon^+ \leq 1$, then both (τ, υ) are arbitrarily fixed.

Definition 3.1. A bipolar valued intuitionistic neutrosophic subset
$\flat = [(\tilde{\Re}^-_{\intercal}, \tilde{\Re}^+_{\intercal}), (\tilde{\Im}^-_{\intercal}, \tilde{\Im}^+_{\intercal}), (\tilde{\mho}^-_{\intercal}, \tilde{\mho}^+_{\intercal})]$ of $\mathscr{O}$ is called a (τ, υ) BIntNS of $\mathscr{O}$ if

(1) $\epsilon \leq \varsigma \Rightarrow \tilde{\Re}^-(\epsilon) \leq \tilde{\Re}^-(\varsigma)$, $\tilde{\Im}^-(\epsilon) \leq \tilde{\Im}^-(\varsigma)$ and $\tilde{\mho}^-(\epsilon) \geq \tilde{\mho}^-(\varsigma)$, $\tilde{\Re}^+(\epsilon) \geq \tilde{\Re}^+(\varsigma)$, $\tilde{\Im}^+(\epsilon) \geq \tilde{\Im}^+(\varsigma)$ and $\tilde{\mho}^+(\epsilon) \leq \tilde{\mho}^+(\varsigma)$,

(2) $\min\{\tilde{\Re}^-(\epsilon\varsigma), \tau^-\} \leq \max\{\tilde{\Re}^-(\epsilon), \tilde{\Re}^-(\varsigma), \upsilon^-\}$,
$\min\{\tilde{\Im}^-(\epsilon\varsigma), \tau^-\} \leq \max\{\tilde{\Im}^-(\epsilon), \tilde{\Im}^-(\varsigma), \upsilon^-\}$,
$\max\{\tilde{\mho}^-(\epsilon\varsigma), \tau^-\} \geq \min\{\tilde{\mho}^-(\epsilon), \tilde{\mho}^-(\varsigma), \upsilon^-\}$,

(3) $\max\{\tilde{\Re}^+(\epsilon\varsigma), \tau^+\} \geq \min\{\tilde{\Re}^+(\epsilon), \tilde{\Re}^+(\varsigma), \upsilon^+\}$,
$\max\{\tilde{\Im}^+(\epsilon\varsigma), \tau^+\} \geq \min\{\tilde{\Im}^+(\epsilon), \tilde{\Im}^+(\varsigma), \upsilon^+\}$,
$\min\{\tilde{\mho}^+(\epsilon\varsigma), \tau^+\} \leq \max\{\tilde{\mho}^+(\epsilon), \tilde{\mho}^+(\varsigma), \upsilon^+\}$. for all $\epsilon, \varsigma \in \mathscr{O}$.

Example 3.2. Let $\mathscr{O} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$ is defined on $\mathscr{O}$ with the following Cayley table:

$\cdot$	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$
$\hbar_1$	$\hbar_1$	$\hbar_1$	$\hbar_1$	$\hbar_1$
$\hbar_2$	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$
$\hbar_3$	$\hbar_1$	$\hbar_3$	$\hbar_3$	$\hbar_3$
$\hbar_4$	$\hbar_1$	$\hbar_3$	$\hbar_3$	$\hbar_3$

$\leq := \{(\hbar_1,\hbar_1),(\hbar_1,\hbar_2),(\hbar_1,\hbar_3),(\hbar_1,\hbar_4),(\hbar_2,\hbar_2),(\hbar_2,\hbar_3),(\hbar_2,\hbar_4),(\hbar_3,\hbar_3),(\hbar_4,\hbar_3),(\hbar_4,\hbar_4)\}$.
The mapping $\mathsf{T} = [(\tilde{\Re}^-_{\mathsf{T}},\tilde{\Re}^+_{\mathsf{T}}),(\tilde{\Im}^-_{\mathsf{T}},\tilde{\Im}^+_{\mathsf{T}}),(\tilde{\mho}^-_{\mathsf{T}},\tilde{\mho}^+_{\mathsf{T}})] : \mathscr{O}\times\mathscr{O}\to[-1,0]\times[0,1]$.

$$(\tilde{\Re}^-,\tilde{\Re}^+)(\hbar)=\begin{cases}(-0.47,44)\ \ if\ \hbar=\hbar_1\\(-0.32,37)\ \ if\ \hbar=\hbar_2\\(-0.02,07)\ \ if\ \hbar=\hbar_3\\(-0.12,17)\ \ if\ \hbar=\hbar_4\end{cases}\quad(\tilde{\Im}^-,\tilde{\Im}^+)(\hbar)=\begin{cases}(-0.27,23)\ \ if\ \hbar=\hbar_1\\(-0.17,12)\ \ if\ \hbar=\hbar_2\\(-0.02,07)\ \ if\ \hbar=\hbar_3\\(-0.07,10)\ \ if\ \hbar=\hbar_4\end{cases}$$

$$(\tilde{\mho}^-,\tilde{\mho}^+)(\hbar)=\begin{cases}(-0.22,27)\ \ if\ \hbar=\hbar_1\\(-0.27,32)\ \ if\ \hbar=\hbar_2\\(-0.37,42)\ \ if\ \hbar=\hbar_3\\(-0.32,37)\ \ if\ \hbar=\hbar_4\end{cases}$$

Then T is a $(0.42,0.48)$ BIntNS of $\mathscr{O}$.

Definition 3.3. A subset $\flat$ of $\mathscr{O}$ is called a (τ,υ)-BIntNBI of $\mathscr{O}$ if

(1) If $\epsilon\leq\varsigma$, then $\tilde{\Re}^-(\epsilon)\leq\tilde{\Re}^-(\varsigma)$, $\tilde{\Im}^-(\epsilon)\leq\tilde{\Im}^-(\varsigma)$ and $\tilde{\mho}^-(\epsilon)\geq\tilde{\mho}^-(\varsigma)$, $\tilde{\Re}^+(\epsilon)\geq\tilde{\Re}^+(\varsigma)$, $\tilde{\Im}^+(\epsilon)\geq\tilde{\Im}^+(\varsigma)$ and $\tilde{\mho}^+(\epsilon)\leq\tilde{\mho}^+(\varsigma)$,

(2) $\min\{\tilde{\Re}^-(\epsilon_1\varsigma),\tau^-\}\leq\max\{\tilde{\Re}^-(\epsilon),\tilde{\Re}^-(\varsigma),\upsilon^-\}$,
$\min\{\tilde{\Im}^-(\epsilon_1\varsigma),\tau^-\}\leq\max\{\tilde{\Im}^-(\epsilon),\tilde{\Im}^-(\varsigma),\upsilon^-\}$,
$\max\{\tilde{\mho}^-(\epsilon_1\varsigma),\tau^-\}\geq\min\{\tilde{\mho}^-(\epsilon),\tilde{\mho}^-(\varsigma),\upsilon^-\}$,
$\min\{\tilde{\Re}^-(\epsilon_1\varsigma_2\kappa),\tau^-\}\leq\max\{\tilde{\Re}^-(\epsilon),\tilde{\Re}^-(\kappa),\upsilon^-\}$,
$\min\{\tilde{\Im}^-(\epsilon_1\varsigma_2\kappa),\tau^-\}\leq\max\{\tilde{\Im}^-(\epsilon),\tilde{\Im}^-(\kappa),\upsilon^-\}$,
$\max\{\tilde{\mho}^-(\epsilon_1\varsigma_2\kappa),\tau^-\}\geq\min\{\tilde{\mho}^-(\epsilon),\tilde{\mho}^-(\kappa),\upsilon^-\}$,

(3) $\max\{\tilde{\Re}^+(\epsilon_1\varsigma),\tau^+\}\geq\min\{\tilde{\Re}^+(\epsilon),\tilde{\Re}^+(\varsigma),\upsilon^+\}$,
$\max\{\tilde{\Im}^+(\epsilon_1\varsigma),\tau^+\}\geq\min\{\tilde{\Im}^+(\epsilon),\tilde{\Im}^+(\varsigma),\upsilon^+\}$,
$\min\{\tilde{\mho}^+(\epsilon_1\varsigma),\tau^+\}\leq\max\{\tilde{\mho}^+(\epsilon),\tilde{\mho}^+(\varsigma),\upsilon^+\}$,
$\max\{\tilde{\Re}^+(\epsilon_1\varsigma_2\kappa),\tau^+\}\geq\min\{\tilde{\Re}^+(\epsilon),\tilde{\Re}^+(\kappa),\upsilon^+\}$,
$\max\{\tilde{\Im}^+(\epsilon_1\varsigma_2\kappa),\tau^+\}\geq\min\{\tilde{\Im}^+(\epsilon),\tilde{\Im}^+(\kappa),\upsilon^+\}$,
$\min\{\tilde{\mho}^+(\epsilon_1\varsigma_2\kappa),\tau^+\}\leq\max\{\tilde{\mho}^+(\epsilon),\tilde{\mho}^+(\kappa),\upsilon^+\}$, for $\epsilon,\varsigma,\kappa\in\mathscr{O}$

Theorem 3.4. *A non-empty subset $\flat_\tau$ is a $\tilde{\Re}_\tau$ is a (τ,υ) BIntNS (BIntNLI, BIntNRI, BIntNBI) of $\mathscr{O}$. Then the lower level set is an BSS $(BVLI, BVRI, BVBI)$ of $\mathscr{O}$, where $\tilde{\Re}_{\tau^-}=\{\epsilon\in\mathscr{O}|\tilde{\Re}(\epsilon)<\tau^-\}$, $\tilde{\Im}_{\tau^-}=\{\epsilon\in\mathscr{O}|\tilde{\Re}(\epsilon)<\tau^-\}$, $\tilde{\mho}_{\tau^-}=\{\epsilon\in\mathscr{O}|\tilde{\Re}(\epsilon)>\tau^-\}$, $\tilde{\Re}_{\tau^+}=\{\epsilon\in\mathscr{O}|\tilde{\Re}(\epsilon)>\tau^+\}$, $\tilde{\Im}_{\tau^+}=\{\epsilon\in\mathscr{O}|\tilde{\Re}(\epsilon)>\tau^+\}$ and $\tilde{\mho}_{\tau^+}=\{\epsilon\in\mathscr{O}|\tilde{\Re}(\epsilon)<\tau^+\}$*

Proof. Suppose that $\flat_{\tau^-}$ is a (τ,υ) BIntNS of $\mathscr{O}$. Let $\epsilon,\varsigma\in\mathscr{O}$ such that $\epsilon,\varsigma\in\tilde{\Re}^-_{\tau^-}$. Then $\tilde{\Re}^-(\epsilon)<\tau^-,\tilde{\Re}^-(\varsigma)<\tau^-$. Therefore $\min\{\tilde{\Re}^-(\epsilon\varsigma),\tau^-\}\leq\max\{\tilde{\Re}^-(\epsilon),\tilde{\Re}^-(\varsigma),\upsilon^-\}<\max\{\tau^-,\tau^-,\upsilon^-\}=\tau^-$. Hence $\tilde{\Re}^-(\epsilon\varsigma)<\tau^-$. It shows that $\epsilon\varsigma\in\tilde{\Re}^-_{\tau^-}$. Therefore $\tilde{\Re}^-_{\tau^-}$ is a

BSS of $\mathscr{O}$. Let $\epsilon, \varsigma \in \mathscr{O}$ such that $\epsilon, \varsigma \in \tilde{\Im}^{-}_{\tau^-}$. Then $\tilde{\Im}^{-}(\epsilon) < \tau^-, \tilde{\Im}^{-}(\varsigma) < \tau^-$. Therefore $\min\{\tilde{\Im}^{-}(\epsilon\varsigma), \tau^-\} \leq \max\{\tilde{\Im}^{-}(\epsilon), \tilde{\Im}^{-}(\varsigma), \upsilon^-\} < \max\{\tau^-, \tau^-, \upsilon^-\} = \tau^-$. Hence $\tilde{\Im}^{-}(\epsilon\varsigma) < \tau^-$. It shows that $\epsilon\varsigma \in \tilde{\Im}^{-}_{\tau^-}$. Therefore $\tilde{\Im}^{-}_{\tau^-}$ is a BSS of $\mathscr{O}$. Let $\epsilon, \varsigma \in \mathscr{O}$ such that $\epsilon, \varsigma \in \tilde{\mho}^{-}_{\tau^-}$. Then $\tilde{\mho}^{-}(\epsilon) > \tau^-, \tilde{\mho}^{-}(\varsigma) > \tau^-$. Therefore $\max\{\tilde{\mho}^{-}(\epsilon\varsigma), \tau^-\} \geq \min\{\tilde{\mho}^{-}(\epsilon), \tilde{\mho}^{-}(\varsigma), \upsilon^-\} > \min\{\tau^-, \tau^-, \upsilon^-\} = \upsilon^-$. Hence $\tilde{\mho}^{-}(\epsilon\varsigma) > \tau^-$. It shows that $\epsilon\varsigma \in \tilde{\mho}^{-}_{\tau^-}$. Therefore $\tilde{\mho}^{-}_{\tau^-}$ is a BSS of $\mathscr{O}$.

Suppose that $\flat_{\tau^+}$ is a (τ, υ) BIntNS of $\mathscr{O}$. Let $\epsilon, \varsigma \in \mathscr{O}$ such that $\epsilon, \varsigma \in \tilde{\Re}^{+}_{\tau^+}$. Then $\tilde{\Re}^{+}(\epsilon) > \tau^+, \tilde{\Re}^{+}(\varsigma) > \tau^+$. Therefore $\max\{\tilde{\Re}^{+}(\epsilon\varsigma), \tau^+\} \geq \min\{\tilde{\Re}^{+}(\epsilon), \tilde{\Re}^{+}(\varsigma), \upsilon^+\} > \min\{\tau^+, \tau^+, \upsilon^+\} = \tau^+$. Hence $\tilde{\Re}^{+}(\epsilon\varsigma) > \tau^+$. It shows that $\epsilon\varsigma \in \tilde{\Re}^{+}_{\tau^+}$. Therefore $\tilde{\Re}^{+}_{\tau^+}$ is a BSS of $\mathscr{O}$. Let $\epsilon, \varsigma \in \mathscr{O}$ such that $\epsilon, \varsigma \in \tilde{\Im}^{+}_{\tau^+}$. Then $\tilde{\Im}^{+}(\epsilon) > \tau^+, \tilde{\Im}^{+}(\varsigma) > \tau^+$. Therefore $\max\{\tilde{\Im}^{+}(\epsilon\varsigma), \tau^+\} \geq \min\{\tilde{\Im}^{+}(\epsilon), \tilde{\Im}^{+}(\varsigma), \upsilon^+\} > \min\{\tau^+, \tau^+, \upsilon^+\} = \tau^+$. Hence $\tilde{\Im}^{+}(\epsilon\varsigma) > \tau^+$. It shows that $\epsilon\varsigma \in \tilde{\Im}^{+}_{\tau^+}$. Therefore $\tilde{\Im}^{+}_{\tau^+}$ is a BSS of $\mathscr{O}$. Let $\epsilon, \varsigma \in \mathscr{O}$ such that $\epsilon, \varsigma \in \tilde{\mho}^{+}_{\tau^+}$. Then $\tilde{\mho}^{+}(\epsilon) < \tau^+, \tilde{\mho}^{+}(\varsigma) < \tau^+$. Therefore $\min\{\tilde{\mho}^{+}(\epsilon\varsigma), \tau^+\} \leq \max\{\tilde{\mho}^{+}(\epsilon), \tilde{\mho}^{+}(\varsigma), \upsilon^+\} < \max\{\tau^+, \tau^+, \upsilon^+\} = \upsilon^+$. Hence $\tilde{\mho}^{+}(\epsilon\varsigma) < \tau^+$. It shows that $\epsilon\varsigma \in \tilde{\mho}^{+}_{\tau^+}$. Therefore $\tilde{\mho}^{+}_{\tau^+}$ is a BSS of $\mathscr{O}$. Therefore $\flat_{\tau^+}$ is a BSS of $\mathscr{O}$. Therefore $\flat_{\tau}$ is a BSS of $\mathscr{O}$.

Theorem 3.5. *A non-empty subset $\daleth$ of $\mathscr{O}$ is a BSS $[BVLI, BVRI, BVBI]$ of $\mathscr{O}$ if and only if the intuitionistic neutrosophic subset $\flat = [(\tilde{\Re}^{-}_{\mathsf{T}}, \tilde{\Re}^{+}_{\mathsf{T}}), (\tilde{\Im}^{-}_{\mathsf{T}}, \tilde{\Im}^{+}_{\mathsf{T}}), (\tilde{\mho}^{-}_{\mathsf{T}}, \tilde{\mho}^{+}_{\mathsf{T}})]$ of $\mathscr{O}$ is defined as*

$$\tilde{\Re}^{-}(\epsilon) = \begin{cases} \leq \upsilon^- \, for\, all \;\; \epsilon \in (\daleth] \\ \tau^- \, for\, all \; \epsilon \notin (\daleth] \end{cases} \quad \tilde{\Im}^{-}(\epsilon) = \begin{cases} \leq \upsilon^- \, for\, all \;\; \epsilon \in (\daleth] \\ \tau^- \, for\, all \; \epsilon \notin (\daleth] \end{cases} \quad \tilde{\mho}^{-}(\epsilon) = \begin{cases} \geq \upsilon^- \, for\, all \;\; \epsilon \in (\daleth] \\ \tau^- \, for\, all \; \epsilon \notin (\daleth] \end{cases}$$

$$\tilde{\Re}^{+}(\epsilon) = \begin{cases} \geq \upsilon^+ \, for\, all \;\; \epsilon \in (\daleth] \\ \tau^+ \, for\, all \; \epsilon \notin (\daleth] \end{cases} \quad \tilde{\Im}^{+}(\epsilon) = \begin{cases} \geq \upsilon^+ \, for\, all \;\; \epsilon \in (\daleth] \\ \tau^+ \, for\, all \; \epsilon \notin (\daleth] \end{cases} \quad \tilde{\mho}^{+}(\epsilon) = \begin{cases} \leq \upsilon^+ \, for\, all \;\; \epsilon \in (\daleth] \\ \tau^+ \, for\, all \; \epsilon \notin (\daleth] \end{cases}$$

is a (τ, υ) BIntNS[BIntNLI, BIntNRI, BIntNBI] of $\mathscr{O}$

Proof. Suppose that $\daleth$ is an BSS of $\mathscr{O}$. Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\daleth]$ then $\epsilon\varsigma \in (\daleth]$. Hence $\tilde{\Re}^{-}(\epsilon\varsigma) \leq \upsilon^-$, $\tilde{\Im}^{-}(\epsilon\varsigma) \leq \upsilon^-$ and $\tilde{\mho}^{-}(\epsilon\varsigma) \geq \upsilon^-$. Thus, $\min\{\tilde{\Re}^{-}(\epsilon\varsigma), \tau^-\} \leq \upsilon^- = \max\{\tilde{\Re}^{-}(\epsilon), \tilde{\Re}^{-}(\varsigma), \upsilon^-\}$, $\min\{\tilde{\Im}^{-}(\epsilon\varsigma), \tau^-\} \leq \upsilon^- = \max\{\tilde{\Im}^{-}(\epsilon), \tilde{\Im}^{-}(\varsigma), \upsilon^-\}$ and $\max\{\tilde{\mho}^{-}(\epsilon\varsigma), \tau^-\} \geq \upsilon^- = \min\{\tilde{\mho}^{-}(\epsilon), \tilde{\mho}^{-}(\varsigma), \upsilon^-\}$.

If $\epsilon \notin (\daleth]$ or $\varsigma \notin (\daleth]$, then $\max\{\tilde{\Re}^{-}(\epsilon), \tilde{\Re}^{-}(\varsigma), \upsilon^-\} = \tau^-$, $\max\{\tilde{\Im}^{-}(\epsilon), \tilde{\Im}^{-}(\varsigma), \upsilon^-\} = \tau^-$ and $\min\{\tilde{\mho}^{-}(\epsilon), \tilde{\mho}^{-}(\varsigma), \upsilon^-\} = \upsilon^-$. That is $\min\{\tilde{\Re}^{-}(\epsilon\varsigma), \tau^-\} \leq \max\{\tilde{\Re}^{-}(\epsilon), \tilde{\Re}^{-}(\varsigma), \upsilon^-\}$, $\min\{\tilde{\Im}^{-}(\epsilon\varsigma), \tau^-\} \leq \max\{\tilde{\Im}^{-}(\epsilon), \tilde{\Im}^{-}(\varsigma), \upsilon^-\}$ and $\max\{\tilde{\mho}^{-}(\epsilon\varsigma), \tau^-\} \geq \min\{\tilde{\mho}^{-}(\epsilon), \tilde{\mho}^{-}(\varsigma), \upsilon^-\}$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\daleth]$ then $\epsilon\varsigma \in (\daleth]$. Hence $\tilde{\Re}^{+}(\epsilon\varsigma) \geq \upsilon^+$, $\tilde{\Im}^{+}(\epsilon\varsigma) \geq \upsilon^+$ and $\tilde{\mho}^{+}(\epsilon\varsigma) \leq \upsilon^+$. Thus, $\max\{\tilde{\Re}^{+}(\epsilon\varsigma), \tau^+\} \geq \upsilon^+ = \min\{\tilde{\Re}^{+}(\epsilon), \tilde{\Re}^{+}(\varsigma), \upsilon^+\}$, $\max\{\tilde{\Im}^{+}(\epsilon\varsigma), \tau^+\} \geq \upsilon^+ = \min\{\tilde{\Im}^{+}(\epsilon), \tilde{\Im}^{+}(\varsigma), \upsilon^+\}$ and $\min\{\tilde{\mho}^{+}(\epsilon\varsigma), \tau^+\} \leq \upsilon^+ = \max\{\tilde{\mho}^{+}(\epsilon), \tilde{\mho}^{+}(\varsigma), \upsilon^+\}$.

If $\epsilon \notin (\daleth]$ or $\varsigma \notin (\daleth]$, then $\min\{\tilde{\Re}^{+}(\epsilon), \tilde{\Re}^{+}(\varsigma), \upsilon^+\} = \tau^+$, $\min\{\tilde{\Im}^{+}(\epsilon), \tilde{\Im}^{+}(\varsigma), \upsilon^+\} = \tau^+$ and $\max\{\tilde{\mho}^{+}(\epsilon), \tilde{\mho}^{+}(\varsigma), \upsilon^+\} = \upsilon^+$.

That is $\max\{\tilde{\Re}^+(\epsilon\varsigma),\tau^+\} \geq \min\{\tilde{\Re}^+(\epsilon),\tilde{\Re}^+(\varsigma),\upsilon^+\}$, $\max\{\tilde{\Im}^+(\epsilon\varsigma),\tau^+\} \geq \min\{\tilde{\Im}^+(\epsilon),\tilde{\Im}^+(\varsigma),\upsilon^+\}$ and $\min\{\tilde{\mho}^+(\epsilon\varsigma),\tau^+\} \leq \max\{\tilde{\mho}^+(\epsilon),\tilde{\mho}^+(\varsigma),\upsilon^+\}$. Therefore $\flat$ is a (τ,υ) BIntNS of $\mathscr{O}$.

Conversely, assume that $\flat = [\tilde{\Re}^-,\tilde{\Im}^-,\tilde{\mho}^-]$ is a (τ,υ) BIntNS of $\mathscr{O}$. Let $\epsilon,\varsigma \in (\daleth]$. Then $\tilde{\Re}^-(\epsilon) \leq \upsilon^-, \tilde{\Re}^-(\varsigma) \leq \upsilon^-$, $\tilde{\Im}^-(\epsilon) \leq \upsilon^-, \tilde{\Im}^-(\varsigma) \leq \upsilon^-$ and $\tilde{\mho}^-(\epsilon) \geq \upsilon^-, \tilde{\mho}^-(\varsigma) \geq \upsilon^-$. Now $\flat = [\tilde{\Re}^-,\tilde{\Im}^-,\tilde{\mho}^-]$ is a (τ,υ) BIntNS of $\mathscr{O}$. Therefore $\min\{\tilde{\Re}^-(\epsilon\varsigma),\tau^-\} \leq \max\{\tilde{\Re}^-(\epsilon),\tilde{\Re}^-(\varsigma),\upsilon^-\} \leq \max\{\upsilon^-,\upsilon^-,\upsilon^-\} = \upsilon^-$, $\min\{\tilde{\Im}^-(\epsilon\varsigma),\tau^-\} \leq \max\{\tilde{\Im}^-(\epsilon),\tilde{\Im}^-(\varsigma),\upsilon^-\} \leq \max\{\upsilon^-,\upsilon^-,\upsilon^-\} = \upsilon^-$ and $\max\{\tilde{\mho}^-(\epsilon\varsigma),\tau^-\} \geq \min\{\tilde{\mho}^-(\epsilon),\tilde{\mho}^-(\varsigma),\upsilon^-\} \geq \min\{\upsilon^-,\upsilon^-,\upsilon^-\} = \upsilon^-$. It follows that $\epsilon\varsigma \in (\daleth]$. Let $\epsilon,\varsigma \in (\daleth]$. Then $\tilde{\Re}^+(\epsilon) \geq \upsilon^+, \tilde{\Re}^+(\varsigma) \geq \upsilon^+$, $\tilde{\Im}^+(\epsilon) \geq \upsilon^+, \tilde{\Im}^+(\varsigma) \geq \upsilon^+$ and $\tilde{\mho}^+(\epsilon) \leq \upsilon^+, \tilde{\mho}^+(\varsigma) \leq \upsilon^+$. Now $\flat = [\tilde{\Re}^+,\tilde{\Im}^+,\tilde{\mho}^+]$ is a (τ,υ) BIntNS of $\mathscr{O}$.
Therefore, $\max\{\tilde{\Re}^+(\epsilon\varsigma),\tau^+\} \geq \min\{\tilde{\Re}^+(\epsilon),\tilde{\Re}^+(\varsigma),\upsilon^+\} \geq \min\{\upsilon^+,\upsilon^+,\upsilon^+\} = \upsilon^+$, $\max\{\tilde{\Im}^+(\epsilon\varsigma),\tau^+\} \geq \min\{\tilde{\Im}^+(\epsilon),\tilde{\Im}^+(\varsigma),\upsilon^+\} \geq \min\{\upsilon^+,\upsilon^+,\upsilon^+\} = \upsilon^+$ and $\min\{\tilde{\mho}^+(\epsilon\varsigma),\tau^+\} \leq \max\{\tilde{\mho}^+(\epsilon),\tilde{\mho}^+(\varsigma),\upsilon^+\} \leq \max\{\upsilon^+,\upsilon^+,\upsilon^+\} = \upsilon^+$. It follows that $\epsilon\varsigma \in (\daleth]$. Therefore $\daleth$ is a BSS of $\mathscr{O}$.

Theorem 3.6. *A subset* $\flat = [(\tilde{\Re}_{\mathsf{T}}^-,\tilde{\Re}_{\mathsf{T}}^+),(\tilde{\Im}_{\mathsf{T}}^-,\tilde{\Im}_{\mathsf{T}}^+),(\tilde{\mho}_{\mathsf{T}}^-,\tilde{\mho}_{\mathsf{T}}^+)]$
is a (τ,υ) *BIntNS[BIntNLI,BIntNRI,BIntNBI] of* $\mathscr{O}$ *if and only if each non-empty level subset* $\flat^{(t_1,t_2)}$ *is a BSS* $[BVLI,BVRI,BVBI]$ *of* $\mathscr{O}$ *for all* $t_1 \in (\tau^-,\upsilon^-]$ *and* $t_2 \in (\tau^+,\upsilon^+]$.

Proof. Assume that $\flat^{(t_1,t_2)}$ is a BSS of $\mathscr{O}$ for each $t_1 \in [-1,0]$ and $t_2 \in [0,1]$.
Let $t_1 = \max\{\tilde{\Re}^-(\epsilon_1),\tilde{\Re}^-(\epsilon_2)\}$. Then $\epsilon_1,\epsilon_2 \in \tilde{\Re}_t^-$ for each $\epsilon_1,\epsilon_2 \in \mathscr{O}$. Thus $\min\{\tilde{\Re}^-(\epsilon\varsigma),\tau^-\} \leq t_1 = \max\{\tilde{\Re}^-(\epsilon_1),\tilde{\Re}^-(\epsilon_2),\upsilon^-\}$. Let $t_1 = \max\{\tilde{\Im}^-(\epsilon_1),\tilde{\Im}^-(\epsilon_2)\}$. Then $\epsilon_1,\epsilon_2 \in \tilde{\Im}_t^-$ for each $\epsilon_1,\epsilon_2 \in \mathscr{O}$. Thus $\min\{\tilde{\Im}^-(\epsilon\varsigma),\tau^-\} \leq t_1 = \max\{\tilde{\Im}^-(\epsilon_1),\tilde{\Im}^-(\epsilon_2),\upsilon^-\}$. Let $t_1 = \min\{\tilde{\mho}^-(\epsilon_1),\tilde{\mho}^-(\epsilon_2)\}$. Then $\epsilon_1,\epsilon_2 \in \tilde{\mho}_t^-$ for each $\epsilon_1,\epsilon_2 \in \mathscr{O}$. Thus $\max\{\tilde{\mho}^-(\epsilon\varsigma),\tau^-\} \geq t_1 = \min\{\tilde{\mho}^-(\epsilon_1),\tilde{\mho}^-(\epsilon_2),\upsilon^-\}$. Let $t_2 = \min\{\tilde{\Re}^+(\epsilon_1),\tilde{\Re}^+(\epsilon_2)\}$. Then $\epsilon_1,\epsilon_2 \in \tilde{\Re}_s^+$ for each $\epsilon_1,\epsilon_2 \in \mathscr{O}$. Thus $\max\{\tilde{\Re}^+(\epsilon\varsigma),\tau^+\} \geq t_2 = \min\{\tilde{\Re}^+(\epsilon_1),\tilde{\Re}^+(\epsilon_2),\upsilon^+\}$. Let $t_2 = \min\{\tilde{\Im}^+(\epsilon_1),\tilde{\Im}^+(\epsilon_2)\}$. Then $\epsilon_1,\epsilon_2 \in \tilde{\Im}_t^+$ for each $\epsilon_1,\epsilon_2 \in \mathscr{O}$. Thus $\max\{\tilde{\Im}^+(\epsilon\varsigma),\tau^+\} \geq t_2 = \min\{\tilde{\Im}^+(\epsilon_1),\tilde{\Im}^+(\epsilon_2),\upsilon^+\}$. Let $t_2 = \max\{\tilde{\mho}^+(\epsilon_1),\tilde{\mho}^+(\epsilon_2)\}$. Then $\epsilon_1,\epsilon_2 \in \tilde{\mho}_{t_2}^+$ for each $\epsilon_1,\epsilon_2 \in \mathscr{O}$. Thus $\min\{\tilde{\mho}^+(\epsilon\varsigma),\tau^+\} \leq t_2 = \max\{\tilde{\mho}^+(\epsilon_1),\tilde{\mho}^+(\epsilon_2),\upsilon^+\}$. This shows that $\flat^{(t_1,t_2)}$ is NS of $\mathscr{O}$.

Conversely, assume that $\flat^{(t_1,t_2)}$ is a BIntNS of $\mathscr{O}$. For each $t_1 \in [-1,0]$ and $t_2 \in [0,1]$ and $\epsilon_1,\epsilon_2 \in \tilde{\Re}_t^-$. We have $\tilde{\Re}^-(\epsilon_1) \leq t_1, \tilde{\Re}^-(\epsilon_2) \leq t_1$.
Since $\tilde{\Re}^-$ is a BSS of $\mathscr{O}$, $\min\{\tilde{\Re}^-(\epsilon_1\epsilon_2),\tau^-\} \leq \max\{\tilde{\Re}^-(\epsilon_1),\tilde{\Re}^-(\epsilon_2),\upsilon^-\} \leq t_1$. This implies that $\epsilon_1\epsilon_2 \in \tilde{\Re}_t^-$. We have $\tilde{\Im}^-(\epsilon_1) \leq t_1, \tilde{\Im}^-(\epsilon_2) \leq t_1$.
Since $\tilde{\Im}^-$ is a BSS of $\mathscr{O}$, $\min\{\tilde{\Im}^-(\epsilon_1\epsilon_2),\tau^-\} \leq \max\{\tilde{\Im}^-(\epsilon_1),\tilde{\Im}^-(\epsilon_2),\upsilon^-\} \leq t_1$. This implies that $\epsilon_1\epsilon_2 \in \tilde{\Im}_t^-$. We have $\tilde{\mho}^-(\epsilon_1) \geq t_1, \tilde{\mho}^-(\epsilon_2) \geq t_1$.
Since $\tilde{\mho}^-$ is a BSS of $\mathscr{O}$, $\max\{\tilde{\mho}^-(\epsilon_1\epsilon_2),\tau^-\} \geq \min\{\tilde{\mho}^-(\epsilon_1),\tilde{\mho}^-(\epsilon_2),\upsilon^-\} \geq t_1$. This implies that $\epsilon_1\epsilon_2 \in \tilde{\mho}_t^-$. We have $\tilde{\Re}^+(\epsilon_1) \geq t_1, \tilde{\Re}^+(\epsilon_2) \geq t_1$.
Since $\tilde{\Re}^+$ is a BSS of $\mathscr{O}$, $\max\{\tilde{\Re}^+(\epsilon_1\epsilon_2),\tau^+\} \geq \min\{\tilde{\Re}^+(\epsilon_1),\tilde{\Re}^+(\epsilon_2),\upsilon^+\} \geq t_2$. This implies

that $\epsilon_1\epsilon_2 \in \tilde{\Re}_t^+$. We have $\tilde{\Im}^+(\epsilon_1) \geq t_2, \tilde{\Im}^+(\epsilon_2) \geq t_2$.
Since $\tilde{\Im}^+$ is a BSS of $\mathscr{O}$, $\max\{\tilde{\Im}^+(\epsilon_1\epsilon_2), \tau^+\} \geq \min\{\tilde{\Im}^+(\epsilon_1), \tilde{\Im}^+(\epsilon_2), \upsilon^+\} \geq t_2$. This implies that $\epsilon_1\epsilon_2 \in \tilde{\Im}_s^+$. We have $\tilde{\mho}^+(\epsilon_1) \leq t_2, \tilde{\mho}^+(\epsilon_2) \leq t_2$.
Since $\tilde{\mho}^+$ is a BSS of $\mathscr{O}$, $\min\{\tilde{\mho}^+(\epsilon_1\epsilon_2), \tau^+\} \leq \max\{\tilde{\mho}^+(\epsilon_1), \tilde{\mho}^+(\epsilon_2), \upsilon^+\} \leq t_2$. This implies that $\epsilon_1\epsilon_2 \in \tilde{\mho}_s^+$. Therefore $\flat_{(t_1,t_2)}$ is a BSS of $\mathscr{O}$. Similar proofs holds.

Example 3.7. Every BIntNS $\flat$ of $\mathscr{O}$ is a (τ, υ) BIntNS of $\mathscr{O}$ and reverse implication may not be true.

For the Example 3.2, we define subset $\flat$ as

$$(\tilde{\Re}^-, \tilde{\Re}^+)(\hbar) = \begin{cases} (-0.41, 44) \ \ if \ \hbar = \hbar_1 \\ (-0.34, 37) \ \ if \ \hbar = \hbar_2 \\ (-0.24, 27) \ \ if \ \hbar = \hbar_3 \\ (-0.29, 32) \ \ if \ \hbar = \hbar_4 \end{cases} \quad (\tilde{\Im}^-, \tilde{\Im}^+)(\hbar) = \begin{cases} (-0.21, 24) \ \ if \ \hbar = \hbar_1 \\ (-0.14, 17) \ \ if \ \hbar = \hbar_2 \\ (-0.04, 07) \ \ if \ \hbar = \hbar_3 \\ (-0.09, 12) \ \ if \ \hbar = \hbar_4 \end{cases}$$

$$(\tilde{\mho}^-, \tilde{\mho}^+)(\hbar) = \begin{cases} (-0.26, 29) \ \ if \ \hbar = \hbar_1 \\ (-0.31, 34) \ \ if \ \hbar = \hbar_2 \\ (-0.41, 44) \ \ if \ \hbar = \hbar_3 \\ (-0.36, 39) \ \ if \ \hbar = \hbar_4 \end{cases}$$

Then $\flat$ is a $(0.35, 0.49)$ BIntNS of $\mathscr{O}$, but not a BIntNS.

Definition 3.8. If $\Lambda_\beth$ is the characteristic function of $\beth$, then $(\Lambda_\beth)_\tau^\upsilon$ is defined as

$$(\Lambda_\beth^{t-})_{\tau^-}^{\upsilon^-}(\epsilon) = \begin{cases} \upsilon^- \, if \, \epsilon \in (\beth] \\ \tau^- \, if \, \epsilon \notin (\beth] \end{cases} \quad (\Lambda_\beth^{i-})_{\tau^-}^{\upsilon^-}(\epsilon) = \begin{cases} \upsilon^- \, if \, \epsilon \in (\beth] \\ \tau^- \, if \, \epsilon \notin (\beth] \end{cases} \quad (\Lambda_\beth^{f-})_{\tau^-}^{\upsilon^-}(\epsilon) = \begin{cases} \tau^- \, if \, \epsilon \in (\beth] \\ \upsilon^- \, if \, \epsilon \notin (\beth] \end{cases}$$

$$(\Lambda_\beth^{t+})_{\tau^+}^{\upsilon^+}(\epsilon) = \begin{cases} \upsilon^+ \, if \, \epsilon \in (\beth] \\ \tau^+ \, if \, \epsilon \notin (\beth] \end{cases} \quad (\Lambda_\beth^{i+})_{\tau^+}^{\upsilon^+}(\epsilon) = \begin{cases} \upsilon^+ \, if \, \epsilon \in (\beth] \\ \tau^+ \, if \, \epsilon \notin (\beth] \end{cases} \quad (\Lambda_\beth^{f+})_{\tau^+}^{\upsilon^+}(\epsilon) = \begin{cases} \tau^+ \, if \, \epsilon \in (\beth] \\ \upsilon^+ \, if \, \epsilon \notin (\beth] \end{cases}$$

Theorem 3.9. *A non empty subset $\beth$ of $\mathscr{O}$ is a BSS $[BVLI, BVRI, BVBI]$ of $\mathscr{O}$ if and only if subset $\Lambda_{(\beth]}$ is a (τ, υ) BIntNS[BIntNLI, BIntNRI, BIntNBI] of $\mathscr{O}$.*

Proof. Assume that $\beth$ is a BSS of $\mathscr{O}$. Then $\Lambda_{(\beth]}$ is a BIntNS of $\mathscr{O}$ and hence $\Lambda_{(\beth]}$ is an (τ, υ) BIntNS of $\mathscr{O}$.
Conversely, Let $\Lambda_{(\beth]}$ is an (τ, υ) BIntNS of $\mathscr{O}$. Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\beth]$. Then $\Lambda_{(\beth]}^{t-}(\epsilon) = \upsilon^- = \Lambda_{(\beth]}^{t-}(\varsigma) = \upsilon^-$ Since $\Lambda_{(\beth]}^{t-}$ is a (τ, υ) BIntNS. Consider

$$\begin{aligned} \min\{\Lambda_{(\beth]}^{t-}(\epsilon\varsigma), \tau^-\} &\leq \max\{\Lambda_{(\beth]}^{t-}(\epsilon), \Lambda_{(\beth]}^{t-}(\varsigma), \upsilon^-\} \\ &= \max\{\upsilon^-, \upsilon^-, \upsilon^-\} \\ &= \upsilon^- \end{aligned}$$

as $\tau^- > \upsilon^-$, this implies that $\Lambda^{t-}_{(\mathfrak{J}]}(\epsilon\varsigma) \leq \upsilon^-$. Thus $\epsilon\varsigma \in (\mathfrak{J}]$. Thus $\epsilon\varsigma \in (\mathfrak{J}]$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\mathfrak{J}]$. Then $\Lambda^{i-}_{(\mathfrak{J}]}(\epsilon) = \upsilon^- = \Lambda^{i-}_{(\mathfrak{J}]}(\varsigma) = \upsilon^-$. Since $\Lambda^{i-}_{(\mathfrak{J}]}$ is a (τ, υ)BIntNS. Consider

$$\begin{aligned}\min\{\Lambda^{i-}_{(\mathfrak{J}]}(\epsilon\varsigma), \tau^-\} &\leq \max\{\Lambda^{i-}_{(\mathfrak{J}]}(\epsilon), \Lambda^{i-}_{(\mathfrak{J}]}(\varsigma), \upsilon^-\} \\ &= \max\{\upsilon^-, \upsilon^-, \upsilon^-\} \\ &= \upsilon^-\end{aligned}$$

as $\tau^- > \upsilon^-$, this implies that $\Lambda^{i-}_{(\mathfrak{J}]}(\epsilon\varsigma) \leq \upsilon^-$. Thus $\epsilon\varsigma \in (\mathfrak{J}]$. Thus $\epsilon\varsigma \in (\mathfrak{J}]$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\mathfrak{J}]$. Then $\Lambda^{f-}_{(\mathfrak{J}]}(\epsilon) = \tau^- = \Lambda^{f-}_{(\mathfrak{J}]}(\varsigma) = \tau^-$. Since $\Lambda^{f-}_{(\mathfrak{J}]}$ is a (τ, υ)BIntNS. Consider

$$\begin{aligned}\max\{\Lambda^{f-}_{(\mathfrak{J}]}(\epsilon\varsigma), \tau^-\} &\geq \min\{\Lambda^{f-}_{(\mathfrak{J}]}(\epsilon), \Lambda^{f-}_{(\mathfrak{J}]}(\varsigma), \upsilon^-\} \\ &= \min\{\tau^-, \tau^-, \upsilon^-\} \\ &= \upsilon^-\end{aligned}$$

as $\tau^- > \upsilon^-$, this implies that $\Lambda^{f-}_{(\mathfrak{J}]}(\epsilon\varsigma) \geq \tau^-$. Thus $\epsilon\varsigma \in (\mathfrak{J}]$. Thus $\epsilon\varsigma \in (\mathfrak{J}]$.

Therefore $\mathfrak{J}$ is a BSS of $\mathscr{O}$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \notin (\mathfrak{J}]$. Then $\Lambda^{t-}_{(\mathfrak{J}]}(\epsilon) = \tau^- = \Lambda^{t-}_{(\mathfrak{J}]}(\varsigma) = \tau^-$. Since $\Lambda^{t-}_{(\mathfrak{J}]}$ is a (τ, υ)BIntNS.

$$\begin{aligned}\min\{\Lambda^{t-}_{(\mathfrak{J}]}(\epsilon\varsigma), \tau^-\} &\leq \max\{\Lambda^{t-}_{(\mathfrak{J}]}(\epsilon), \Lambda^{t-}_{(\mathfrak{J}]}(\varsigma), \upsilon^-\} \\ &= \max\{\tau^-, \tau^-, \upsilon^-\} \\ &= \tau^-\end{aligned}$$

as $\tau^- > \upsilon^-$, this implies that $\Lambda^{t-}_{(\mathfrak{J}]}(\epsilon\varsigma) \leq \tau^-$. Thus $\epsilon\varsigma \notin (\mathfrak{J}]$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \notin (\mathfrak{J}]$. Then $\Lambda^{i-}_{(\mathfrak{J}]}(\epsilon) = \tau^- = \Lambda^{i-}_{(\mathfrak{J}]}(\varsigma) = \tau^-$. Since $\Lambda^{i-}{}_{(\mathfrak{J}]}$ is a (τ, υ)BIntNS.

$$\begin{aligned}\min\{\Lambda^{i-}_{(\mathfrak{J}]}(\epsilon\varsigma), \tau^-\} &\leq \max\{\Lambda^{i-}_{(\mathfrak{J}]}(\epsilon), \Lambda^{i-}_{(\mathfrak{J}]}(\varsigma), \upsilon^-\} \\ &= \max\{\tau^-, \tau^-, \upsilon^-\} \\ &= \tau^-\end{aligned}$$

as $\tau^- > \upsilon^-$, this implies that $\Lambda^{i-}_{(\mathfrak{J}]}(\epsilon\varsigma) \leq \tau^-$. Thus $\epsilon\varsigma \notin (\mathfrak{J}]$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \notin (\mathfrak{J}]$. Then $\Lambda^{f-}_{(\mathfrak{J}]}(\epsilon) = \upsilon^- = \Lambda^{f-}_{(\mathfrak{J}]}(\varsigma) = \upsilon^-$. Since $\Lambda^{f-}{}_{(\mathfrak{J}]}$ is a (τ, υ)BIntNS.

$$\begin{aligned}\max\{\Lambda^{f-}_{(\mathfrak{J}]}(\epsilon\varsigma), \tau^-\} &\geq \min\{\Lambda^{f-}_{(\mathfrak{J}]}(\epsilon), \Lambda^{f-}_{(\mathfrak{J}]}(\varsigma), \upsilon^-\} \\ &= \min\{\upsilon^-, \upsilon^-, \upsilon^-\} \\ &= \upsilon^-\end{aligned}$$

as $\tau^- > \upsilon^-$, this implies that $\Lambda^{f-}_{(\mathfrak{J}]}(\epsilon\varsigma) \geq \upsilon^-$. Thus $\epsilon\varsigma \notin (\mathfrak{J}]$.

Let $\Lambda_{(\beth]}$ is an (τ, υ) BIntNS of $\mathscr{O}$. Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\beth]$. Then $\Lambda^{t+}_{(\beth]}(\epsilon) = \upsilon^+ = \Lambda^{t+}_{(\beth]}(\varsigma) = \upsilon^+$. Since $\Lambda^{t+}_{(\beth]}$ is a (τ, υ)BIntNS. Consider

$$\begin{aligned}\max\{\Lambda^{t+}_{(\beth]}(\epsilon\varsigma), \tau^+\} &\geq \min\{\Lambda^{t+}_{(\beth]}(\epsilon), \Lambda^{t+}_{(\beth]}(\varsigma), \upsilon^+\} \\ &= \min\{\upsilon^+, \upsilon^+, \upsilon^+\} \\ &= \upsilon^+\end{aligned}$$

as $\tau^+ < \upsilon^+$, this implies that $\Lambda^{t+}_{(\beth]}(\epsilon\varsigma) \geq \upsilon^+$. Thus $\epsilon\varsigma \in (\beth]$. Thus $\epsilon\varsigma \in (\beth]$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\beth]$. Then $\Lambda^{i+}_{(\beth]}(\epsilon) = \upsilon^+ = \Lambda^{i+}_{(\beth]}(\varsigma) = \upsilon^+$. Since $\Lambda^{i+}_{(\beth]}$ is a (τ, υ)BIntNS. Consider

$$\begin{aligned}\max\{\Lambda^{i+}_{(\beth]}(\epsilon\varsigma), \tau^+\} &\geq \min\{\Lambda^{i+}_{(\beth]}(\epsilon), \Lambda^{i+}_{(\beth]}(\varsigma), \upsilon^+\} \\ &= \min\{\upsilon^+, \upsilon^+, \upsilon^+\} \\ &= \upsilon^+\end{aligned}$$

as $\tau^+ < \upsilon^+$, this implies that $\Lambda^{i+}_{(\beth]}(\epsilon\varsigma) \geq \upsilon^+$. Thus $\epsilon\varsigma \in (\beth]$. Thus $\epsilon\varsigma \in (\beth]$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\beth]$. Then $\Lambda^{f+}_{(\beth]}(\epsilon) = \tau^+ = \Lambda^{f+}_{(\beth]}(\varsigma) = \tau^+$. Since $\Lambda^{f+}_{(\beth]}$ is a (τ, υ)BIntNS. Consider

$$\begin{aligned}\min\{\Lambda^{f+}_{(\beth]}(\epsilon\varsigma), \tau^+\} &\leq \max\{\Lambda^{f+}_{(\beth]}(\epsilon), \Lambda^{f+}_{(\beth]}(\varsigma), \upsilon^+\} \\ &= \max\{\tau^+, \tau^+, \upsilon^+\} \\ &= \upsilon^+\end{aligned}$$

as $\tau^+ < \upsilon^+$, this implies that $\Lambda^{f+}_{(\beth]}(\epsilon\varsigma) \leq \tau^+$. Thus $\epsilon\varsigma \in (\beth]$. Thus $\epsilon\varsigma \in (\beth]$.

Therefore $\beth$ is a BSS of $\mathscr{O}$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \notin (\beth]$. Then $\Lambda^{t+}_{(\beth]}(\epsilon) = \tau^+ = \Lambda^{t+}_{(\beth]}(\varsigma) = \tau^+$. Since $\Lambda^{t+}_{(\beth]}$ is a (τ, υ)BIntNS.

$$\begin{aligned}\max\{\Lambda^{t+}_{(\beth]}(\epsilon\varsigma), \tau^+\} &\geq \min\{\Lambda^{t+}_{(\beth]}(\epsilon), \Lambda^{t+}_{(\beth]}(\varsigma), \upsilon^+\} \\ &= \min\{\tau^+, \tau^+, \upsilon^+\} \\ &= \tau^+\end{aligned}$$

as $\tau^+ < \upsilon^+$, this implies that $\Lambda^{t+}_{(\beth]}(\epsilon\varsigma) \geq \tau^+$. Thus $\epsilon\varsigma \notin (\beth]$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \notin (\beth]$. Then $\Lambda^{i+}_{(\beth]}(\epsilon) = \tau^+ = \Lambda^{i+}_{(\beth]}(\varsigma) = \tau^+$. Since $\Lambda^{i+}_{(\beth]}$ is a (τ, υ)BIntNS.

$$\begin{aligned}\max\{\Lambda^{i+}_{(\beth]}(\epsilon\varsigma), \tau^+\} &\geq \min\{\Lambda^{i+}_{(\beth]}(\epsilon), \Lambda^{i+}_{(\beth]}(\varsigma), \upsilon^+\} \\ &= \min\{\tau^+, \tau^+, \upsilon^+\} \\ &= \tau^+\end{aligned}$$

as $\tau^+ < \upsilon^+$, this implies that $\Lambda^{i+}_{(\beth]}(\epsilon\varsigma) \geq \tau^+$. Thus $\epsilon\varsigma \notin (\beth]$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \notin (\beth]$. Then $\Lambda^{f+}_{(\beth]}(\epsilon) = \upsilon^+ = \Lambda^{f+}_{(\beth]}(\varsigma) = \upsilon^+$. Since $\Lambda^{f+}_{(\beth]}$ is a

(τ, υ)BIntNS.

$$\begin{aligned}\min\{\Lambda^{f+}_{(\beth]}(\epsilon\varsigma), \tau^+\} &\leq \max\{\Lambda^{f+}_{(\beth]}(\epsilon), \Lambda^{f+}_{(\beth]}(\varsigma), \upsilon^+\} \\ &= \max\{\upsilon^+, \upsilon^+, \upsilon^+\} \\ &= \upsilon^+\end{aligned}$$

as $\tau^+ < \upsilon^+$, this implies that $\Lambda^{f+}_{(\beth]}(\epsilon\varsigma) \leq \upsilon^+$. Thus $\epsilon\varsigma \notin (\beth]$. Therefore $\beth$ is a BSS of $\mathscr{O}$. Similar to proof holds.

Definition 3.10. For two intuitionistic neutrosophic subsets χ and ℓ of $\mathscr{O}$, their product $\chi \cdot \ell$ is defined as

$$(\chi^{t-} \cdot \ell^{t-})(\epsilon) = \begin{cases} \inf\limits_{(s,t)\in\beth_\epsilon} \{\chi^{t-}(s) \veebar \ell^{t-}(t)\} \ if \ \beth_\epsilon \neq 0 \\ 0 \ otherwise \end{cases}$$

$$(\chi^{i-} \cdot \ell^{i-})(\epsilon) = \begin{cases} \inf\limits_{(s,t)\in\beth_\epsilon} \{\chi^{i-}(s) \veebar \ell^{i-}(t)\} \ if \ \beth_\epsilon \neq 0 \\ 0 \ otherwise \end{cases}$$

$$(\chi^{f-} \cdot \ell^{f-})(\epsilon) = \begin{cases} \sup\limits_{(s,t)\in\beth_\epsilon} \{\chi^{f-}(s) \barwedge \ell^{f-}(t)\} \ if \ \beth_\epsilon \neq 0 \\ 1 \ otherwise \end{cases}$$

$$(\chi^{t+} \cdot \ell^{t+})(\epsilon) = \begin{cases} \sup\limits_{(s,t)\in\beth_\epsilon} \{\chi^{t+}(s) \barwedge \ell^{t+}(t)\} \ if \ \beth_\epsilon \neq 0 \\ 0 \ otherwise \end{cases}$$

$$(\chi^{i+} \cdot \ell^{i+})(\epsilon) = \begin{cases} \sup\limits_{(s,t)\in\beth_\epsilon} \{\chi^{i+}(s) \barwedge \ell^{i+}(t)\} \ if \ \beth_\epsilon \neq 0 \\ 0 \ otherwise \end{cases}$$

$$(\chi^{f+} \cdot \ell^{f+})(\epsilon) = \begin{cases} \inf\limits_{(s,t)\in\beth_\epsilon} \{\chi^{f+}(s) \veebar \ell^{f+}(t)\} \ if \ \beth_\epsilon \neq 0 \\ -1 \ otherwise \end{cases}$$

Definition 3.11. Let $\flat$ be subset of $\mathscr{O}$, we define the subset $(\tilde{\Re}^-)^{\upsilon^-}_{\tau^-}(\epsilon) = \{\tilde{\Re}^-(\epsilon) \veebar \upsilon^-\} \barwedge \tau^-$, $(\tilde{\Im}^-)^{\upsilon^-}_{\tau^-}(\epsilon) = \{\tilde{\Im}^-(\epsilon) \veebar \upsilon^-\} \barwedge \tau^-$ and $(\tilde{\mho}^-)^{\upsilon^-}_{\tau^-}(\epsilon) = \{\tilde{\mho}^-(\epsilon) \barwedge \upsilon^-\} \veebar \tau^-$, $(\tilde{\Re}^+)^{\upsilon^+}_{\tau^+}(\epsilon) = \{\tilde{\Re}^+(\epsilon) \barwedge \upsilon^+\} \veebar \tau^+$, $(\tilde{\Im}^+)^{\upsilon^+}_{\tau^+}(\epsilon) = \{\tilde{\Im}^+(\epsilon) \barwedge \upsilon^+\} \veebar \tau^+$ and $(\tilde{\mho}^+)^{\upsilon^+}_{\tau^+}(\epsilon) = \{\tilde{\mho}^+(\epsilon) \veebar \upsilon^+\} \barwedge \tau^+$, for all $\epsilon \in \mathscr{O}$.

Lemma 3.12. *Let $\beth$ and $\daleth$ be non-empty subsets of $\mathscr{O}$. Then the following hold:*

(1) $(\Lambda_{(\beth]} \veebar^{\upsilon}_{\tau} \Lambda_{(\daleth]}) = (\Lambda_{(\beth \barwedge \daleth]})^{\upsilon}_{\tau}$,
(2) $(\Lambda_{(\beth]} \barwedge^{\upsilon}_{\tau} \Lambda_{(\daleth]}) = (\Lambda_{(\beth \veebar \daleth]})^{\upsilon}_{\tau}$,
(3) $(\Lambda_{(\beth]} \cdot^{\upsilon}_{\tau} \Lambda_{(\daleth]}) = (\Lambda_{(\beth\daleth]})^{\upsilon}_{\tau}$.

Proof. (3) Let $\epsilon \in \mathscr{O}$. If $\epsilon \in (\mathfrak{I}\mathfrak{T}]$, then $(\Lambda_{(\mathfrak{I}\mathfrak{T}]})(\epsilon) = \upsilon^-$.

Since $\epsilon \leq a_1a_2$ for some $a_1 \in (\mathfrak{I}]$,$a_2 \in (\mathfrak{T}]$,we have $(a_1, a_2) \in \mathfrak{I}_\epsilon$ and $\mathfrak{I}_\epsilon \neq 0$.

$$\begin{aligned}(\Lambda^{t-}_{(\mathfrak{I}]} \cdot \Lambda^{t-}_{(\mathfrak{T}]})(\epsilon) &= \inf_{\epsilon=yz} \max\{\Lambda^{t-}_{(\mathfrak{I}]}(y), \Lambda^{t-}_{(\mathfrak{T}]}(z)\} \\ &\leq \max\{\Lambda^{t-}_{(\mathfrak{I}]}(a_1), \Lambda^{t-}_{(\mathfrak{T}]}(a_2)\} \\ &= \upsilon^-\end{aligned}$$

$$\begin{aligned}(\Lambda^{i-}_{(\mathfrak{I}]} \cdot \Lambda^{i-}_{(\mathfrak{T}]})(\epsilon) &= \inf_{\epsilon=yz} \max\{\Lambda^{i-}_{(\mathfrak{I}]}(y), \Lambda^{i-}_{(\mathfrak{T}]}(z)\} \\ &\leq \max\{\Lambda^{i-}_{(\mathfrak{I}]}(a_1), \Lambda^{i-}_{(\mathfrak{T}]}(a_2)\} \\ &= \upsilon^-\end{aligned}$$

$$\begin{aligned}(\Lambda^{f-}_{(\mathfrak{I}]} \cdot \Lambda^{f-}_{(\mathfrak{T}]})(\epsilon) &= \sup_{\epsilon=yz} \min\{\Lambda^{f-}_{(\mathfrak{I}]}(y), \Lambda^{f-}_{(\mathfrak{T}]}(z)\} \\ &\geq \min\{\Lambda^{f-}_{(\mathfrak{I}]}(a_1), \Lambda^{f-}_{(\mathfrak{T}]}(a_2)\} \\ &= \tau^-\end{aligned}$$

Therefore $(\Lambda_{(\mathfrak{I}]} \cdot \Lambda_{(\mathfrak{T}]})(\epsilon) = (\Lambda_{(\mathfrak{I}\mathfrak{T}]})(\epsilon)$.

If $\epsilon \in (\mathfrak{I}\mathfrak{T}]$, then $(\Lambda_{(\mathfrak{I}\mathfrak{T}]})(\epsilon) = \upsilon^+$.

Since $\epsilon \leq a_1a_2$ for some $a_1 \in (\mathfrak{I}]$,$a_2 \in (\mathfrak{T}]$,we have $(a_1, a_2) \in \mathfrak{I}_\epsilon$ and $\mathfrak{I}_\epsilon \neq 0$.

$$\begin{aligned}(\Lambda^{t+}_{(\mathfrak{I}]} \cdot \Lambda^{t+}_{(\mathfrak{T}]})(\epsilon) &= \sup_{\epsilon=yz} \min\{\Lambda^{t+}_{(\mathfrak{I}]}(y), \Lambda^{t+}_{(\mathfrak{T}]}(z)\} \\ &\geq \min\{\Lambda^{t+}_{(\mathfrak{I}]}(a_1), \Lambda^{t+}_{(\mathfrak{T}]}(a_2)\} \\ &= \upsilon^+\end{aligned}$$

$$\begin{aligned}(\Lambda^{i+}_{(\mathfrak{I}]} \cdot \Lambda^{i+}_{(\mathfrak{T}]})(\epsilon) &= \sup_{\epsilon=yz} \min\{\Lambda^{i+}_{(\mathfrak{I}]}(y), \Lambda^{i+}_{(\mathfrak{T}]}(z)\} \\ &\geq \min\{\Lambda^{i+}_{(\mathfrak{I}]}(a_1), \Lambda^{i+}_{(\mathfrak{T}]}(a_2)\} \\ &= \upsilon^+\end{aligned}$$

$$\begin{aligned}(\Lambda^{f+}_{(\mathfrak{I}]} \cdot \Lambda^{f+}_{(\mathfrak{T}]})(\epsilon) &= \inf_{\epsilon=yz} \max\{\Lambda^{f+}_{(\mathfrak{I}]}(y), \Lambda^{f+}_{(\mathfrak{T}]}(z)\} \\ &\leq \max\{\Lambda^{f+}_{(\mathfrak{I}]}(a_1), \Lambda^{f+}_{(\mathfrak{T}]}(a_2)\} \\ &= \tau^+\end{aligned}$$

Therefore $(\Lambda_{(\mathfrak{I}]} \cdot \Lambda_{(\mathfrak{T}]})(\epsilon) = (\Lambda_{(\mathfrak{I}\mathfrak{T}]})(\epsilon)$.

If $\epsilon \notin (\mathfrak{I}\mathfrak{T}]$ then $(\Lambda^{t-}_{(\mathfrak{I}\mathfrak{T}]})(\epsilon) = \tau^-$, $(\Lambda^{i-}_{(\mathfrak{I}\mathfrak{T}]})(\epsilon) = \tau^-$ and $(\Lambda^{f-}_{(\mathfrak{I}\mathfrak{T}]})(\epsilon) = \upsilon^-$. Since $\epsilon \leq a_1a_2$ for some $a_1 \notin (\mathfrak{I}]$,$a_2 \notin (\mathfrak{T}]$. We have

$$\begin{aligned}(\Lambda^{t-}_{(\mathfrak{I}]} \cdot \Lambda^{t-}_{(\mathfrak{T}]})(\epsilon) &= \inf_{\epsilon=yz} \max\{\Lambda^{t-}_{(\mathfrak{I}]}(y), \Lambda^{t-}_{(\mathfrak{T}]}(z)\} \\ &\leq \max\{\Lambda^{t-}_{(\mathfrak{I}]}(a_1), \Lambda^{t-}_{(\mathfrak{T}]}(a_2)\} \\ &= \tau^-\end{aligned}$$

$$(\Lambda^{i-}_{(\beth]} \cdot \Lambda^{i-}_{(\daleth]})(\epsilon) = \inf_{\epsilon=yz} \max\{\Lambda^{i-}_{(\beth]}(y), \Lambda^{i-}_{(\daleth]}(z)\}$$
$$\leq \max\{\Lambda^{i-}_{(\beth]}(a_1), \Lambda^{i-}_{(\daleth]}(a_2)\}$$
$$= \tau^-$$

$$(\Lambda^{f-}_{(\beth]} \cdot \Lambda^{f-}_{(\daleth]})(\epsilon) = \sup_{\epsilon=yz} \min\{\Lambda^{f-}_{(\beth]}(y), \Lambda^{f-}_{(\daleth]}(z)\}$$
$$\geq \min\{\Lambda^{f-}_{(\beth]}(a_1), \Lambda^{f-}_{(\daleth]}(a_2)\}$$
$$= \upsilon^-$$

If $\epsilon \notin (\beth\daleth]$ then $(\Lambda^{t+}_{(\beth\daleth]})(\epsilon) = \tau^+$, $(\Lambda^{i+}_{(\beth\daleth]})(\epsilon) = \tau^+$ and $(\Lambda^{f+}_{(\beth\daleth]})(\epsilon) = \upsilon^+$. Since $\epsilon \leq a_1 a_2$ for some $a_1 \notin (\beth]$,$a_2 \notin (\daleth]$. We have

$$(\Lambda^{t+}_{(\beth]} \cdot \Lambda^{t+}_{(\daleth]})(\epsilon) = \sup_{\epsilon=yz} \min\{\Lambda^{t+}_{(\beth]}(y), \Lambda^{t+}_{(\daleth]}(z)\}$$
$$\geq \min\{\Lambda^{t+}_{(\beth]}(a_1), \Lambda^{t+}_{(\daleth]}(a_2)\}$$
$$= \tau^+$$

$$(\Lambda^{i+}_{(\beth]} \cdot \Lambda^{i+}_{(\daleth]})(\epsilon) = \sup_{\epsilon=yz} \min\{\Lambda^{i+}_{(\beth]}(y), \Lambda^{i+}_{(\daleth]}(z)\}$$
$$\geq \min\{\Lambda^{i+}_{(\beth]}(a_1), \Lambda^{i+}_{(\daleth]}(a_2)\}$$
$$= \tau^+$$

$$(\Lambda^{f+}_{(\beth]} \cdot \Lambda^{f+}_{(\daleth]})(\epsilon) = \inf_{\epsilon=yz} \max\{\Lambda^{f+}_{(\beth]}(y), \Lambda^{f+}_{(\daleth]}(z)\}$$
$$\leq \max\{\Lambda^{f+}_{(\beth]}(a_1), \Lambda^{f+}_{(\daleth]}(a_2)\}$$
$$= \upsilon^+$$

Hence $(\Lambda_{(\beth]} \cdot \Lambda_{(\daleth]})(\epsilon) = (\Lambda_{(\beth\daleth]})(\epsilon)$.

Theorem 3.13. *For $\beth, \daleth \subseteq \mathscr{O}$ and $\{\beth_i | i \in I\}$ be a family of subsets of $\mathscr{O}$ then*
(i) $(\beth] \subseteq (\daleth]$ if and only if $(\Lambda_{(\beth]})^{\upsilon}_{\tau} \leq (\Lambda_{(\daleth]})^{\upsilon}_{\tau}$.
(ii) $(\overline{\wedge}_{i\in I}\Lambda_{(\beth_i]})^{\upsilon}_{\tau} = (\Lambda_{\overline{\wedge}_{i\in I}(\beth_i]})^{\upsilon}_{\tau}$.
(iii) $(\underline{\vee}_{i\in I}\Lambda_{(\beth_i]})^{\upsilon}_{\tau} = (\Lambda_{\underline{\vee}_{i\in I}(\beth_i]})^{\upsilon}_{\tau}$.

Theorem 3.14. *If $\beth$ is a (τ, υ)-BIntNLI[BIntNS, BIntNRI] of $\mathscr{O}$, then $(\beth)^{\upsilon}_{\tau}$ is a BIntNLI[BIntNS, BIntNRI] of $\mathscr{O}$.*

Proof. Assume that $\beth$ is a (τ, υ)BIntNLI of $\mathscr{O}$. If there exist $\epsilon, \varsigma \in \mathscr{O}$. Now

$$\begin{aligned}
\min\{(\tilde{\Re}^{-})^{\upsilon^{-}}_{\tau^{-}}(\epsilon\varsigma), \tau^{-}\} &= \min\{(\{\tilde{\Re}^{-}(\epsilon\varsigma) \veebar \upsilon^{-}\} \barwedge \tau^{-}), \tau^{-}\} \\
&= \{\tilde{\Re}^{-}(\epsilon\varsigma) \veebar \upsilon^{-}\} \barwedge \tau^{-} \\
&= \{\tilde{\Re}^{-}(\epsilon\varsigma) \barwedge \tau^{-}\} \veebar \{\upsilon^{-} \barwedge \tau^{-}\} \\
&= \{(\tilde{\Re}^{-}(\epsilon\varsigma) \barwedge \tau^{-}) \barwedge \tau^{-}\} \veebar \upsilon^{-} \\
&\leq \{(\tilde{\Re}^{-}(\varsigma) \veebar \upsilon^{-}) \barwedge \tau^{-}\} \veebar \upsilon^{-} \\
&= \{(\tilde{\Re}^{-}(\varsigma) \veebar \upsilon^{-}) \veebar \upsilon^{-}\} \barwedge (\tau^{-} \veebar \upsilon^{-}) \\
&= \{(\tilde{\Re}^{-}(\varsigma) \veebar \upsilon^{-}) \veebar \upsilon^{-}\} \barwedge \tau^{-} \\
&\leq (\tilde{\Re}^{-})^{\upsilon^{-}}_{\tau^{-}}(\varsigma) \veebar \upsilon^{-}.
\end{aligned}$$

$$\begin{aligned}
\min\{(\tilde{\Im}^{-})^{\upsilon^{-}}_{\tau^{-}}(\epsilon\varsigma), \tau^{-}\} &= \min\{(\{\tilde{\Im}^{-}(\epsilon\varsigma) \veebar \upsilon^{-}\} \barwedge \tau^{-}), \tau^{-}\} \\
&= \{\tilde{\Im}^{-}(\epsilon\varsigma) \veebar \upsilon^{-}\} \barwedge \tau^{-} \\
&= \{\tilde{\Im}^{-}(\epsilon\varsigma) \barwedge \tau^{-}\} \veebar \{\upsilon^{-} \barwedge \tau^{-}\} \\
&= \{(\tilde{\Im}^{-}(\epsilon\varsigma) \barwedge \tau^{-}) \barwedge \tau^{-}\} \veebar \upsilon^{-} \\
&\leq \{(\tilde{\Im}^{-}(\varsigma) \veebar \upsilon^{-}) \barwedge \tau^{-}\} \veebar \upsilon^{-} \\
&= \{(\tilde{\Im}^{-}(\varsigma) \veebar \upsilon^{-}) \veebar \upsilon^{-}\} \barwedge (\tau^{-} \veebar \upsilon^{-}) \\
&= \{(\tilde{\Im}^{-}(\varsigma) \veebar \upsilon^{-}) \veebar \upsilon^{-}\} \barwedge \tau^{-} \\
&\leq (\tilde{\Im}^{-})^{\upsilon^{-}}_{\tau^{-}}(\varsigma) \veebar \upsilon^{-}.
\end{aligned}$$

$$\begin{aligned}
\max\{(\tilde{\mho}^{-})^{\upsilon^{-}}_{\tau^{-}}(\epsilon\varsigma), \tau^{-}\} &= \max\{(\{\tilde{\mho}^{-}(\epsilon\varsigma) \barwedge \upsilon^{-}\} \veebar \tau^{-}), \tau^{-}\} \\
&= \{\tilde{\mho}^{-}(\epsilon\varsigma) \barwedge \upsilon^{-}\} \veebar \tau^{-} \\
&= \{\tilde{\mho}^{-}(\epsilon\varsigma) \veebar \tau^{-}\} \barwedge \{\upsilon^{-} \veebar \tau^{-}\} \\
&= \{(\tilde{\mho}^{-}(\epsilon\varsigma) \veebar \tau^{-}) \veebar \tau^{-}\} \barwedge \tau^{-} \\
&\geq \{(\tilde{\mho}^{-}(\varsigma) \barwedge \upsilon^{-}) \veebar \tau^{-}\} \barwedge \tau^{-} \\
&= \{(\tilde{\mho}^{-}(\varsigma) \barwedge \upsilon^{-}) \barwedge \upsilon^{-}\} \veebar \tau^{-} \\
&\geq (\tilde{\mho}^{-})^{\upsilon^{-}}_{\tau^{-}}(\varsigma) \barwedge \upsilon^{-}
\end{aligned}$$

If there exist $\epsilon, \varsigma \in \mathscr{O}$. Now

$$\begin{aligned}
\max\{(\tilde{\Re}^{+})^{\upsilon^{+}}_{\tau^{+}}(\epsilon\varsigma), \tau^{+}\} &= \max\{(\{\tilde{\Re}^{+}(\epsilon\varsigma) \barwedge \upsilon^{+}\} \veebar \tau^{+}), \tau^{+}\} \\
&= \{\tilde{\Re}^{+}(\epsilon\varsigma) \barwedge \upsilon^{+}\} \veebar \tau^{+} \\
&= \{\tilde{\Re}^{+}(\epsilon\varsigma) \veebar \tau^{+}\} \barwedge \{\upsilon^{+} \veebar \tau^{+}\} \\
&= \{(\tilde{\Re}^{+}(\epsilon\varsigma) \veebar \tau^{+}) \veebar \tau^{+}\} \barwedge \upsilon^{+} \\
&\geq \{(\tilde{\Re}^{+}(\varsigma) \barwedge \upsilon^{+}) \veebar \tau^{+}\} \barwedge \upsilon^{+} \\
&= \{(\tilde{\Re}^{+}(\varsigma) \barwedge \upsilon^{+}) \barwedge \upsilon^{+}\} \veebar (\tau^{+} \barwedge \upsilon^{+}) \\
&= \{(\tilde{\Re}^{+}(\varsigma) \barwedge \upsilon^{+}) \barwedge \upsilon^{+}\} \veebar \tau^{+} \\
&\geq (\tilde{\Re}^{+})^{\upsilon^{+}}_{\tau^{+}}(\varsigma) \barwedge \upsilon^{+}.
\end{aligned}$$

$$\begin{aligned}
\max\{(\tilde{\Im}^{+})^{\upsilon^{+}}_{\tau^{+}}(\epsilon\varsigma), \tau^{+}\} &= \max\{(\{\tilde{\Im}^{+}(\epsilon\varsigma) \barwedge \upsilon^{+}\} \veebar \tau^{+}), \tau^{+}\} \\
&= \{\tilde{\Im}^{+}(\epsilon\varsigma) \barwedge \upsilon^{+}\} \veebar \tau^{+} \\
&= \{\tilde{\Im}^{+}(\epsilon\varsigma) \veebar \tau^{+}\} \barwedge \{\upsilon^{+} \veebar \tau^{+}\} \\
&= \{(\tilde{\Im}^{+}(\epsilon\varsigma) \veebar \tau^{+}) \veebar \tau^{+}\} \barwedge \upsilon^{+} \\
&\geq \{(\tilde{\Im}^{+}(\varsigma) \barwedge \upsilon^{+}) \veebar \tau^{+}\} \barwedge \upsilon^{+} \\
&= \{(\tilde{\Im}^{+}(\varsigma) \barwedge \upsilon^{+}) \barwedge \upsilon^{+}\} \veebar (\tau^{+} \barwedge \upsilon^{+}) \\
&= \{(\tilde{\Im}^{+}(\varsigma) \barwedge \upsilon^{+}) \barwedge \upsilon^{+}\} \veebar \tau^{+} \\
&\geq (\tilde{\Im}^{+})^{\upsilon^{+}}_{\tau^{+}}(\varsigma) \barwedge \upsilon^{+}.
\end{aligned}$$

$$\begin{aligned}
\min\{(\tilde{\mho}^{+})^{\upsilon^{+}}_{\tau^{+}}(\epsilon\varsigma), \tau^{+}\} &= \min\{(\{\tilde{\mho}^{+}(\epsilon\varsigma) \veebar \upsilon^{+}\} \barwedge \tau^{+}), \tau^{+}\} \\
&= \{\tilde{\mho}^{+}(\epsilon\varsigma) \veebar \upsilon^{+}\} \barwedge \tau^{+} \\
&= \{\tilde{\mho}^{+}(\epsilon\varsigma) \barwedge \tau^{+}\} \veebar \{\upsilon^{+} \barwedge \tau^{+}\} \\
&= \{(\tilde{\mho}^{+}(\epsilon\varsigma) \barwedge \tau^{+}) \barwedge \tau^{+}\} \veebar \tau^{+} \\
&\leq \{(\tilde{\mho}^{+}(\varsigma) \veebar \upsilon^{+}) \barwedge \tau^{+}\} \veebar \tau^{+} \\
&= \{(\tilde{\mho}^{+}(\varsigma) \veebar \upsilon^{+}) \veebar \upsilon^{+}\} \barwedge \tau^{+} \\
&\leq (\tilde{\mho}^{+})^{\upsilon^{+}}_{\tau^{+}}(\varsigma) \veebar \upsilon^{+}
\end{aligned}$$

Hence $\beth = [(\tilde{\Re})^{\upsilon}_{\tau}, (\tilde{\Im})^{\upsilon}_{\tau}, (\tilde{\mho})^{\upsilon}_{\tau}]$ is a BIntNLI of $\mathscr{O}$.

Theorem 3.15. *Let $\beth$ be an (τ, υ)BIntNRI and $\daleth$ be an (τ, υ)BIntNLI of $\mathscr{O}$ then $((\beth \cdot \daleth])^{\upsilon}_{\tau} \subseteq (\beth \barwedge^{\upsilon}_{\tau} \daleth]$.*

Proof. Let $\beth = [(\tilde{\Re}^{-}_{\beth}, \tilde{\Re}^{+}_{\beth}), (\tilde{\Im}^{-}_{\beth}, \tilde{\Im}^{+}_{\beth}), (\tilde{\mho}^{-}_{\beth}, \tilde{\mho}^{+}_{\beth})]$ be an (τ, υ)BIntNRI and $\daleth = [(\tilde{\Re}^{-}_{\daleth}, \tilde{\Re}^{+}_{\daleth}), (\tilde{\Im}^{-}_{\daleth}, \tilde{\Im}^{+}_{\daleth}), (\tilde{\mho}^{-}_{\daleth}, \tilde{\mho}^{+}_{\daleth})]$ be an (τ, υ)BIntNLI of $\mathscr{O}$. Let $(\epsilon, \varsigma) \in I_{\kappa}$. If $I_{\kappa} \neq \emptyset$, then $\kappa \leq \epsilon\varsigma$. Thus $\tilde{\Re}^{-}_{\beth}(\kappa) \leq \tilde{\Re}^{-}_{\beth}(\epsilon\varsigma) \leq \tilde{\Re}^{-}_{\beth}(\epsilon)$, $\tilde{\Im}^{-}_{\beth}(\kappa) \leq \tilde{\Im}^{-}_{\beth}(\epsilon\varsigma) \leq \tilde{\Im}^{-}_{\beth}(\epsilon)$ and $\tilde{\mho}^{-}_{\beth}(\kappa) \geq \tilde{\mho}^{-}_{\beth}(\epsilon\varsigma) \geq$

$\tilde{\mho}^{-}_{\rfloor}(\epsilon)$. Similarly $\tilde{\Re}^{-}_{\intercal}(\kappa) \leq \tilde{\Re}^{-}_{\intercal}(\epsilon\varsigma) \leq \tilde{\Re}^{-}_{\intercal}(\epsilon)$, $\tilde{\Im}^{-}_{\intercal}(\kappa) \leq \tilde{\Im}^{-}_{\intercal}(\epsilon\varsigma) \leq \tilde{\Im}^{-}_{\intercal}(\epsilon)$ and $\tilde{\mho}^{-}_{\intercal}(\kappa) \geq \tilde{\mho}^{-}_{\intercal}(\epsilon\varsigma) \geq \tilde{\mho}^{-}_{\intercal}(\epsilon)$. Let $(\epsilon, \varsigma) \in I_\kappa$. If $I_\kappa \neq \emptyset$, then $\kappa \leq \epsilon\varsigma$. Thus $\tilde{\Re}^{+}_{\rfloor}(\kappa) \geq \tilde{\Re}^{+}_{\rfloor}(\epsilon\varsigma) \geq \tilde{\Re}^{+}_{\rfloor}(\epsilon)$, $\tilde{\Im}^{+}_{\rfloor}(\kappa) \geq \tilde{\Im}^{+}_{\rfloor}(\epsilon\varsigma) \geq \tilde{\Im}^{+}_{\rfloor}(\epsilon)$ and $\tilde{\mho}^{+}_{\rfloor}(\kappa) \leq \tilde{\mho}^{+}_{\rfloor}(\epsilon\varsigma) \leq \tilde{\mho}^{+}_{\rfloor}(\epsilon)$. Similarly $\tilde{\Re}^{+}_{\intercal}(\kappa) \geq \tilde{\Re}^{+}_{\intercal}(\epsilon\varsigma) \geq \tilde{\Re}^{+}_{\intercal}(\epsilon)$, $\tilde{\Im}^{+}_{\intercal}(\kappa) \geq \tilde{\Im}^{+}_{\intercal}(\epsilon\varsigma) \geq \tilde{\Im}^{+}_{\intercal}(\epsilon)$ and $\tilde{\mho}^{+}_{\intercal}(\kappa) \leq \tilde{\mho}^{+}_{\intercal}(\epsilon\varsigma) \leq \tilde{\mho}^{+}_{\intercal}(\epsilon)$. We have

$$
\begin{aligned}
(\tilde{\Re}^{-}_{(\rfloor\cdot\intercal]})^{\upsilon^-}_{\tau^-}(\kappa) &= (\tilde{\Re}^{-}_{(\rfloor\cdot\intercal]}(\kappa) \veebar \upsilon^-) \barwedge \tau^- \\
&= \Big[[\inf_{\kappa\leq\epsilon\varsigma}\{\tilde{\Re}^{-}_{\rfloor}(\epsilon) \veebar \tilde{\Re}^{-}_{\intercal}(\varsigma)\} \veebar \upsilon^-]\Big] \barwedge \tau^- \\
&= \Big[\inf_{\kappa\leq\epsilon\varsigma}\{\tilde{\Re}^{-}_{\rfloor}(\epsilon) \veebar \tilde{\Re}^{-}_{\intercal}(\varsigma)\} \veebar \upsilon^- \veebar \upsilon^-\Big] \barwedge \tau^- \\
&= \Big[\inf_{\kappa\leq\epsilon\varsigma}\{(\tilde{\Re}^{-}_{\rfloor}(\epsilon) \veebar \upsilon^-) \veebar (\tilde{\Re}^{-}_{\intercal}(\varsigma) \veebar \upsilon^-)\} \veebar \upsilon^-\Big] \barwedge \tau^- \\
&\geq (\{(\tilde{\Re}^{-}_{\rfloor}(\kappa) \barwedge \tau^-) \veebar (\tilde{\Re}^{-}_{\intercal}(\kappa) \barwedge \tau^-)\} \veebar \upsilon^-) \barwedge \tau^- \\
&= \{((\tilde{\Re}^{-}_{\rfloor}(\kappa) \veebar \tilde{\Re}^{-}_{\intercal}(\kappa)) \barwedge \tau^-) \veebar \upsilon^-\} \barwedge \tau^- \\
&= \{((\tilde{\Re}^{-}_{\rfloor} \veebar \tilde{\Re}^{-}_{\intercal})(\kappa) \veebar \upsilon^-\} \barwedge \tau^- \\
&= (\tilde{\Re}^{-}_{\rfloor \barwedge^{\upsilon^-}_{\tau^-} \intercal})(\kappa)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\Im}^{-}_{(\rfloor\cdot\intercal]})^{\upsilon^-}_{\tau^-}(\kappa) &= (\tilde{\Im}^{-}_{(\rfloor\cdot\intercal]}(\kappa) \veebar \upsilon^-) \barwedge \tau^- \\
&= \Big[[\inf_{\kappa\leq\epsilon\varsigma}\{\tilde{\Im}^{-}_{\rfloor}(\epsilon) \veebar \tilde{\Im}^{-}_{\intercal}(\varsigma)\} \veebar \upsilon^-]\Big] \barwedge \tau^- \\
&= \Big[\inf_{\kappa\leq\epsilon\varsigma}\{\tilde{\Im}^{-}_{\rfloor}(\epsilon) \veebar \tilde{\Im}^{-}_{\intercal}(\varsigma)\} \veebar \upsilon^- \veebar \upsilon^-\Big] \barwedge \tau^- \\
&= \Big[\inf_{\kappa\leq\epsilon\varsigma}\{(\tilde{\Im}^{-}_{\rfloor}(\epsilon) \veebar \upsilon^-) \veebar (\tilde{\Im}^{-}_{\intercal}(\varsigma) \veebar \upsilon^-)\} \veebar \upsilon^-\Big] \barwedge \tau^- \\
&\geq (\{(\tilde{\Im}^{-}_{\rfloor}(\kappa) \barwedge \tau^-) \veebar (\tilde{\Im}^{-}_{\intercal}(\kappa) \barwedge \tau^-)\} \veebar \upsilon^-) \barwedge \tau^- \\
&= \{((\tilde{\Im}^{-}_{\rfloor}(\kappa) \veebar \tilde{\Im}^{-}_{\intercal}(\kappa)) \barwedge \tau^-) \veebar \upsilon^-\} \barwedge \tau^- \\
&= \{((\tilde{\Im}^{-}_{\rfloor} \veebar \tilde{\Im}^{-}_{\intercal})(\kappa) \veebar \upsilon^-\} \barwedge \tau^- \\
&= (\tilde{\Im}^{-}_{\rfloor \barwedge^{\upsilon^-}_{\tau^-} \intercal})(\kappa)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\mho}^{-}_{(\lrcorner\cdot\urcorner]})^{\upsilon^-}_{\tau^-}(\kappa) &= (\tilde{\mho}^{-}_{(\lrcorner\cdot\urcorner]}(\kappa) \barwedge \upsilon^-) \veebar \tau^- \\
&= \Big[[\sup_{\kappa\le\epsilon\varsigma}\{\tilde{\mho}^{-}_{\lrcorner}(\epsilon) \barwedge \tilde{\mho}^{-}_{\urcorner}(\varsigma)\} \barwedge \upsilon^-]\Big] \veebar \tau^- \\
&= \Big[\sup_{\kappa\le\epsilon\varsigma}\{\tilde{\mho}^{-}_{\lrcorner}(\epsilon) \barwedge \tilde{\mho}^{-}_{\urcorner}(\varsigma)\} \barwedge \upsilon^- \barwedge \upsilon^-\Big] \veebar \tau^- \\
&= \Big[\sup_{\kappa\le\epsilon\varsigma}\{(\tilde{\mho}^{-}_{\lrcorner}(\epsilon) \barwedge \upsilon^-) \barwedge (\tilde{\mho}^{-}_{\urcorner}(\varsigma) \barwedge \upsilon^-)\} \barwedge \upsilon^-\Big] \veebar \tau^- \\
&\le (\{(\tilde{\mho}^{-}_{\lrcorner}(\kappa) \veebar \tau^-) \barwedge (\tilde{\mho}^{-}_{\urcorner}(\kappa) \veebar \tau^-)\} \barwedge \upsilon^-) \veebar \tau^- \\
&= \{((\tilde{\mho}^{-}_{\lrcorner}(\kappa) \barwedge \tilde{\mho}^{-}_{\urcorner}(\kappa)) \veebar \tau^-) \barwedge \upsilon^-\} \veebar \tau^- \\
&= \{((\tilde{\mho}^{-}_{\lrcorner} \barwedge \tilde{\mho}^{-}_{\urcorner})(\kappa) \barwedge \upsilon^-\} \veebar \tau^- \\
&= (\tilde{\mho}^{-}_{\lrcorner_{\veebar_{\tau^-}^{\upsilon^-}}\urcorner})(\kappa)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\Re}^{+}_{(\lrcorner\cdot\urcorner]})^{\upsilon^+}_{\tau^+}(\kappa) &= (\tilde{\Re}^{+}_{(\lrcorner\cdot\urcorner]}(\kappa) \barwedge \upsilon^+) \veebar \tau^+ \\
&= \Big[[\sup_{\kappa\le\epsilon\varsigma}\{\tilde{\Re}^{+}_{\lrcorner}(\epsilon) \barwedge \tilde{\Re}^{+}_{\urcorner}(\varsigma)\} \barwedge \upsilon^+]\Big] \veebar \tau^+ \\
&= \Big[\sup_{\kappa\le\epsilon\varsigma}\{\tilde{\Re}^{+}_{\lrcorner}(\epsilon) \barwedge \tilde{\Re}^{+}_{\urcorner}(\varsigma)\} \barwedge \upsilon^+ \barwedge \upsilon^+\Big] \veebar \tau^+ \\
&= \Big[\sup_{\kappa\le\epsilon\varsigma}\{(\tilde{\Re}^{+}_{\lrcorner}(\epsilon) \barwedge \upsilon^+) \barwedge (\tilde{\Re}^{+}_{\urcorner}(\varsigma) \barwedge \upsilon^+)\} \barwedge \upsilon^+\Big] \veebar \tau^+ \\
&\le (\{(\tilde{\Re}^{+}_{\lrcorner}(\kappa) \veebar \tau^+) \barwedge (\tilde{\Re}^{+}_{\urcorner}(\kappa) \veebar \tau^+)\} \barwedge \upsilon^+) \veebar \tau^+ \\
&= \{((\tilde{\Re}^{+}_{\lrcorner}(\kappa) \barwedge \tilde{\Re}^{+}_{\urcorner}(\kappa)) \veebar \tau^+) \barwedge \upsilon^+\} \veebar \tau^+ \\
&= \{((\tilde{\Re}^{+}_{\lrcorner} \barwedge \tilde{\Re}^{+}_{\urcorner})(\kappa) \barwedge \upsilon^+\} \veebar \tau^+ \\
&= (\tilde{\Re}^{+}_{\lrcorner_{\barwedge_{\tau^+}^{\upsilon^+}}\urcorner})(\kappa)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\Im}^{+}_{(\lrcorner\cdot\urcorner]})^{\upsilon^+}_{\tau^+}(\kappa) &= (\tilde{\Im}^{+}_{(\lrcorner\cdot\urcorner]}(\kappa) \barwedge \upsilon^+) \veebar \tau^+ \\
&= \Big[[\sup_{\kappa\le\epsilon\varsigma}\{\tilde{\Im}^{+}_{\lrcorner}(\epsilon) \barwedge \tilde{\Im}^{+}_{\urcorner}(\varsigma)\} \barwedge \upsilon^+]\Big] \veebar \tau^+ \\
&= \Big[\sup_{\kappa\le\epsilon\varsigma}\{\tilde{\Im}^{+}_{\lrcorner}(\epsilon) \barwedge \tilde{\Im}^{+}_{\urcorner}(\varsigma)\} \barwedge \upsilon^+ \barwedge \upsilon^+\Big] \veebar \tau^+ \\
&= \Big[\sup_{\kappa\le\epsilon\varsigma}\{(\tilde{\Im}^{+}_{\lrcorner}(\epsilon) \barwedge \upsilon^+) \barwedge (\tilde{\Im}^{+}_{\urcorner}(\varsigma) \barwedge \upsilon^+)\} \barwedge \upsilon^+\Big] \veebar \tau^+ \\
&\le (\{(\tilde{\Im}^{+}_{\lrcorner}(\kappa) \veebar \tau^+) \barwedge (\tilde{\Im}^{+}_{\urcorner}(\kappa) \veebar \tau^+)\} \barwedge \upsilon^+) \veebar \tau^+ \\
&= \{((\tilde{\Im}^{+}_{\lrcorner}(\kappa) \barwedge \tilde{\Im}^{+}_{\urcorner}(\kappa)) \veebar \tau^+) \barwedge \upsilon^+\} \veebar \tau^+ \\
&= \{((\tilde{\Im}^{+}_{\lrcorner} \barwedge \tilde{\Im}^{+}_{\urcorner})(\kappa) \barwedge \upsilon^+\} \veebar \tau^+ \\
&= (\tilde{\Im}^{+}_{\lrcorner_{\barwedge_{\tau^+}^{\upsilon^+}}\urcorner})(\kappa)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\mho}^{+}_{(\lrcorner\cdot\urcorner]})^{\upsilon^+}_{\tau^+}(\kappa) &= (\tilde{\mho}^{+}_{(\lrcorner\cdot\urcorner]}(\kappa) \veebar \upsilon^+) \barwedge \tau^+ \\
&= \Big[[\inf_{\kappa\le\epsilon\varsigma} \{\tilde{\mho}^{+}_{\lrcorner}(\epsilon) \veebar \tilde{\mho}^{+}_{\urcorner}(\varsigma)\} \veebar \upsilon^+] \Big] \barwedge \tau^+ \\
&= \Big[\inf_{\kappa\le\epsilon\varsigma} \{\tilde{\mho}^{+}_{\lrcorner}(\epsilon) \veebar \tilde{\mho}^{+}_{\urcorner}(\varsigma)\} \veebar \upsilon^+ \veebar \upsilon^+ \Big] \barwedge \tau^+ \\
&= \Big[\inf_{\kappa\le\epsilon\varsigma} \{(\tilde{\mho}^{+}_{\lrcorner}(\epsilon) \veebar \upsilon^+) \veebar (\tilde{\mho}^{+}_{\urcorner}(\varsigma) \veebar \upsilon^+)\} \veebar \upsilon^+ \Big] \barwedge \tau^+ \\
&\ge (\{(\tilde{\mho}^{+}_{\lrcorner}(\kappa) \barwedge \tau^+) \veebar (\tilde{\mho}^{+}_{\urcorner}(\kappa) \barwedge \tau^+)\} \veebar \upsilon^+) \barwedge \tau^+ \\
&= \{((\tilde{\mho}^{+}_{\lrcorner}(\kappa) \veebar \tilde{\mho}^{+}_{\urcorner}(\kappa)) \barwedge \tau^+) \veebar \upsilon^+\} \barwedge \tau^+ \\
&= \{((\tilde{\mho}^{+}_{\lrcorner} \veebar \tilde{\mho}^{+}_{\urcorner})(\kappa) \veebar \upsilon^+\} \barwedge \tau^+ \\
&= (\tilde{\mho}^{+}_{\lrcorner_{\veebar_{\tau^+}}\urcorner})(\kappa)
\end{aligned}
$$

Let $\epsilon, \varsigma \notin I_\kappa$. If $I_\kappa = \emptyset$, then $(\tilde{\Re}^{-}_{\lrcorner} \cdot \tilde{\Re}^{-}_{\urcorner})(\kappa) = 0$, $(\tilde{\Im}^{-}_{\lrcorner} \cdot \tilde{\Im}^{-}_{\urcorner})(\kappa) = 0$ and $(\tilde{\mho}^{-}_{\lrcorner} \cdot \tilde{\mho}^{-}_{\urcorner})(\kappa) = -1$ and such that $\kappa \le \epsilon\varsigma$.

$$
\begin{aligned}
(\tilde{\Re}^{-}_{(\lrcorner\cdot\urcorner]})^{\upsilon^-}_{\tau^-}(\kappa) &= (\tilde{\Re}^{-}_{(\lrcorner\cdot\urcorner]}(\kappa) \veebar \upsilon^-) \barwedge \tau^- \\
&= 0 \barwedge \tau^- \\
&\ge (\tilde{\Re}^{-}_{\lrcorner_{\barwedge}\urcorner}(\kappa) \veebar \upsilon^-) \barwedge \tau^- \\
&= (\tilde{\Re}^{-}_{\lrcorner_{\barwedge}\urcorner}(\kappa) \veebar \upsilon^-)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\Im}^{-}_{(\lrcorner\cdot\urcorner]})^{\upsilon^-}_{\tau^-}(\kappa) &= (\tilde{\Im}^{-}_{(\lrcorner\cdot\urcorner]}(\kappa) \veebar \upsilon^-) \barwedge \tau^- \\
&= 0 \barwedge \tau^- \\
&\ge (\tilde{\Im}^{-}_{\lrcorner_{\barwedge}\urcorner}(\kappa) \veebar \upsilon^-) \barwedge \tau^- \\
&= (\tilde{\Im}^{-}_{\lrcorner_{\barwedge}\urcorner}(\kappa) \veebar \upsilon^-)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\mho}^{-}_{(\lrcorner\cdot\urcorner]})^{\upsilon^-}_{\tau^-}(\kappa) &= (\tilde{\mho}^{-}_{(\lrcorner\cdot\urcorner]}(\kappa) \barwedge \upsilon^-) \veebar \tau^- \\
&= -1 \veebar \tau^- \\
&= \tau^- \\
&\le (\tilde{\mho}^{-}_{\lrcorner_{\veebar}\urcorner}(\kappa) \barwedge \upsilon^-) \veebar \tau^- \\
&= (\tilde{\mho}^{-}_{\lrcorner_{\veebar}\urcorner}(\kappa) \barwedge \upsilon^-)
\end{aligned}
$$

Let $\epsilon, \varsigma \notin I_\kappa$. If $I_\kappa = \emptyset$, then $(\tilde{\Re}^{+}_{\lrcorner} \cdot \tilde{\Re}^{+}_{\urcorner})(\kappa) = 0$, $(\tilde{\Im}^{+}_{\lrcorner} \cdot \tilde{\Im}^{+}_{\urcorner})(\kappa) = 0$ and $(\tilde{\mho}^{+}_{\lrcorner} \cdot \tilde{\mho}^{+}_{\urcorner})(\kappa) = 1$ and such that $\kappa \le \epsilon\varsigma$.

$$
\begin{aligned}
(\tilde{\Re}^{+}_{(\lrcorner\cdot\urcorner]})^{\upsilon^+}_{\tau^+}(\kappa) &= (\tilde{\Re}^{+}_{(\lrcorner\cdot\urcorner]}(\kappa) \barwedge \upsilon^+) \veebar \tau^+ \\
&= 0 \veebar \tau^+ \\
&\le (\tilde{\Re}^{+}_{\lrcorner_{\barwedge}\urcorner}(\kappa) \barwedge \upsilon^+) \veebar \tau^+ \\
&= (\tilde{\Re}^{+}_{\lrcorner_{\barwedge}\urcorner}(\kappa) \barwedge \upsilon^+)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\Im}^{+}_{(\mathbb{J}\cdot\mathbb{T}]})^{\upsilon^+}_{\tau^+}(\kappa) &= (\tilde{\Im}^{+}_{(\mathbb{J}\cdot\mathbb{T}]}(\kappa) \barwedge \upsilon^+) \veebar \tau^+ \\
&= 0 \veebar \tau^+ \\
&\leq (\tilde{\Im}^{+}_{\mathbb{J}\barwedge\mathbb{T}}(\kappa) \barwedge \upsilon^+) \veebar \tau^+ \\
&= (\tilde{\Im}^{+}_{\mathbb{J}\barwedge\mathbb{T}}(\kappa) \barwedge \upsilon^+) \\
(\tilde{\mho}^{+}_{(\mathbb{J}\cdot\mathbb{T}]})^{\upsilon^+}_{\tau^+}(\kappa) &= (\tilde{\mho}^{+}_{(\mathbb{J}\cdot\mathbb{T}]}(\kappa) \veebar \upsilon^+) \barwedge \tau^+ \\
&= 1 \barwedge \tau^+ \\
&= \tau^+ \\
&\geq (\tilde{\mho}^{+}_{\mathbb{J}\veebar\mathbb{T}}(\kappa) \veebar \upsilon^+) \barwedge \tau^+ \\
&= (\tilde{\mho}^{+}_{\mathbb{J}\veebar\mathbb{T}}(\kappa) \veebar \upsilon^+)
\end{aligned}
$$

Therefore $((\mathbb{J}\cdot\mathbb{T}])^{\upsilon}_{\tau} \subseteq (\mathbb{J} \barwedge^{\upsilon}_{\tau} \mathbb{T}]$.

4. **Conclusion and future direction**

The notions of (τ, υ)-BIntNS, BIntNLI, BIntNRI, BIntNI, and BIntNBI were introduced, along with a discussion of some of the characteristics of an ordered semigroup. Furthermore, an analysis of the properties of various transformations is conducted. We are attempting to handle novel fuzzy structures with cubic and interval values. Therefore, in the future, we should think about adopting advanced, soft settings.

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Measuring the Efficiency of Competitive Sports Talent Cultivation in Universities through the Synergy of Sports and Education using SuperHyperSoft Set

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Abstract: This work suggests the decision-making process for evaluating the cultivating competitive sports talent in universities under the integration of sports and education. We used the neutrosophic sets to deal with the uncertain and vague information. Two methods are used under the neutrosophic sets such as MEREC and TOPSIS. We used the MEREC method to compute the criteria weights and the TOPSIS method to rank the alternatives. These methods are used with the SuperHyperSoft set to treat the various criteria and sub criteria. We proposed nine groups of the SuperHyperSoft set to rank the alternatives. The ranks of alternatives of these groups are stable. We used seven criteria and seven alternatives.

Keywords: Neutrosophic Set; SuperHyperSoft Set; HyperSoft Set; Sports Talent; Education.

1. Introduction and Related Works

The cultivation of competitive sports talent in universities has become a crucial aspect of higher education, particularly under the integration of sports and education policies. These policies aim to strike a balance between academic success and athletic excellence, ensuring that student-athletes receive structured training while also achieving educational milestones[1], [2]. However, evaluating the efficiency of such programs requires a multi-criteria decision-making (MCDM) approach to account for the diverse and complex factors influencing sports talent development.

Universities implement various sports talent development models, ranging from elite sports academies to comprehensive universities with strong sports programs. The success of these models depends on multiple interrelated factors, including training facilities, coaching expertise, financial support, academic flexibility, competition exposure, athlete development programs, and sports science investments[3], [4]. Traditional evaluation methods often fail to provide an objective and quantifiable assessment of these factors. Therefore, MCDM methodologies are essential to systematically analyze, and rank university programs based on multiple conflicting criteria.

Fuzzy sets (FS) were popular when Zadeh [5] introduced the idea. However, the FS only uses the membership degree to indicate the uncertainty of decision information. By adding a non-membership degree, Atanassov [6] introduced intuitionistic fuzzy sets (IFS), which may successfully address issues that FS is unable to resolve. Interval IFS (IIFS) was proposed by Atanassov[7] , who expanded the IFS to interval numbers. Hesitant fuzzy sets (HFS), as defined by Torra [8], are a good way to handle the ambiguity brought on by the decision maker's hesitancy. Generalized HFS and double HFS were defined by Qian et al[9].

Information that is inconsistent and discontinuous cannot be handled by FS and its extension sets. This shortcoming is simply compensated for by the rise of the Neutrosophic sets (NS). Smarandache [10] introduced the idea of the NS, which represents fuzzy information using the truth-membership function, the indeterminacy-membership function, and the falsity-membership function—all of which are independent of one another.

In real-world applications, the NS is very inconvenient, even though it broadens the representation of ambiguous information. Ye [11] introduced the idea of simplified Neutrosophic sets (SNS) to streamline the NS, pointing out that SNS include both interval Neutrosophic sets (INS) and single-valued Neutrosophic sets (SVNS). Ye [12]suggested cosine similarity assessment of SNS and aggregation operators.

SNS operations were enhanced in the literature by Peng, Wang et al. [13]introduced single-valued triangular Neutrosophic sets (SVTNS) by combining triangular fuzzy numbers with SVNS. Wang et al. [14]considered the competing criteria, expanded the VIKOR model to SVTNS, and provided the precise procedures for using the approach.

The single-valued triangular Neutrosophic number (SVTNN), which is an extension of the single-valued Neutrosophic number, can successfully manage the enormous quantity of erroneous, partial, and inconsistent information involved in choosing green suppliers. To address the issue of MCDM, Fan et al. [15] suggested a novel aggregate operator that considers the benefits of the single-valued triangular Neutrosophic number.

The criteria's interrelationships and priority connection are taken into consideration by the new aggregate operator. Their presented the Dombi operations to increase the flexibility of the new aggregate operator. The SVTN Dombi prioritized normalized Bonferroni mean operator is proposed by combining the Dombi operations with the prioritized average operator and the Bonferroni mean operator.

The remainder of the document is organized as follows: The second portion provides a detailed introduction to the SVTNS, the suggested approach to computing the criteria weights and ranks the alternatives are suggested in the third part. A model for choosing appropriate alternatives is constructed in the fourth segment. The conclusion is the final section.

2. Single-Valued Triangular Neutrosophic Sets (SVNSs)

The SVTNS is used to deal with uncertainties and vague information in the decision-making process. This section shows the definitions of SVTNSs[15].

Definition 1

We can define the SVNSs by three functions such as truth, indeterminacy, and falsity.

$$A = \{(x, T_A(x), I_A(x), F_A(x)\ )|x \in X\} \tag{1}$$

$$T_A(x) = \left(T_A^1(x), T_A^2(x), T_A^3(x)\right) \tag{2}$$

$$I_A(x) = \left(I_A^1(x), I_A^2(x), I_A^3(x)\right) \tag{3}$$

$$F_A(x) = \left(F_A^1(x), F_A^2(x), F_A^3(x)\right) \tag{4}$$

$$0 \le T_A^3(x) + I_A^3(x) + F_A^3(x) \le 3 \tag{5}$$

We can define the SVTN number as:

$$A = \left(\left(T_A^1(x), T_A^2(x), T_A^3(x)\right), \left(I_A^1(x), I_A^2(x), I_A^3(x)\right), \left(F_A^1(x), F_A^2(x), F_A^3(x)\right)\right)$$

Definition 2

Let two SVTNNs as:

$$A_1 = \left(\left(T_{A_1}^1(x), T_{A_1}^2(x), T_{A_1}^3(x)\right), \left(I_{A_1}^1(x), I_{A_1}^2(x), I_{A_1}^3(x)\right), \left(F_{A_1}^1(x), F_{A_2}^2(x), F_{A_1}^3(x)\right)\right)$$

$$A_2 = \left(\left(T_{A_2}^1(x), T_{A_2}^2(x), T_{A_2}^3(x)\right), \left(I_{A_2}^1(x), I_{A_2}^2(x), I_{A_2}^3(x)\right), \left(F_{A_2}^1(x), F_{A_2}^2(x), F_{A_2}^3(x)\right)\right)$$

and their operations can be defined as:

$$A_1 \oplus A_2 = \begin{pmatrix} \begin{pmatrix} T_{A_1}^1(x) + T_{A_2}^1(x) - T_{A_1}^1(x)T_{A_2}^1(x), \\ T_{A_1}^2(x) + T_{A_2}^2(x) - T_{A_1}^2(x)T_{A_2}^2(x), \\ T_{A_1}^3(x) + T_{A_2}^3(x) - T_{A_1}^3(x)T_{A_2}^3(x) \end{pmatrix}, \\ \left(I_{A_1}^1(x)I_{A_2}^1(x), I_{A_1}^2(x)I_{A_2}^2(x), I_{A_1}^3(x)I_{A_2}^3(x)\right), \\ \left(F_{A_1}^1(x)F_{A_2}^1(x), F_{A_1}^2(x)F_{A_2}^2(x), F_{A_1}^3(x)F_{A_2}^3(x)\right) \end{pmatrix} \tag{6}$$

$$A_1 \oplus A_2 = \begin{pmatrix} \left(T_{A_1}^1(x)T_{A_2}^1(x), T_{A_1}^2(x)T_{A_2}^2(x), T_{A_1}^3(x)T_{A_2}^3(x)\right), \\ \begin{pmatrix} I_{A_1}^1(x) + I_{A_2}^1(x) - I_{A_1}^1(x)I_{A_2}^1(x), \\ I_{A_1}^2(x) + I_{A_2}^2(x) - I_{A_1}^2(x)I_{A_2}^2(x), \\ I_{A_1}^3(x) + I_{A_2}^3(x) - I_{A_1}^3(x)I_{A_2}^3(x) \end{pmatrix}, \\ \begin{pmatrix} F_{A_1}^1(x) + F_{A_2}^1(x) - F_{A_1}^1(x)F_{A_2}^1(x), \\ F_{A_1}^2(x) + F_{A_2}^2(x) - F_{A_1}^2(x)F_{A_2}^2(x), \\ F_{A_1}^3(x) + F_{A_2}^3(x) - F_{A_1}^3(x)F_{A_2}^3(x) \end{pmatrix} \end{pmatrix} \tag{7}$$

$$\nabla A_1 = \begin{pmatrix} \left(1-\left(1-T_{A_1}^1(x)\right)^{\nabla}, 1-\left(1-T_{A_1}^2(x)\right)^{\nabla}, 1-\left(1-T_{A_1}^3(x)\right)^{\nabla}\right), \\ \left(\left(I_{A_1}^1(x)\right)^{\nabla}, \left(I_{A_1}^2(x)\right)^{\nabla}, \left(I_{A_1}^3(x)\right)^{\nabla}\right), \\ \left(\left(F_{A_1}^1(x)\right)^{\nabla}, \left(F_{A_1}^2(x)\right)^{\nabla}, \left(F_{A_1}^3(x)\right)^{\nabla}\right) \end{pmatrix} \tag{8}$$

$$A_1^{\nabla} = \begin{pmatrix} \left(\left(T_{A_1}^1(x)\right)^{\nabla}, \left(T_{A_1}^2(x)\right)^{\nabla}, \left(T_{A_1}^3(x)\right)^{\nabla}\right), \\ \left(1-\left(1-I_{A_1}^1(x)\right)^{\nabla}, 1-\left(1-I_{A_1}^2(x)\right)^{\nabla}, 1-\left(1-I_{A_1}^3(x)\right)^{\nabla}\right), \\ \left(1-\left(1-F_{A_1}^1(x)\right)^{\nabla}, 1-\left(1-F_{A_1}^2(x)\right)^{\nabla}, 1-\left(1-F_{A_1}^3(x)\right)^{\nabla}\right) \end{pmatrix} \tag{9}$$

Definition 3

The score function can be defined as:

$$S(A) = \frac{1}{12}\begin{bmatrix} 8+\left(T_{A_1}^1(x)+2T_{A_1}^2(x)+T_{A_1}^3(x)\right)- \\ \left(I_{A_1}^1(x)+2I_{A_1}^2(x)+I_{A_1}^3(x)\right)- \\ \left(F_{A_1}^1(x)+2F_{A_1}^2(x)+F_{A_1}^3(x)\right) \end{bmatrix} \tag{10}$$

3. Suggested Approach

This section shows the steps of the suggested approach. We used the MEREC method to compute the criteria weights and the TOPSIS method to rank the alternatives. These methods are used under the SVTNSs to deal with uncertainty information.

SuperHyperSoft Set (SHSS)

SHSS is an extension of the hyper soft set. It is used to obtain the relations between the criteria and sub criteria[16], [17]. Let the universe set $U = \{A_1, A_2, \dots . A_n\}$. The power set of U is a $P(U)$ and R_1, R_2, R_3 are select as a criteria. $P(R_1) \times P(R_2)\ and\ P(R_3)\ are\ powersets\ of\ R_1, R_2, R_3$

Let F: $P(R_1) \times P(R_2) \times P(R_3) \to P(R)\ where\ \times refers\ to\ cartesian\ product$, and this called SHSS over R.

$$P(R_1) \times P(R_2) \times P(R_3) = \left\{ \begin{matrix} \{R_{11}\}, \{R_{12}\}, \{R_{11}, R_{12}\} \times \\ \{R_{21}\}, \{R_{22}\}, \{R_{21}, R_{22}\} \times \\ \{R_{31}\}, \{R_{32}\}, \{R_{33}\}, \{R_{31}, R_{32}\}, \{R_{31}, R_{33}\}, \\ \{R_{32}, R_{33}\}, \{R_{31}, R_{32}, R_{33}\} \end{matrix} \right\}$$

1. Build the decision matrix.

2. Normalize the decision matrix.

$$y_{ij} = \begin{cases} \frac{\min x_{ij}}{x_{ij}} \ if \ j \in B \\ \frac{x_{ij}}{\max x_{ij}} \ if \ j \in C \end{cases} \tag{11}$$

3. Compute the overall performance of the alternatives.

$$R_i = \ln\left(1 + \left(\frac{1}{m}\sum_j \left|\ln(y_{ij})\right|\right)\right) \tag{12}$$

4. Compute the performance of the alternatives by deleting each criterion.

$$R_{ij} = \ln\left(1 + \left(\frac{1}{m}\sum_{k,k\neq j} \left|\ln(y_{ik})\right|\right)\right) \tag{13}$$

5. Obtain the summation of absolute deviations.

$$G_j = \sum_i \left|R_{ij} - R_i\right| \tag{14}$$

6. Compute the criteria weights.

$$w_j = \frac{G_j}{\sum_j G_j} \tag{15}$$

TOPSIS Method

This part applies the steps of the TOPIS method to rank the alternatives.

1. Compute the normalization matrix

$$r_{ij} = \frac{x_{ij}}{\sum_{i=1}^{i} x_{ij}^2} \tag{16}$$

2. Compute the weighted normalized matrix

$$v_{ij} = w_j r_{ij} \tag{17}$$

3. Compute the ideal and non-ideal solution

$$A^+ = \begin{Bmatrix} \max v_{ij} \ for \ positive \ criterion, \\ \left(\min v_{ij} \ for \ cost \ criterion\right) \end{Bmatrix} \tag{18}$$

$$A^- = \begin{Bmatrix} \min v_{ij} \ for \ positive \ criterion, \\ \left(\max v_{ij} \ for \ cost \ criterion\right) \end{Bmatrix} \tag{19}$$

4. Compute the separation measures

$$Q_i^+ = \sum_{j=1}^{n} \left(v_{ij} - v_i^+\right)^2 \tag{20}$$

$$Q_i^- = \sum_{j=1}^{n} \left(v_{ij} - v_i^-\right)^2$$

5. Compute the relative closeness values

$$F_i = \frac{Q_i^-}{Q_i^- + Q_i^+} \tag{21}$$

4. Results and Discussion

This section shows the results of the proposed approach to compute the criteria and ranks the alternatives. We invited three experts to evaluate the criteria and alternatives. We gathered seven criteria and seven alternatives.

The criteria: Financial & Scholarship Support {Low, Medium, High}

Coaching Quality & Expertise {less than 0.5, between 0.5 and 0.7, more than 0.7}

Competition Exposure & Performance {medals are less than 2, between 2 and 6, more than 6}

Athlete Development Programs {adjustable, non-adjustable}

Sports Science & Injury Prevention {sufficient and in sufficient}

Academic-Sports Balance {optimal, non-optimal}

Training Facilities & Infrastructure {low quality and high quality}

The alternatives: Private Universities with Sports Excellence Programs, Public Universities with Government Support, Specialized Sports Academies, Comprehensive Universities with Strong Sports Programs, Community-Based Sports Talent Development Programs, International University Exchange Sports Programs, Elite Sports Universities

1. We build the decision matrix between the criteria and the alternatives as shown in Table 1 using the SVTNNs. Then we obtain crisp values by score function. Then we combine these values into one matrix.

2. Eq. (11) is used to normalize the decision matrix as shown in Table 2.

3. Eq. (12) is used to compute the overall performance of the alternatives.

4. Eq. (13) is used to compute the performance of the alternatives by deleting each criterion.

5. Eq. (14) is used to obtain the summation of absolute deviations.

6. Eq. (15) is used to compute the criteria weights as shown in Table 3.

Table 1. The decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>
A_2	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>
A_3	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>
A_4	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>

A_5	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>
A_6	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>
A_7	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>
A_2	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>
A_3	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>
A_4	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>
A_5	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>
A_6	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>
A_7	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>
A_2	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>
A_3	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>
A_4	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>
A_5	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.42,0.53,0.71), (0.38,0.45,0.53), (0.21,0.34,0.46)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>
A_6	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.54,0.68,0.72), (0.45,0.48,0.51), (0.35,0.39,0.42)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>
A_7	<(0.50,0.55,0.64), (0.25,0.37,0.42), (0.20,0.29,0.31)>	<(0.85,0.89,0.94), (0.65,0.67,0.75), (0.61,0.69,0.76)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.42,0.52,0.81), (0.41,0.45,0.58), (0.34,0.42,0.61)>	<(0.64,0.69,0.85), (0.34,0.45,0.52), (0.21,0.29,0.38)>	<(0.75,0.84,0.91), (0.54,0.62,0.67), (0.47,0.57,0.61)>

Table 2. The normalized decision matrix by MEREC method.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	1.041365	1.205607	1.097039	1	1.083882	1.190405	1.088832
A_2	1.078077	1.113707	1.305921	1.055921	1	1	1.022335
A_3	1.210962	1.057632	1	1.088268	1.203947	1.16042	1.094416
A_4	1.018614	1.063863	1.234101	1.195175	1.143092	1.053973	1.134518
A_5	1.024819	1.043614	1.123355	1.195175	1.201754	1.065967	1

A_6	1	1	1.101974	1.145285	1.212719	1.057971	1.006091
A_7	1.155636	1.074247	1.092654	1.305921	1.129934	1.05947	1.14467

Table 3. The criteria weights.

Criteria	Weights
C_1	0.062754
C_2	0.068514
C_3	0.048278
C_4	0.047335
C_5	0.047185
C_6	0.06656
C_7	0.659374

TOPSIS Method

In this part, we rank the alternatives by 8 groups of the SHSS. We proposed 8 groups of SHSS values to rank the alternatives as:

{Low, Medium, High}

{less than 0.5, between 0.5 and 0.7, more than 0.7}

{medals are less than 2, between 2 and 6, more than 6}

{adjustable, non-adjustable}

{suffice and in sufficient}

{optimal, non-optimal}

{low quality and high quality}

Group 1: {Low}, {less than 0.5}, {medals are more than 6}, {adjustable}, {sufficient}, {optimal}, {high quality}

Group 2: {Medium}, {less than 0.5}, {medals are more than 6}, {adjustable}, {sufficient}, {optimal}, {high quality}

Group 3: {High}, {less than 0.5}, {medals are more than 6}, {adjustable}, {sufficient}, {optimal}, {high quality}

Group 4: {Low}, {between 0.5 and 0.7}, {medals are more than 6}, {adjustable}, {sufficient}, {optimal}, {high quality}

Group 5: {Medium}, {between 0.5 and 0.7}, {medals are more than 6}, {adjustable}, {sufficient}, {optimal}, {high quality}

Group 6: {High}, {between 0.5 and 0.7}, {medals are more than 6}, {adjustable}, {sufficient}, {optimal}, {high quality}

Group 7: {Low}, {more than 0.7}, {medals are more than 6}, {adjustable}, {sufficient}, {optimal}, {high quality}

Group 8: {Medium}, {more than 0.7}, {medals are more than 6}, {adjustable}, {sufficient}, {optimal}, {high quality}

Group 9: {High}, {more than 0.7}, {medals are more than 6}, {adjustable}, {sufficient}, {optimal}, {high quality}

Group 1

1. Eq. (16) is used to compute the normalization matrix as in Fig 1.

2. Eq. (17) is used to compute the weighted normalized matrix as in Fig 2.

3. Eqs. (18 and 19) are used to compute the ideal and non-ideal solution

4.Eq. (20) is used to compute the separation measures

5. Eq. (21) is used to compute the relative closeness values

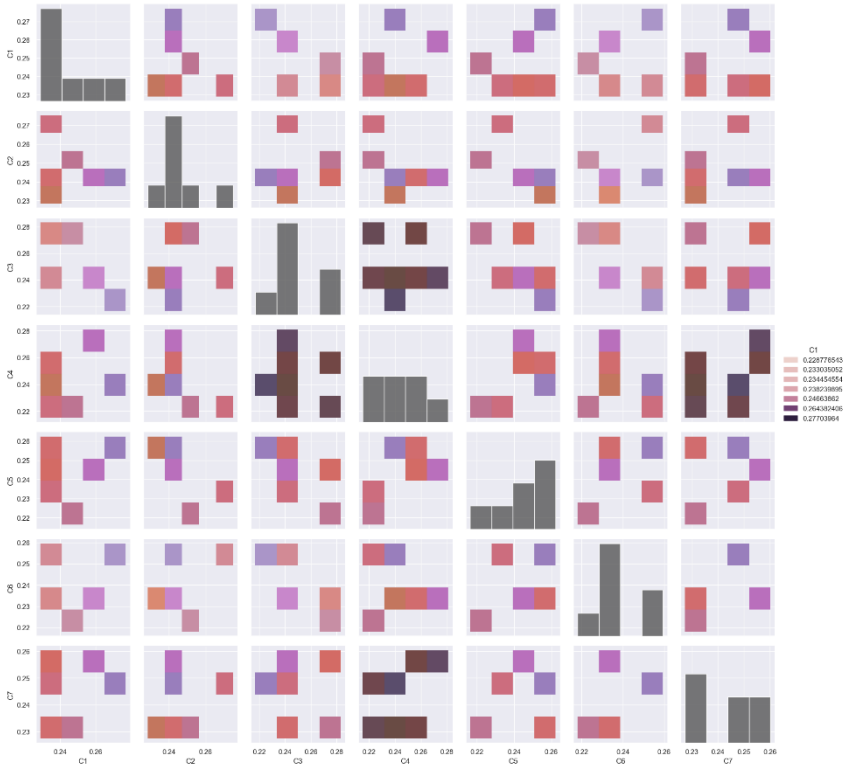

Fig 1. The normalization matrix.

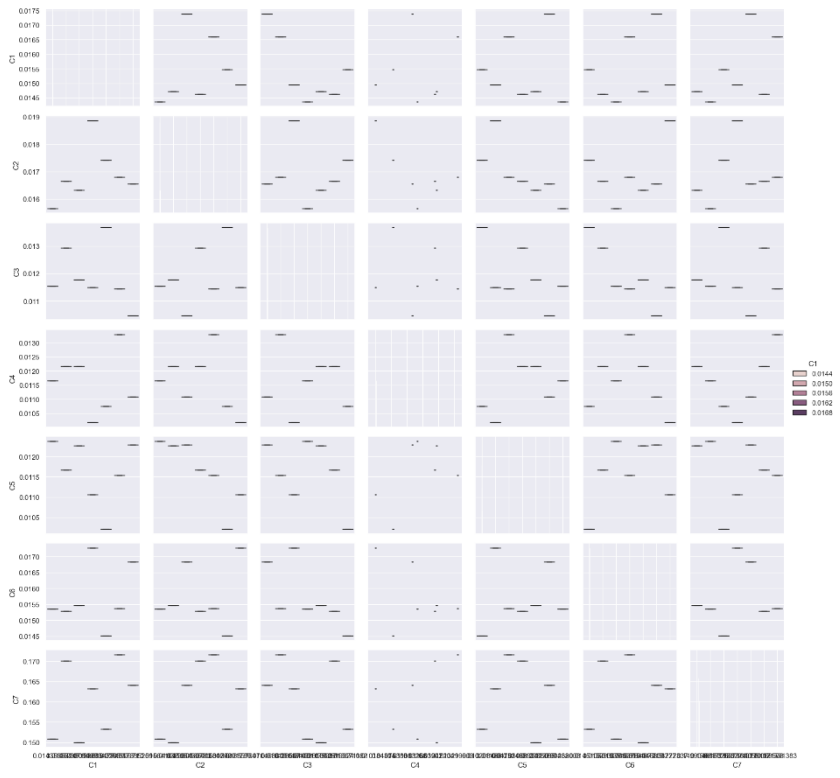

Fig 2. The weighted normalization matrix.

Group 2

The normalization matrix as shown in Fig 3.

The weighted normalized matrix as shown in Fig 4.

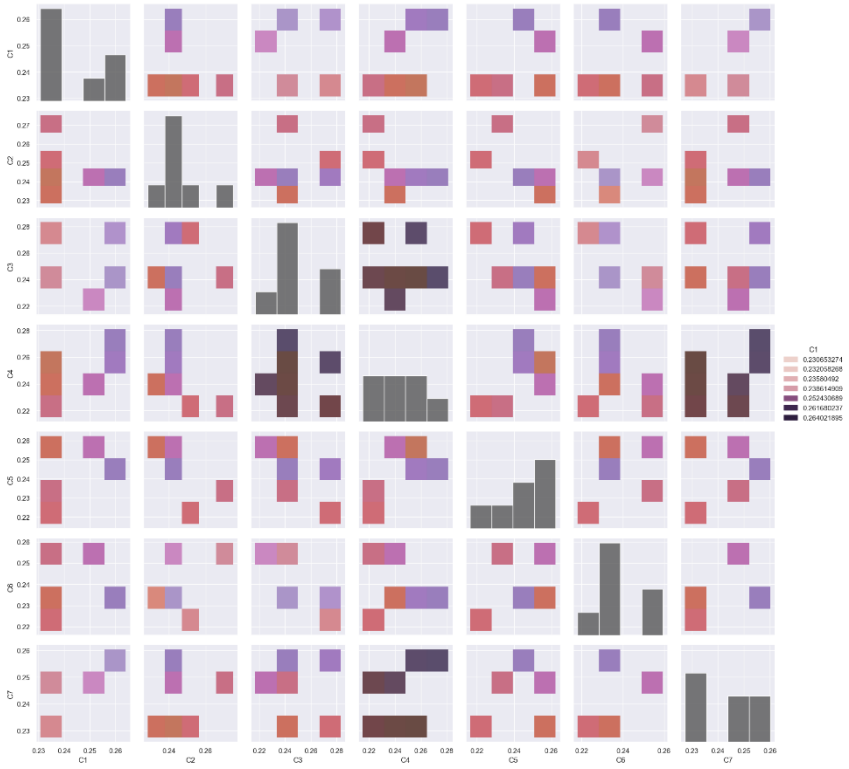

Fig 3. The normalization matrix.

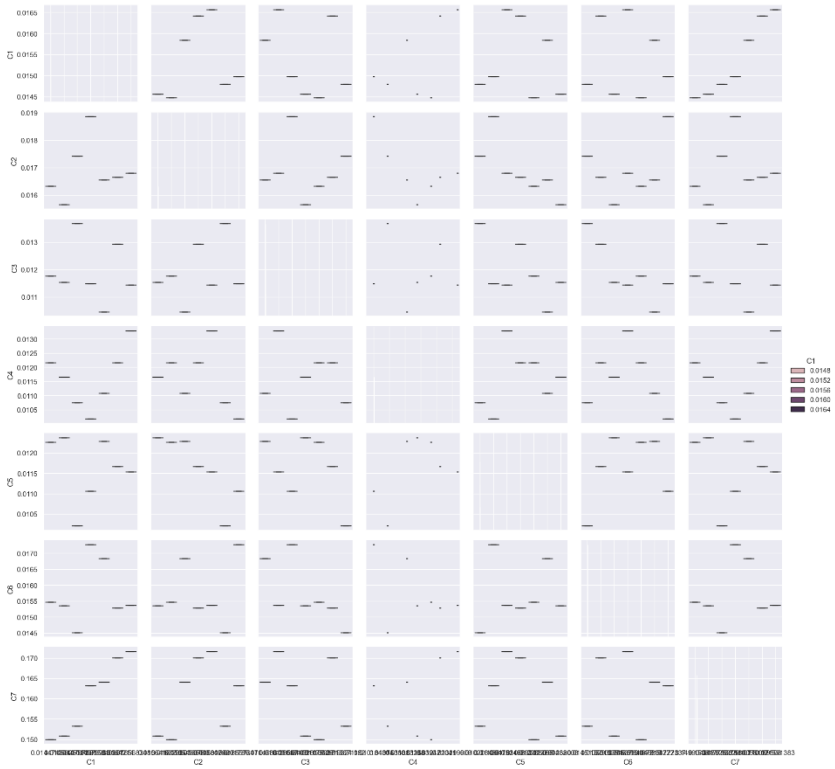

Fig 4. The weighted normalization matrix.

Group 3

The normalization matrix as shown in Fig 5.

The weighted normalized matrix as shown in Fig 6.

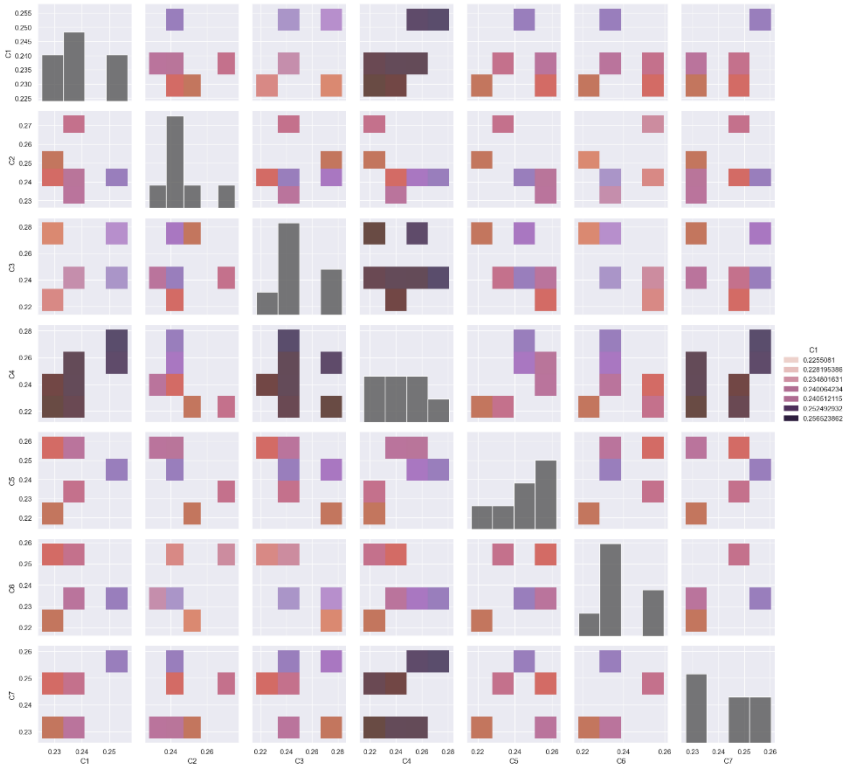

Fig 5. The normalization matrix.

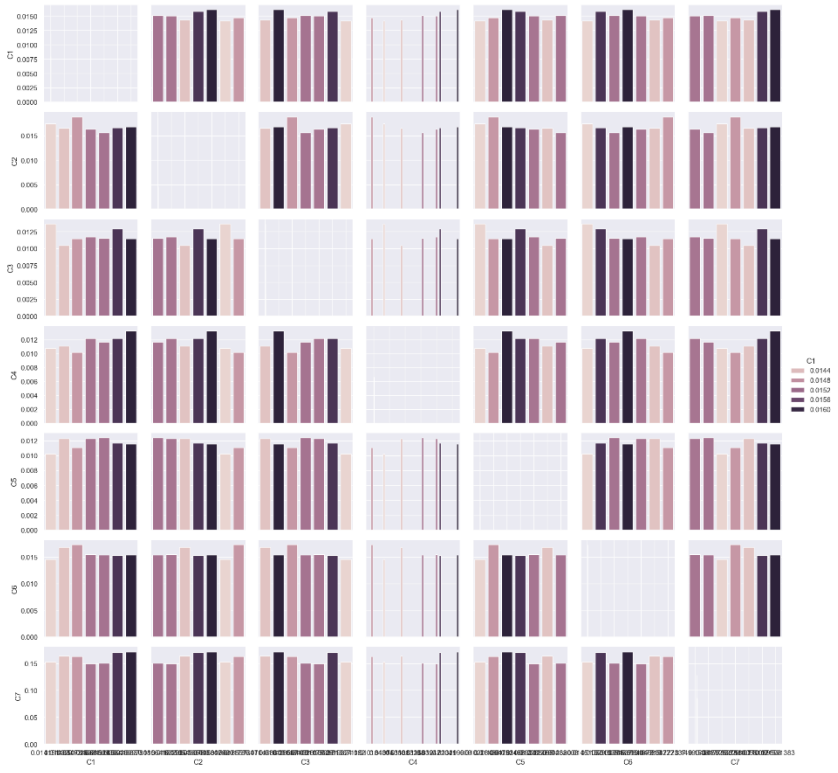

Fig 6. The weighted normalization matrix.

Group 4

The normalization matrix as shown in Fig 7.

The weighted normalized matrix as shown in Fig 8.

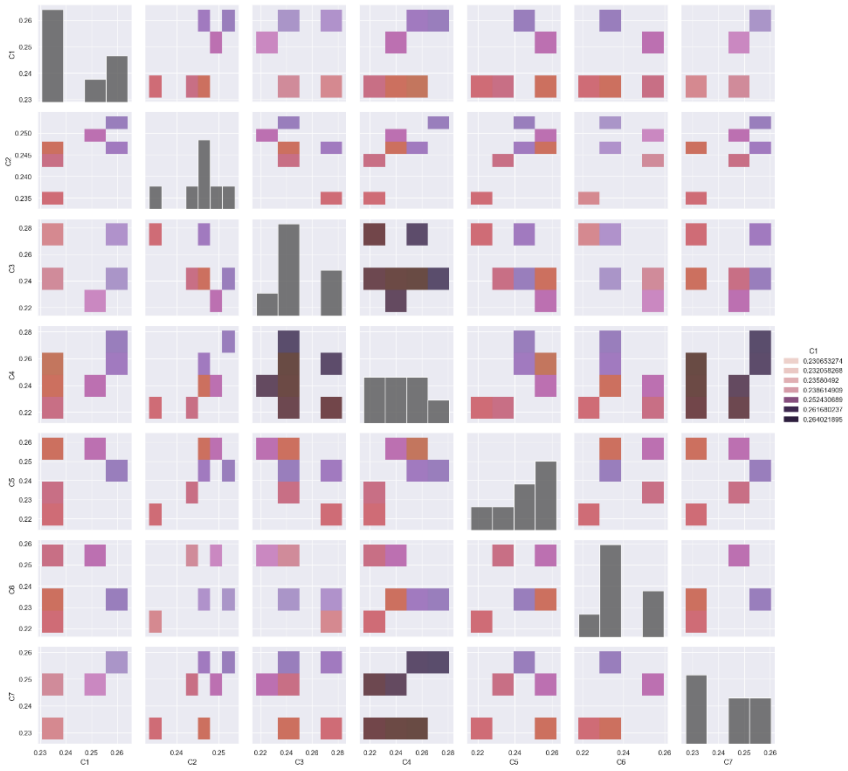

Fig 7. The normalization matrix.

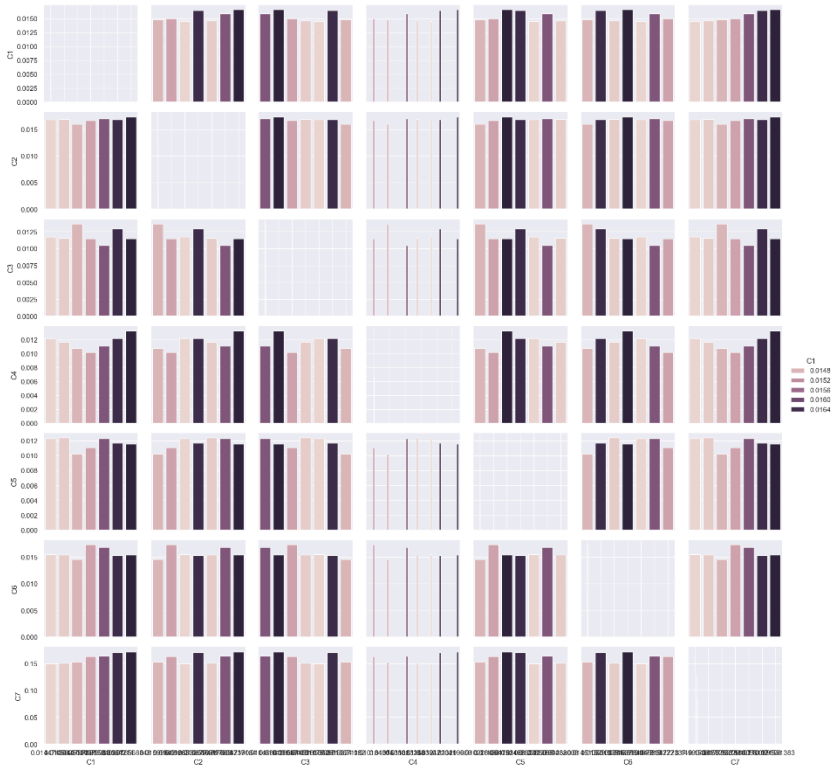

Fig 8. The weighted normalization matrix.

Group 5

The normalization matrix as shown in Fig 9.

The weighted normalized matrix as shown in Fig 10.

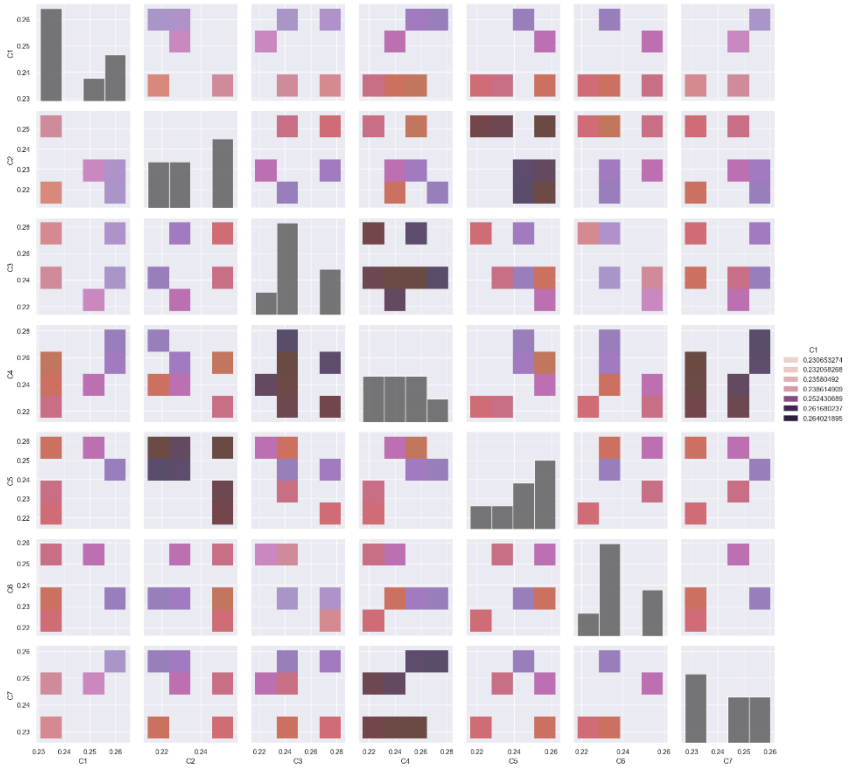

Fig 9. The normalization matrix.

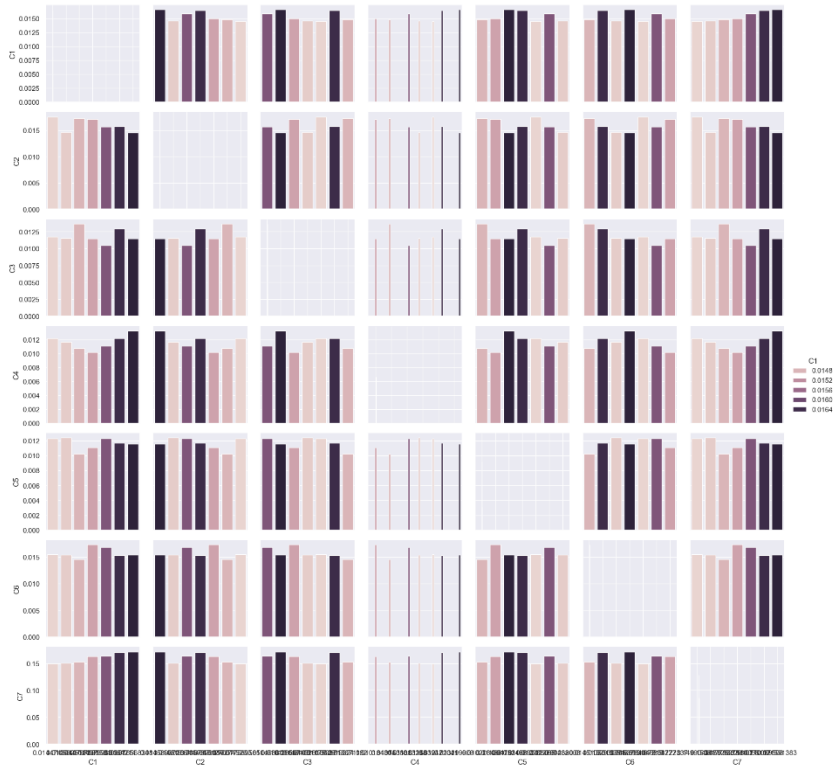

Fig 10. The weighted normalization matrix.

Group 6

The normalization matrix as shown in Fig 11.

The weighted normalized matrix as shown in Fig 12.

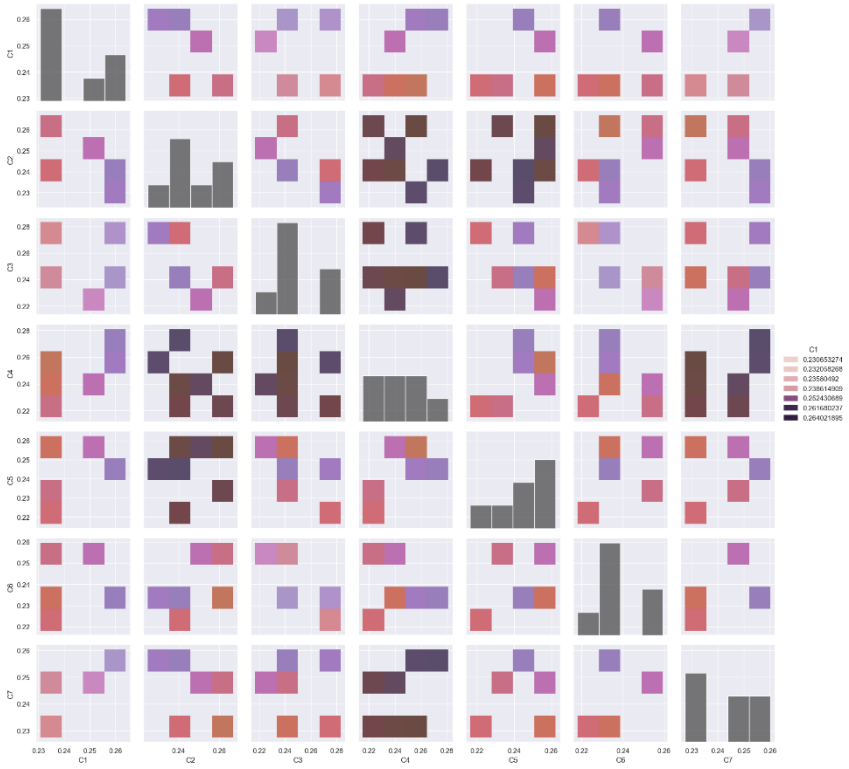

Fig 11. The normalization matrix.

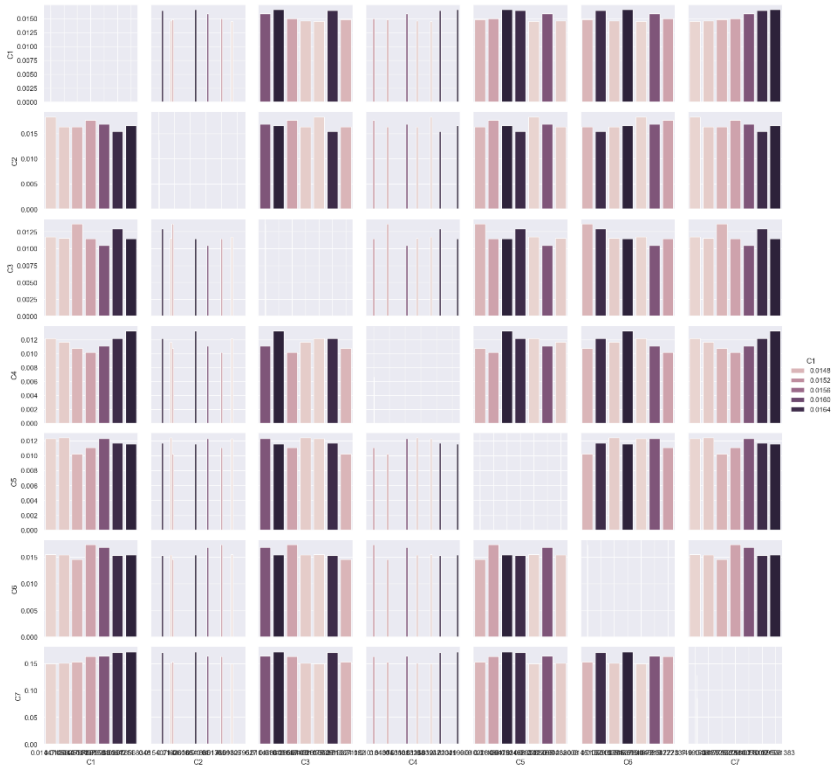

Fig 12. The weighted normalization matrix.

Group 7

The normalization matrix as shown in Fig 13.

The weighted normalized matrix as shown in Fig 14.

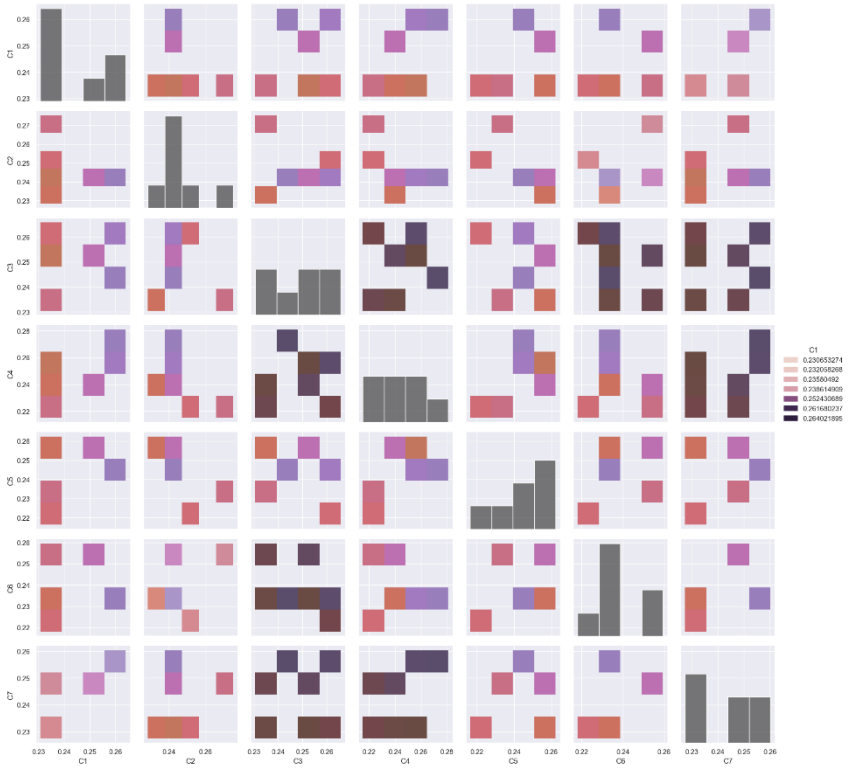

Fig 13. The normalization matrix.

Fig 14. The weighted normalization matrix.

Group 8

The normalization matrix as shown in Fig 15.

The weighted normalized matrix as shown in Fig 16.

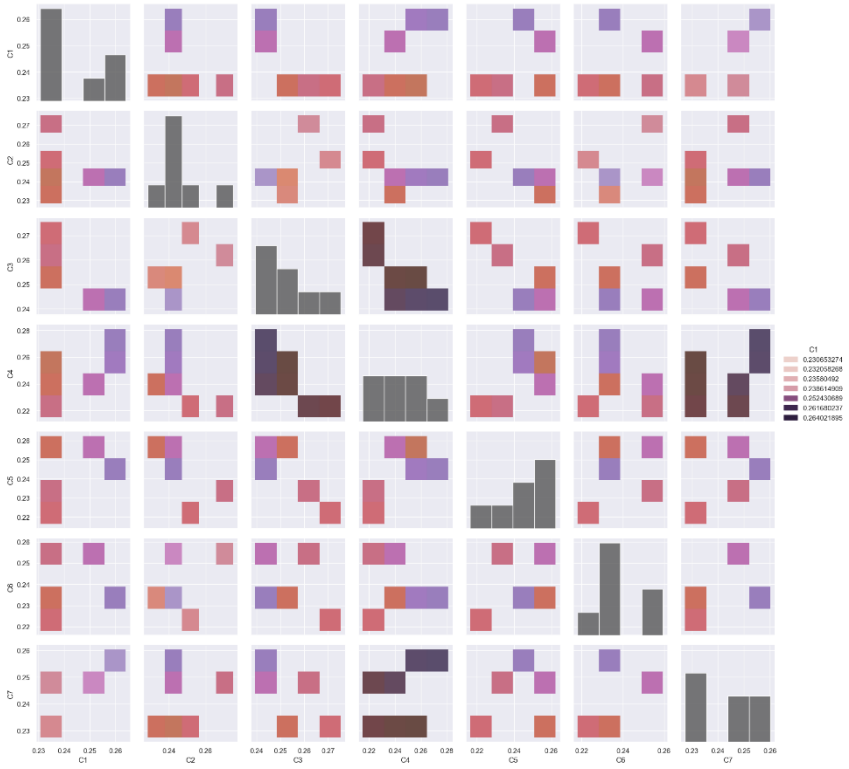

Fig 15. The normalization matrix.

Fig 16. The weighted normalization matrix.

Group 9

The normalization matrix as shown in Fig 17.

The weighted normalized matrix as shown in Fig 18.

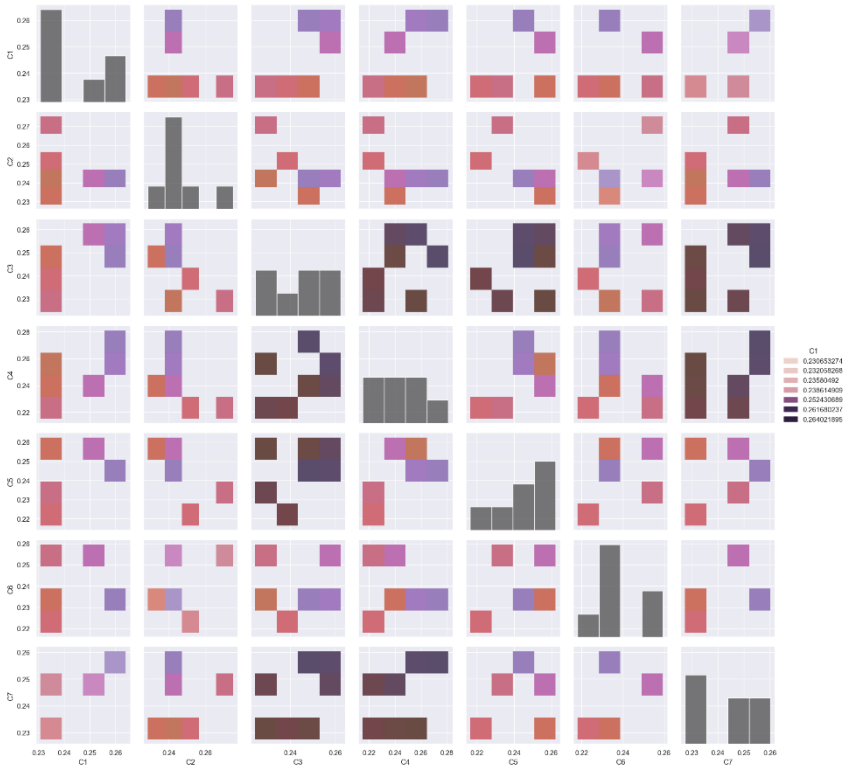

Fig 17. The normalization matrix.

Fig 18. The weighted normalization matrix.

Then we show the final ranks of alternatives as shown in Fig 19. We show alternative 7 is the best and alternative 6 is the worst.

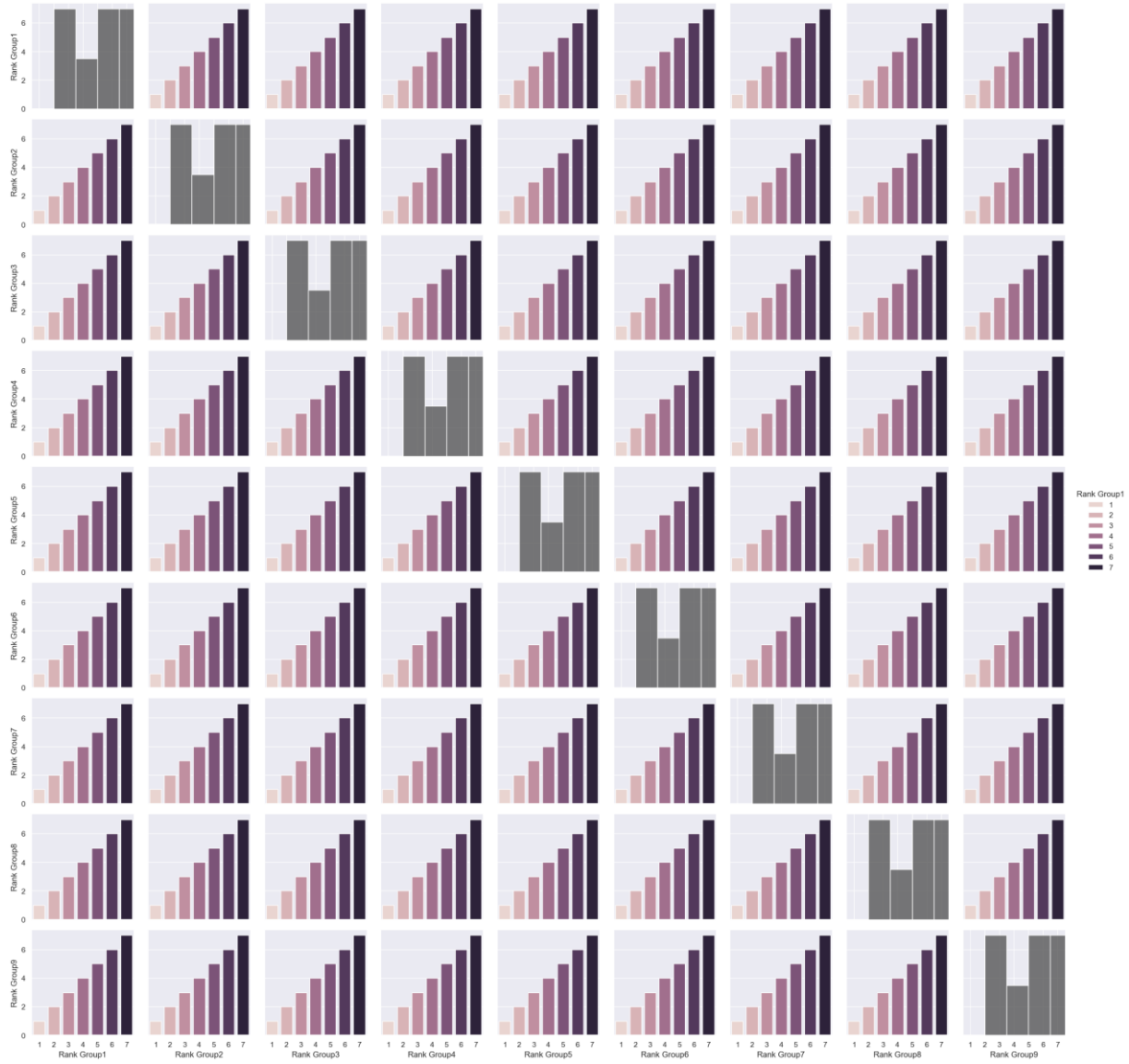

Fig 19. The final ranks of alternatives.

5. Conclusions

This study proposed a uncertainty model to evaluate the criteria and alternatives. Two methods are used in this study named MEREC methodology to compute the criteria weights and the TOPSIS methodology to rank the alternatives. These methods are used under the single valued triangular neutrosophic set to deal with uncertainty and vague information. We used the SuperHyperSoft set to treat the criteria and sub criteria. We proposed 8 groups in the SuperHyperSoft set, then ranked the alternatives under these groups. The ranks of alternatives are stable under these groups.

Acknowledgements

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Bridging Ancient Wisdom and Modern Logic: Neutrosophic Perspectives on Body, Mind, Soul and Spirit

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Abstract: This study explores the application of neutrosophic logic to the interrelation of body, mind, soul, and spirit, proposing a fluid model that moves beyond traditional dualistic frameworks. Unlike rigid categorizations, the Body-Mind-Soul-Spirit Fluidity model introduces a non-binary approach where these elements interact dynamically, existing in varying degrees of truth, falsehood, and indeterminacy. By integrating n-valued refined neutrosophic logic, this framework enhances the representation of uncertainty, contradiction, and partial beliefs, particularly in stance detection. A computational model is developed to classify stances in academic literature, leveraging a zero-shot classifier based on BART-large-MNLI. The model assigns neutrosophic values across multiple dimensions, enabling a nuanced understanding of support, opposition, and neutrality with refined degrees such as partial support and partial opposition. Furthermore, the study establishes philosophical connections between neutrosophy and indigenous worldviews, such as those of the Yoruba, Mayan, and Amazonian traditions, emphasizing the rejection of rigid dichotomies. These insights suggest that neutrosophic logic can be extended to ethics, AI, and cognitive science, providing a robust framework for reasoning under uncertainty and dynamic identity structures.

Keywords: Neutrosophic logic, Body-mind-soul-spirit fluidity, Stance detection, Indeterminacy modeling, Artificial intelligence ethics

1. Introduction

The article "Neutrosophic Perspectives on the Body-Mind-Soul-Spirit Fluidity" by Florentin Smarandache (2025) explores how neutrosophy can be applied to the relationship between body, mind, soul, and spirit [1]. Unlike traditional models that present these components as separate or hierarchical entities, this approach proposes a quadruple fluidity where each element is interconnected and can exist in varying degrees of truth, indeterminacy, and falsehood.

Florentin Smarandache introduces a neutrosophic framework that transcends Cartesian dualism and other traditional dichotomous models in the philosophy of mind and the study of human beings. This is achieved through the concept of quadruple fluidity, which replaces the rigid separation between body and mind with a dynamic and ever-changing interaction among body, mind, soul, and spirit [1]. By incorporating neutrosophic logic, we move beyond binary oppositions (e.g., mind vs. body, true vs. false) to a framework where truth, falsehood, and indeterminacy coexist[3]. This could offer new insights into the philosophy of mind [4], cognitive science [5], and even artificial intelligence [6], where rigid categorizations often fail to capture the complexity of human experience.

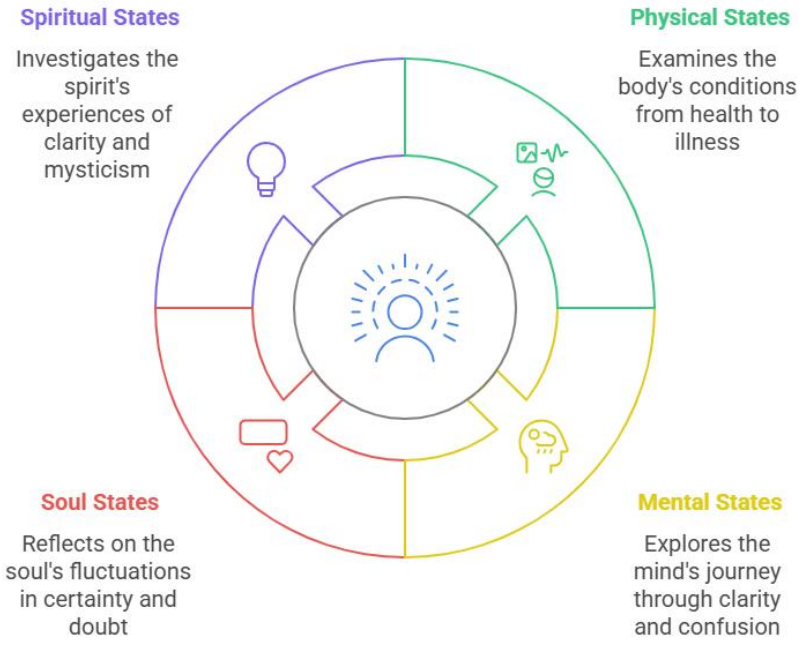
Spiritual States
Investigates the spirit's experiences of clarity and mysticism
Physical States
Examines the body's conditions from health to illness
Soul States
Reflects on the soul's fluctuations in certainty and doubt
Mental States
Explores the mind's journey through clarity and confusion

Figure 1. Neutrosophic Perspectives on the Human Body-Mind-Soul-Spirit Fluidity

The Body-Mind-Soul-Spirit Fluidity model surpasses this dichotomy by not reducing human existence to two static elements (mind and body) but instead proposes a fluid interconnectivity among four dimensions that exist in dynamic and non-binary states. In the neutrosophic model, the four components of the human being are not separate but rather [7]:

- Spirituality is an inherent dimension of the human being, not only linked to religions but also to the search for purpose, transcendence, and connection with something greater.
- The body is not just physical; it also exists in neutral states of transition (e.g., sick, healthy).
- The mind is not purely rational; it can be in states of confusion or indeterminacy (e.g., emotional processing, altered states of consciousness).
- The soul is not a fixed essence; it can fluctuate between states of certainty and uncertainty about identity, morality, or life purpose.
- The spirit is not entirely transcendental; it can oscillate between clarity and confusion, between deep mystical experiences and states of doubt.

These components do not operate separately but affect each other, allowing for a dynamic model where the boundaries between body, mind, soul, and spirit are permeable and not absolute.

Furthermore, the Body-Mind-Soul-Spirit Fluidity model aligns with various indigenous and philosophical worldviews, particularly those found in Latin American cosmologies[, where identity and reality are perceived as interconnected and constantly evolving. Traditions such as Yoruba philosophy, Mayan Nahualism, and Amazonian shamanic practices similarly reject rigid dichotomies, instead embracing a continuum of existence where spiritual, physical, and cognitive states dynamically interact [8-10]. This perspective reinforces the neutrosophic notion that truth, indeterminacy, and falsehood coexist in non-binary, context-dependent relationships rather than fixed categories.

Beyond its philosophical implications, this framework presents practical applications in areas such as stance detection in artificial intelligence (AI) [11, 12], cognitive modeling[13], and algorithmic ethics [14, 15]. By leveraging n-valued refined neutrosophic logic [16], computational models can

more accurately capture ambiguity, contradiction, and partial truths, which are crucial for domains requiring nuanced decision-making. This study proposes an advanced stance classification model based on zero-shot learning and semantic analysis, demonstrating how AI systems can integrate uncertainty and dynamic identity structures to enhance automated reasoning and ethical AI frameworks.

2 Preliminaries

2.1 Relation to Latin American and Modern Cosmovisions

The Yoruba philosophy, which has significantly influenced spiritual and cultural traditions in Cuba, the Caribbean, and Brazil, holds that the human being is composed of ara (physical body), emi (soul or vital energy), and ori (consciousness and destiny), where Ori acts as an intermediary between free will and determinism, allowing variations in a person's destiny over time[17]. Similarly, neutrosophy, developed by Florentin Smarandache, introduces a model where truth, identity, and reality are not absolute but exist on a spectrum of truth (T), falsehood (F), and indeterminacy (I) [18]. This conceptual alignment allows for understanding both Ori and personal development from a dynamic perspective, where identity is not fixed but flows and evolves based on experience and perception.

Figure 2. Yoruba Head Sculpture [19]

Both models reject the rigid dichotomy between determinism and free will, offering a more flexible and contextual approach. In Yoruba philosophy, a person chooses their destiny before birth but can modify it through knowledge and decisions [20]. In the neutrosophy model, existence is conceived in terms of degrees of certainty and uncertainty, allowing for a more nuanced interpretation of human development, morality, and decision-making. Thus, an individual is not entirely bound to their destiny, nor do they have absolute freedom, but rather navigate between states of possibility and change[1,7].

In the Mayan worldview, existence is governed by the concept of Nahualism, where each human has a nahual (complementary energy or spirit) [21]. This fluid relationship between the physical

and spiritual worlds suggests that a person's identity is not static but changes over time and circumstances, depending on their nahual and connection with nature. The Mayans also conceived reality in terms of cycles and transformation, which aligns with the neutrosophic idea that truth and identity fluctuate in degrees of certainty and uncertainty. An individual is not entirely "good" or "bad," nor is their destiny completely determined, but can change according to their interaction with their nahual and cosmic energy.

Amazonian peoples, such as the Shipibo-Konibo or Yanomami, conceive reality as an interaction between multiple dimensions[22]. Shamans access intermediate states of consciousness, resembling neutrosophic indeterminacy: the perception of reality changes according to the connection with spirits and sacred plants. In these traditions, the body is not just physical, the mind is not just rational, and the spirit is not just transcendental, but these elements are in constant interaction and transformation. Rituals and visionary experiences alter states of perception, fitting with the neutrosophic idea that there is no single state of truth or reality, but multiple coexisting interpretations[23].

Neutrosophy and its Body-Mind-Soul-Spirit Fluidity model find parallels in multiple Latin American worldviews. All these traditions reject the absolute separation between matter and spirit, certainty and falsehood, destiny and free will, proposing instead an interconnected system where identity and truth exist in dynamic states. Neutrosophy, developed by Florentin Smarandache, and the logic presented in Scientology (Logic 6 and Logic 7) share a central idea: absolutes are unattainable, and truth must be evaluated on a spectrum or gradient rather than in binary terms[24].

The recognition of spirituality as an integral part provides important elements. The Neutrosophic Body-Mind-Soul-Spirit model could revolutionize how we design and interpret AI, moving away from deterministic approaches and towards systems that are more flexible, intuitive, and closer to human experience. Its application would be crucial in areas such as algorithmic ethics, uncertain reasoning, digital identity, and potential artificial consciousness [25].

Mind-body dualism has profoundly shaped ethics throughout history, generating dichotomies that influence the understanding of good and evil, moral responsibility, and decision-making. Models like Kantian ethics [26] have privileged reason over emotions, while currents like hedonism and utilitarianism have prioritized physical well-being and material influences. The Aristotelian tradition and Hans Jonas's ethics[27] of responsibility have attempted to balance reason and corporeality but still operate within dichotomous frameworks.

Neutrosophy, on the other hand, proposes an ethics of continuity and fluidity, where moral decisions are not reduced to absolute categories of right or wrong but can include states of uncertainty. Reason and emotion are not opposites but interdependent dimensions of morality. Likewise, ethical values are not fixed but dynamic and contextual, evolving based on culture and knowledge. This approach allows for a more nuanced understanding of bioethical, technological, and social dilemmas, overcoming the limitations of traditional views. Mind-body dualism, by splitting the rational from the physical, forged ethical dichotomies that divided morality between reason and emotion, universality and contingency. Kant elevated the mind to an infallible judge; hedonism reduced ethics to the calculation of pleasure; Aristotle sought balance, but still within static frameworks. Neutrosophy, however, dissolves these boundaries: it accepts uncertainty (I) as part of the moral, reconciles compassion with duty, and understands ethics not as an immutable code but as a dialectical flow between mind, body, soul, spirit, and context. Thus, virtue is no longer a choice between poles but a dynamic integration where the ambiguous, the cultural, and the evolutionary redefine what it means to act well [1, 14, 15].

2.1 Extending Stance Detection with Refined Neutrosophic Logic

The n-valued neutrosophic set, with its capacity to represent truth, falsity, and indeterminacy across multiple refined dimensions, offers a robust framework for addressing the complexities of stance detection, where opinions often exist in nuanced, uncertain, or contradictory states[28].

Stance detection is the task of identifying expressed beliefs, opinions, or attitudes within a text. It involves determining an author's viewpoint or attitude toward a specific topic, claim, or target. This process requires analyzing textual data to infer whether the author supports, opposes, or remains neutral regarding the given proposition [29, 30].

A text sample T is said to entail a stance S of an author A when, given a specific context C, an evaluator—whether a human or an algorithm—can infer that T expresses support, opposition, or neutrality toward S.

The stance categories can be expanded beyond the primary labels (support, opposition, neutrality) to include finer distinctions such as partial support and partial opposition, depending on the granularity required for the study [3, 31].

From a computational perspective, stance detection requires[32]:

1. Context Awareness ($\boldsymbol{C}$) – Understanding the external information or assumptions influencing the interpretation of T.
2. Semantic Analysis of the Text ($\boldsymbol{T}$) – Extracting the underlying meaning using contextual embeddings and sentiment analysis techniques.
3. Inference of the Author's Stance ($\boldsymbol{S}$) – Classifying T into an appropriate stance category using machine learning models, pre-trained language models, or rule-based approaches.

This framework provides a foundation for integrating stance detection into both human-driven qualitative analyses and automated natural language processing (NLP) systems.

Incorporating n-valued refined neutrosophic logic into stance detection allows for a more nuanced representation of uncertainty, indeterminacy, and partial beliefs. By assigning degrees of truth (T), falsity (F), and indeterminacy (I) to the stance expressed in a text, this approach can model complex and ambiguous stances that go beyond binary classifications. For example, a stance may simultaneously exhibit partial support (T), partial opposition (F), and neutrality (I) due to the inherent vagueness or contradictions in the text. This aligns with the principles of neutrosophic logic, which generalizes classical and fuzzy logic to handle such complexities[33].

"The neutrosophic components (T, I, F) can be refined into n neutrosophic subcomponents (T_1, T_2, ..., T_p; I_1, I_2, ..., I_r; F_1, F_2, ..., F_s), where p, r, s, n are positive integers and p + r + s = n. This refined structure enables a nuanced representation of truth, indeterminacy, and falsity, making it particularly suitable for stance detection, where expressed beliefs often exhibit overlapping, uncertain, or contradictory dimensions[3].

2. Materials and Methods.

The research methodology employs a refined neutrosophic set structure where the classical neutrosophic components Truth (T), Indeterminacy (I), and Falsity (F) are refined into five subcomponents to model stance detection more precisely. The refined neutrosophic set A is defined as:

$$\boldsymbol{A} = \{\boldsymbol{x}, \boldsymbol{TA}(\boldsymbol{x}), \boldsymbol{IPS}(\boldsymbol{x}), \boldsymbol{IN}(\boldsymbol{x}), \boldsymbol{IPO}(\boldsymbol{x}), \boldsymbol{FA}(\boldsymbol{x}) \mid \boldsymbol{x} \in \boldsymbol{X}\} \quad (1)$$

Where each subcomponent represents:

$\boldsymbol{TA}(\boldsymbol{x})$: Complete support

$\boldsymbol{IPS}(\boldsymbol{x})$: Partial support

$\boldsymbol{IN}(\boldsymbol{x})$: Neutrality

$\boldsymbol{IPO}(\boldsymbol{x})$: Partial opposition

$\boldsymbol{FA}(\boldsymbol{x})$: Complete opposition

For each element x, the subcomponents are normalized according to:

$$\boldsymbol{TA}(\boldsymbol{x}) + \boldsymbol{IPS}(\boldsymbol{x}) + \boldsymbol{IN}(\boldsymbol{x}) + \boldsymbol{IPO}(\boldsymbol{x}) + \boldsymbol{FA}(\boldsymbol{x}) = \boldsymbol{1} \quad (2)$$

Implementation Process

The methodology consists of the following steps:

Data Collection: Research papers are retrieved from Semantic Scholar's API [38] using specific search queries related to the research topic. The API call is configured to return title, abstract, and related metadata for up to 15 papers.

Stance Classification: A zero-shot learning classifier[39] based on the BART-large-MNLI model [40] is employed to classify the stance of each paper into five categories matching the neutrosophic subcomponents:

- Support (T)
- Partial Support (IPS)
- Neutral (IN)
- Partial Opposition (IPO)
- Opposition (F)

Neutrosophic Value Assignment: The classifier's confidence scores for each category are normalized to ensure they sum to 1, aligning with the neutrosophic normalization requirement. These normalized scores become the neutrosophic values for each subcomponent.

Analysis and Aggregation: The frequency distribution of stance classifications is calculated and presented as both raw counts and percentages, providing insights into the overall stance distribution across the analyzed papers.

This refined neutrosophic approach allows for a more nuanced analysis of stance in academic literature, capturing degrees of support and opposition rather than forcing binary classifications.

3. Results

This case study examines the interrelationship between the fundamental components of human existence—Body, Mind, Soul, and Spirit—through a neutrosophic perspective. This analysis aims to understand how these elements interact and fluctuate within different worldviews, including the pre-Columbian worldview, the Yoruba worldview, and other philosophical and cultural perspectives. Hypotheses will be formulated to assess these interrelations and their impact on the perception of human existence.

Tabla 1. Neutrosophic Body-Mind-Soul-Spirit Model components

Name	Dimension Body-Mind-Soul	T, I, or F	Description
Physical health	Body	T	Refers to the overall condition and functioning of the body, including fitness and absence of illness.
Mental clarity	Mind	T	The state of having a clear, focused, and sharp mind, enabling effective thinking and decision-making.
Intolerance of uncertainty	Mind	I	Difficulty in accepting or dealing with uncertain situations, often leading to anxiety or stress.
Emotional well-being	Soul	T	A state of positive mental health where one feels balanced, content, and able to manage emotions effectively.
Spirituality	Spirit	T	The sense of connection to something greater than oneself, often involving beliefs,

			values, and practices.

Different hypotheses are analyzed to examine the Pre-Columbian worldview, the Yoruba worldview, and other perspectives using the proposed stance detection model. The following section presents these interrelations and analyzes them through the proposed stance detection framework.

Body and Mind

Table 2. Body and Mind Relationships

Hipotesis	**T**	**IPS**	**IN**	**IPO**	**F**
Does physical health influence mental clarity?	**0.1**	**0.6**	**0**	**0.3**	**0**

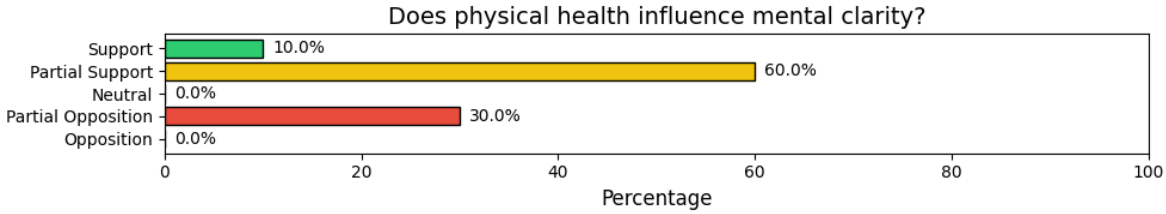

Figure 3. Does physical health influence mental clarity?

From the analysis of papers studying the hypothesis "Does physical health influence mental clarity?", it was found that 60% of studies partially support this relationship, 30% partially oppose it, while only 10% fully support it, suggesting a general trend supporting the connection between physical health and mental clarity, although with some level of controversy in the scientific literature.

Pre-Colombian Worldview

In pre-Colombian cultures, such as those of the Maya and Aztecs, there was a holistic understanding of health that integrated the body, mind, and spirit. Physical health was seen as essential for maintaining mental clarity and spiritual balance. These cultures practiced rituals and used medicinal herbs to ensure the well-being of the body, which in turn was believed to support mental and spiritual health. The interconnectedness of physical and mental states was a fundamental aspect of their worldview, emphasizing that a healthy body contributed to a clear and focused mind.

Yoruba Worldview

The Yoruba worldview also emphasizes the interconnection between physical health and mental clarity. In Yoruba culture, health is viewed as a balance between internal and external forces, and maintaining this balance is crucial for overall well-being. The Yoruba believe that physical health is influenced by spiritual forces and that a healthy body is necessary for mental clarity and spiritual harmony. Practices such as rituals, music, and dance are used to maintain this balance, highlighting the importance of physical well-being in achieving mental clarity. The concept of "Aché," or life force, is central to this belief, indicating that physical health is vital for sustaining mental and spiritual clarity.

Modern Perspective

Modern evidence supports the idea that physical health significantly influences mental clarity. Regular exercise and healthy lifestyle choices enhance cognitive function, improve mood, and reduce symptoms of mental health disorders. This underscores the importance of integrating physical activity into daily routines to support both physical and mental well-being. The consensus is that maintaining good physical health positively impacts mental clarity, aligning with both pre-Colombian and Yoruba perspectives on the interconnectedness of body and mind.

Mind-Soul

Table 2. Mind and Soul Relationships

Hipotesis	**T**	**IPS**	**IN**	**IPO**	**F**
Does intolerance of uncertainty influence emotional well-being?	**0**	**0.846**	**0**	**0.154**	**0**

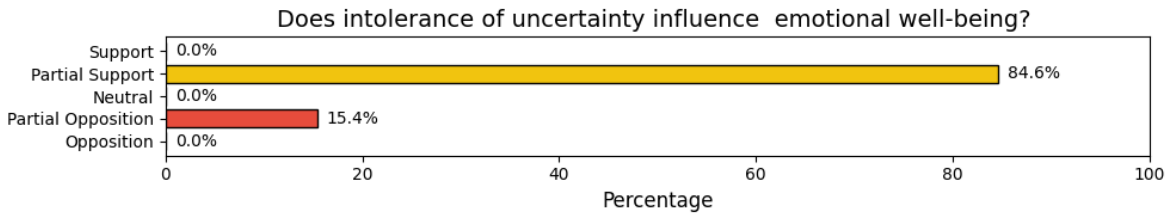

Figure 4. Does intolerance of uncertainty influence emotional well-being?

The results show that 84.6% of studies partially support this relationship, while 15.4% partially oppose it, with no studies either fully supporting, remaining neutral, or fully opposing. This strong majority of partial support suggests a significant consensus in the scientific literature about the influence of uncertainty intolerance on emotional well-being, though some debate remains.

Pre-Columbian Cosmology

In pre-Columbian cosmology, particularly among the indigenous peoples of El Salvador, such as the Pipiles and Lencas (Figure 5), the concept of balance and harmony with nature and the divine was central. Intolerance of uncertainty might have been seen as a disruption to this balance, as these cultures often relied on rituals and divination to understand and predict future events. For example, the Pipiles practiced rituals to honor deities like Tlaloc, the god of rain and agriculture, to ensure favorable conditions and reduce uncertainty in their agricultural practices. This suggests that a high intolerance of uncertainty could negatively impact emotional well-being by creating anxiety and disharmony.

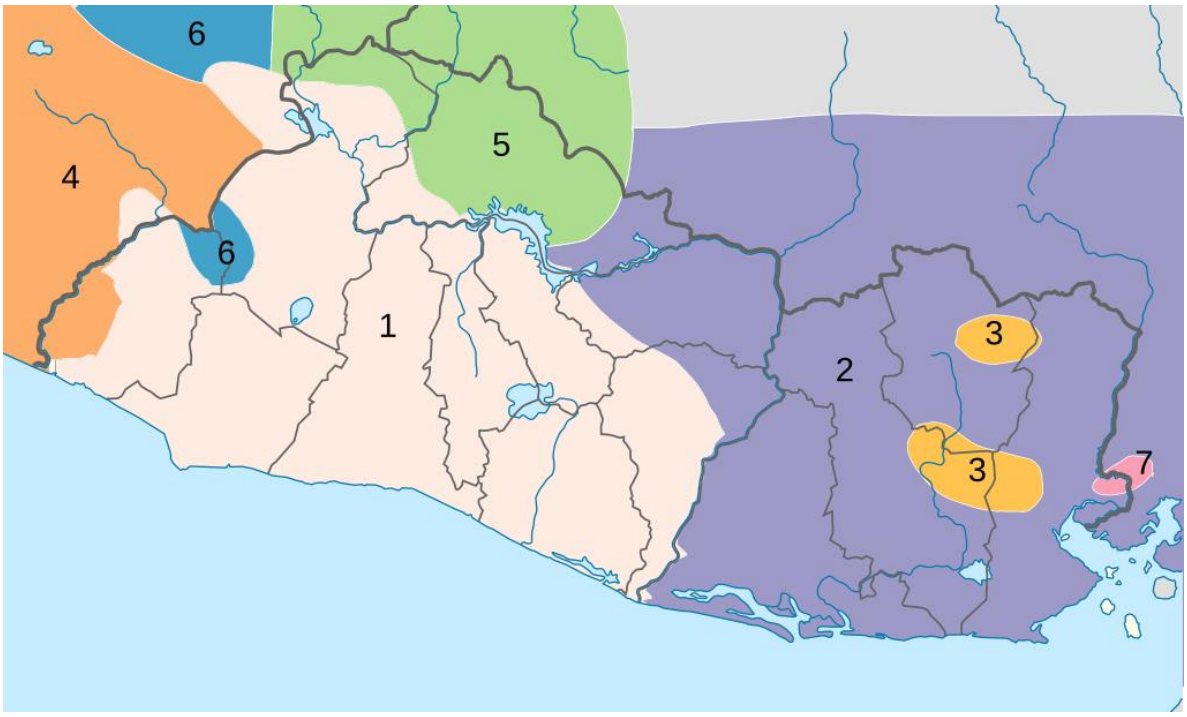

Figure 5. Map of El Salvador's Indigenous Peoples at the time of the Spanish conquest: 1. Pipil (Nahua), 2. Lenca, 3. Kakawira o Cacaopera, 4. Xinca, 5. Maya Ch'orti' people, 6. Maya Poqomam people, 7. Mangue o Chorotega [41].

Yoruba Worldview

The Yoruba worldview also emphasizes balance and harmony between known and unknown forces. Intolerance of uncertainty could be seen as a challenge to emotional well-being, as it disrupts the balance necessary for spiritual and emotional health. The Yoruba use divination, particularly through the Ifá oracle, to gain insights and guidance, which helps reduce uncertainty and promote emotional stability. This suggests that managing uncertainty through spiritual practices is crucial for maintaining emotional well-being in the Yoruba tradition.

Modern Perspective

From a modern perspective, intolerance of uncertainty is generally associated with negative impacts on emotional well-being. It can lead to increased anxiety, stress, and difficulty coping with life's unpredictability. People who struggle with uncertainty may experience heightened emotional distress, which can affect their overall mental health. Therefore, both pre-Columbian and Yoruba perspectives align with the modern understanding that managing uncertainty is important for emotional well-being.

Spirit and Soul Relationships

Table 3. Spirit and Soul Relationships

Hipotesis	**T**	**IPS**	**IN**	**IPO**	**F**
Does spirituality enhance emotional well-being?	**0.77**	**0.769**	**0**	**0.154**	**0**

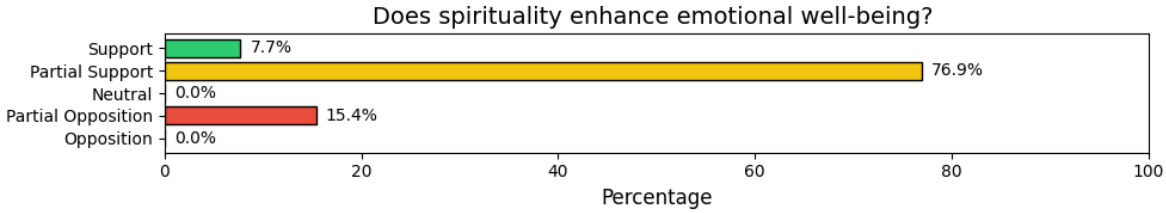

Figure 6. Does spirituality enhance emotional well-being?

The data shows that 76.9% of studies partially support this relationship, 15.4% partially oppose it, and 7.7% fully support it. There were no studies that remained neutral or fully opposed (0%). This distribution suggests a strong consensus in the scientific literature favoring a positive relationship between spirituality and emotional well-being, with the majority taking a moderately supportive stance while acknowledging potential complexities in the relationship.

Pre-Columbian Cosmology

In pre-Columbian cosmology, spirituality was deeply intertwined with daily life and was considered essential for maintaining emotional well-being. The indigenous peoples of regions like Mesoamerica, including the Pipiles and Lencas of El Salvador, practiced rituals and ceremonies to connect with the divine and the natural world. These spiritual practices were believed to bring balance and harmony, which were crucial for emotional health. The emphasis on community rituals and the veneration of deities and ancestors helped individuals feel connected to a larger cosmic order, providing a sense of purpose and emotional stability.

Yoruba Worldview

The Yoruba worldview similarly emphasizes the importance of spirituality for emotional well-being. Yoruba spirituality is centered around the belief in a complex interplay between the physical and spiritual realms, with deities known as Orishas playing a significant role in guiding and protecting individuals. Engaging in spiritual practices, such as rituals, divination, and offerings to the Orishas, is believed to enhance emotional well-being by fostering a sense of connection to the divine and the community. The concept of aché, or the life force, is central to Yoruba spirituality, signifying the energy that flows through all things and supports emotional and spiritual health.

Modern Perspective

From a modern perspective, spirituality is often associated with improved emotional well-being. Engaging in spiritual practices can provide individuals with a sense of purpose, community, and connection to something greater than themselves. This can lead to increased resilience, reduced stress, and a greater sense of peace and contentment. Both pre-Columbian and Yoruba perspectives align with this understanding, highlighting the role of spirituality in fostering emotional health and balance.

5. Conclusions

This study explored the application of neutrosophic logic to the interconnected dynamics of body, mind, soul, and spirit, emphasizing the fluidity of these dimensions rather than treating them as rigid, separate entities. The proposed Body-Mind-Soul-Spirit Fluidity model extends traditional dualistic frameworks, allowing for a non-binary, dynamic understanding of human existence where truth, falsehood, and indeterminacy coexist in varying degrees. By integrating n-valued refined neutrosophic logic, the model enhances the representation of uncertainty, contradiction, and partial truths in different domains, particularly in stance detection. The study demonstrated how refined neutrosophic components (T, I, F) can be effectively used to classify and quantify complex stances, addressing real-world ambiguity where opinions or beliefs often overlap rather than conforming to absolute categories.

The parallels drawn between neutrosophy and Latin American cosmologies, including Yoruba, Mayan, and Amazonian worldviews, further support the relevance of this model in understanding

identity, free will, and moral reasoning. These indigenous perspectives align with neutrosophy in rejecting rigid dichotomies, instead emphasizing interconnectedness, transformation, and fluidity in human experience. From an applied perspective, the refined neutrosophic stance detection model, implemented through zero-shot classification with Semantic Scholar data, revealed promising results. The multi-valued classification system enabled a more granular and accurate representation of support, opposition, neutrality, and partial beliefs, surpassing the limitations of traditional binary or ternary classification schemes. The results indicated that the proposed approach could improve stance detection in academic literature, offering a more nuanced assessment of scientific consensus. Moreover, this study suggests that adopting neutrosophic logic in AI, ethics, and cognitive science could foster systems that better handle uncertainty, adaptive reasoning, and evolving identity structures. By moving beyond deterministic and binary paradigms, future research can leverage this framework in areas such as algorithmic ethics, digital identity, human cognition modeling, and artificial consciousness.

. This study lays the groundwork for further exploration into how non-binary reasoning can reshape scientific inquiry, ethical frameworks, and intelligent systems.Additional future research should explore how neutrosophic logic intersects with indigenous pedagogical approaches to develop more inclusive educational frameworks that honor diverse ways of knowing. Additionally, studies should investigate how combining neutrosophic principles with experiential learning methodologies could create more effective and culturally responsive teaching practices that bridge traditional and contemporary educational paradigms.

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Quadripartitioned Bipolar Neutrosophic Competition Graph with Novel Application

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Abstract. This paper introduces the concept of quadripartitioned neutrosophic bipolar competition graphs. The properties of operations like Cartesian and direct products are investigated within these graphs. Furthermore, this work presents an algorithm that applies the competition model to quadripartitioned bipolar neutrosophic competition graphs for depicting applicant competition. Finally, we compare the results with existing literature, emphasizing the novelty and contributions of the proposed method.

Keywords: Quadripartitioned Bipolar Neutrosophic; Competition Graph; Uncertainty.

1. Introduction

Cohen [1] developed the concept of competition graphs in an ecological context to depict the link between species in the food chain. Competition graphs are frequently utilised in a variety of applications, including channel assignment, complex economic modelling, phytogenetic tree reconstruction, coding, and energy systems. Much work has been done on competition graphs and their variations since the initial inspiration of ecological application of species competition. Each of these representations necessitates the explicit definition of vertices and edges within the networks, a methodology deemed inadequate for fully encapsulating all competitions within real-world challenges. The study of intuitionistic fuzzy graphs involves developing algorithms, methodologies, and techniques to analyze and manipulate these structures effectively in [30, 31]. Researchers explore properties, operations, and algorithms specific to intuitionistic fuzzy graphs to extend the reach of multi polar graph theory into the realm of uncertainty and ambiguity in [29].

The single-valued neutrosophic set, an expansion of intuitionistic fuzzy sets, presents a rapid resolution for real-world challenges, notably in decision support [3], as investigated in [4]. Few authors choose to frame the behavior of indeterminacy in the same manner that truth-membership is modelled, while others infer to frame it in the similar fashion with falsity-membership. Authors in [3] introduced the notion of a single-valued neutrosophic set and described its features. Now it is applied in many domains such as sensors [5], decision making [6], image processing [7], control theory [8], medical diagnosis [9] and graph theory [10–12]. The following Figure 1 represents the distinction of the quadripartioned bipolar neutrosophic competition graphs with previously introduced graphs.

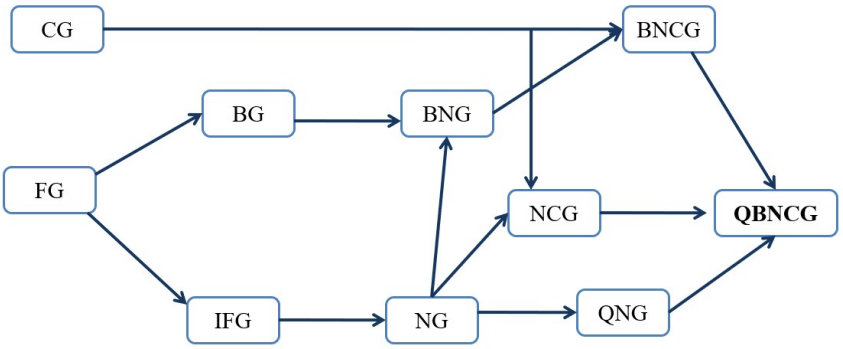


FIGURE 1. CG-Competition Graphs, FG-Fuzzy Graphs, BG-Bipolar Graphs, IFG-Intuitionistic Fuzzy Graphs, NG-Neutrosophic Graphs, BNG-Bipolar Neutrosophic Graphs, NCG-Neutrosophic Competition Graphs, BNCG-Bipolar Neutrosophic Competition Graphs, QNG-Quadripartitioned Neutrosophic Graphs, QBNCG-Quadripartitioned Neutrosophic Competition Graphs

Nowadays, advanced level of neutrosophic sets and graphs received much attention in the application viewpoint. The notion of bipolar neutrosophic competition graphs is insufficient in real-world applications involving competitions and relationships with opposing side facts. This approach pushes us to use bipolar neutrosophic quadripartitioned sets [15] to model bipolar neutrosophic quadripartitioned competition graphs. When we separate the indeterminacy section of the neutrosophic set into two sections, we get four parts: 'Contradiction' (both true and false) and 'Unknown' (neither true nor false), that is $\mathcal{T}, \mathcal{C}, \mathcal{U}$ and $\mathcal{F}$ which defines a new set called 'quadripartitioned single valued neutrosophic set', introduced by Chatterjee., et al. [16]. This study is fundamentally grounded in "Belnap's four-valued logic" [17] and Smarandache's "Four Numerical valued neutrosophic logic" [18]. It introduces quadripartitioned neutrosophic competition graphs by incorporating the concepts of Quadripartitioned Neutrosophic Set (QNS) and competition graphs. In [10], researchers established bipolar

quadripartitioned single-valued neutrosophic graphs. Recently, there has been significant attention given to the quadripartitioned neutrosophic domain, with authors further developing concepts related to graph properties [10–12].

Some operations, regular and irregular neutrosophic graphs and significant real-life applications are discussed in [13]. A lot of works have been done on competition graphs in various domains. In that one can assume that the nodes and arcs of the graphs are defined perfectly, but in reality, one can observe that sometimes the vertices and edges of graphs are not defined precisely. Algorithms are constructed in this domain to model the real-life application [14]. For instance, in predator-prey model, specious and prey are assumed as fuzzy sets and the preys are demonstrated as a fuzzy graphs [1]. Certain bipolar neutrosophic competition graphs are studied in [19] with characterizations of bipolar neutrosophic out-neighborhoods, bipolar neutrosophic in-neighborhoods, bipolar neutrosophic open neighborhood graphs, bipolar neutrosophic closed neighborhood graphs, bipolar neutrosophic p-competition graphs. The quadripartitioned neutrosophic soft graphs for treating neutrosophic soft information by employing the theory of quadripartitioned neutrosophic soft sets with graphs. Operations like strong product, Cartesian product, cross product and lexicographic product, of quadripartitioned neutrosophic soft graphs are established in [11]. In [2], the matrix representation and generalised neutrosophic competition graph are examined. Additionally, with practical applications, the minimal graph and competition number corresponding to the generalised neutrosophic competition graph are defined. Based on the above motivations, to deal with all the competitions of world, quadripartitioned bipolar neutrosophic competition graphs are introduced in this paper. Predicting a suitable job by the applicant and selecting a suitable applicant is the common issue. Based on this motivation, we established the job competition model in quadripartitioned bipolar neutrosophic competition graphs to represent competition between the applicants along with algorithm.

In recent years, neutrosophic theories have been widely explored for their ability to address uncertainty and imprecision in real-world problems. Chakraborty et al. (2020) introduced the concept of cylindrical neutrosophic single-valued numbers, applying them to networking problems, multi-criterion group decision-making, and graph theory, showcasing their versatility in complex systems [20]. Similarly, Chakraborty, Mondal, and Broumi (2019) developed a deneutrosophication technique for pentagonal neutrosophic numbers, demonstrating its utility in solving minimal spanning tree problems [21]. These advancements underline the growing relevance of neutrosophic models in various domains, motivating further exploration of their applications. Building on this foundation, we propose the Quadripartitioned Bipolar Neutrosophic Competition Graph (QBNCG), which enhances existing models by dividing the indeterminacy component into contradiction and ignorance, thereby enabling more precise

modeling of competitive scenarios such as job applicant competition. More concepts related to our study in different fields have been studied in [22–26]

To the best of the authors' knowledge, no research has been done on the notions of quadripartioned neutrosophic bipolar competition graphs with the application, as a result of the above-mentioned works.

The significant contributions in this article are provided as follows:

(1) Discussion is made regarding quadripartitioned bipolar neutrosophic competition graphs. The application in quadripartitioned bipolar neutrosophic field is covered for the first time in this work of literature.
(2) The operations like a Cartesian product and direct product of quadripartitioned neutrosophic bipolar competition graphs with their properties are discussed.
(3) The proposed work is illustrated with an application of quadripartitioned neutrosophic bipolar competition graphs in selection of designation for applicant.

The main advantages for the proposed results are listed as follows:

(1) In many domains, including algebra, geometry, topology, and optimization, the stated results in the quadripartitioned neutrosophic field are utilised as a significant tool for conceiving and solving combinatorial and decision-making problems.
(2) The proposed graph is the generalization of neutrosophic competition graphs and it will generalize the existing works in the literature (see [19, 27]).
(3) In this paper, modelling of job competition problem is discussed. The positive membership function denotes the applicant's eligibility and the negative membership function represents the ineligible percentage of applicants.
(4) Accurate finding of suitable jobs by the applicants can be done by employing the quadripartitioned bipolar neutrosophic competition graphs.
(5) One can apply the proposed work in various quadripartitioned neutrosophic fields.

The following describes the article: Preliminaries Section 2 provides a few necessary definitions. In Section 3, the operations and quadripartioned neutrosophic bipolar competition graphs are introduced. In Section 4, applications are described for the developed results.

2. **Preliminaries**

Definition 2.1. [4] A neutrosophic set Ψ in Λ is defined by three functions: the truth-membership function (TMF) $\alpha_\Psi(\varrho)$, the indeterminacy-membership function (IMF) $\omega_\Psi(\varrho)$, and the falsity-membership function (FMF) $\delta_\Psi(\varrho)$.

For every point ϱ in Λ, $\alpha_\Psi(\varrho)$, $\omega_\Psi(\varrho)$, and $\delta_\Psi(\varrho)$ fall within the range of [0,1].

$\alpha_\Psi(\varrho), \omega_\Psi(\varrho), \delta_\Psi(\varrho) \in [0, 1]$. Also

$$\Psi = \{\varrho, \alpha_\Psi(\varrho), \omega_\Psi(\varrho), \delta_\Psi(\varrho)\} \text{ and } 0 \leq \alpha_\Psi(\varrho) + \omega_\Psi(\varrho) + \delta_\Psi(\varrho) \leq 3.$$

Definition 2.2. [4] A neutrosophic graph is represent as a pair $\mathbb{G}^* = (\mathbb{Y}, \mathbb{D})$ where
(i) $\mathbb{Y} = \{y_1, y_2, .., y_n\}$ such that $\alpha_\Psi : \mathbb{Y} \to [0,1]$, $\omega_\Psi : \mathbb{Y} \to [0,1]$ and $\delta_\Psi : \mathbb{Y} \to [0,1]$ denote the degrees of TMF, IMF and FMF

$$0 \leq \alpha_\Psi(y) + \omega_\Psi(y) + \delta_\Psi(y) \leq 3, \ \forall y \in \mathbb{Y}.$$

(ii) $\mathbb{D} \subseteq \mathbb{Y} \times \mathbb{Y}$ where $\alpha_\mathbb{D} : \mathbb{D} \to [0,1]$, $\omega_\mathbb{D} : \mathbb{D} \to [0,1]$ and $\delta_\mathbb{D} : \mathbb{D} \to [0,1]$ are such that

$$\begin{gathered}\alpha_\Upsilon(pq) \leq \alpha_\Psi(p) \wedge \alpha_\Psi(q),\\ \omega_\Upsilon(pq) \leq \omega_\Psi(p) \wedge \omega_\Psi(q),\\ \delta_\Upsilon(pq) \leq \delta_\Psi(p) \vee \delta_\Psi(q),\\ \text{and } 0 \leq \alpha_\Upsilon(pq) + \omega_\Upsilon(pq) + \delta_\Upsilon(pq) \leq 3, \ \forall pq \in \mathbb{D}.\end{gathered}$$

See article [16] for further information on the following definitions and outcomes.

Definition 2.3. [16] Let Λ be the universal set. A Quadripartitioned Neutrosophic Set (QNS) Ψ on Λ is represented as

$$\Psi = \{\langle p, \alpha_\Psi(p), \beta_\Psi(p), \gamma_\Psi(p), \delta_\Psi(p) : p \in \Lambda\rangle\}$$

where $\alpha_\Psi, \beta_\Psi, \gamma_\Psi, \delta_\Psi : \varrho \to [0,1]$ and $0 \leq \alpha_\Psi(p) + \beta_\Psi(p) + \gamma_\Psi(p) + \delta_\Psi(p) \leq 4$. Here, $\alpha_\Psi(p)$-Truth Membership Function, $\beta_\Psi(p)$-Contradiction Membership Function, $\gamma_\Psi(p)$-Ignorance Membership Function, $\delta_\Psi(p)$-False Membership Function.

Definition 2.4. [15] Let Λ be the universal set. A Quadripartitioned Bipolar Neutrosophic Set (QBNS) Ψ on Λ is defined as

$$\Psi = \{\langle p, \alpha^P_\Psi(p), \beta^P_\Psi(p), \gamma^P_\Psi(p), \delta^P_\Psi(p), \alpha^N_\Psi(p), \beta^N_\Psi(p), \gamma^N_\Psi(p), \delta^N_\Psi(p) : p \in \Lambda\rangle\}$$

where $\alpha^P_\Psi, \beta^P_\Psi, \gamma^P_\Psi, \delta^P_\Psi : \Lambda \to [0,1]$, $\alpha^N_\Psi, \beta^N_\Psi, \gamma^N_\Psi, \delta^N_\Psi : \Lambda \to [-1,0]$ and $0 \leq \alpha^P_\Psi(p) + \beta^P_\Psi(p) + \gamma^P_\Psi(p) + \delta^P_\Psi(p) \leq 4$ and $-4 \leq \alpha^N_\Psi(p) + \beta^N_\Psi(p) + \gamma^N_\Psi(p) + \delta^N_\Psi(p) \leq 0$.

Definition 2.5. [10] A QBSVNG of a crisp graph $\mathbb{G} = (\mathbb{Y}, \mathbb{D})$ is said to be a pair $\mathtt{G} = (\mathcal{C}, \mathcal{H})$ with $\mathcal{C} = (\mathcal{C}^P, \mathcal{C}^N)$ and $\mathcal{H} = (\mathcal{H}^P, \mathcal{H}^N)$, where
(i) the functions $(\alpha_\mathcal{C})^P, (\beta_\mathcal{C})^P, (\gamma_\mathcal{C})^P, (\delta_\mathcal{C})^P : \mathbb{Y} \to [0,1]$ and $(\alpha_\mathcal{C})^N, (\beta_\mathcal{C})^N, (\gamma_\mathcal{C})^N, (\delta_\mathcal{C})^N : \mathbb{Y} \to [-1,0]$ represent the degree of TMF, CMF, IMF and FMF of the element $p \in \mathbb{Y}$, respectively, there is no restriction on the sum $0 \leq (\alpha_\mathcal{C})^P(p) + (\beta_\mathcal{C})^P(p) + (\gamma_\mathcal{C})^P(p) + (\delta_\mathcal{C})^P(p) \leq 4$, $-4 \leq (\alpha_\mathcal{C})^N(p) + (\beta_\mathcal{C})^N(p) + (\gamma_\mathcal{C})^N(p) + (\delta_\mathcal{C})^N(p) \leq 0$ for each $p \in \mathbb{Y}$,
(ii) the functions $(\alpha_\mathcal{H})^P, (\beta_\mathcal{H})^P, (\gamma_\mathcal{H})^P, (\delta_\mathcal{H})^P : \mathbb{D} \subseteq \mathbb{Y} \times \mathbb{Y} \to [0,1]$ and

$(\alpha_{\mathcal{H}})^N, (\beta_{\mathcal{H}})^N, (\gamma_{\mathcal{H}})^N, (\delta_{\mathcal{H}})^N : \mathbb{D} \subseteq \mathbb{Y} \times \mathbb{Y} \to [-1, 0]$ are defined as:

$$
\begin{aligned}
(\alpha_{\mathcal{H}})^P(pq) &\leq \{(\alpha_{\mathcal{C}})^P(p) \wedge (\alpha_{\mathcal{C}})^P(q)\} \\
(\beta_{\mathcal{H}})^P(pq) &\leq \{(\beta_{\mathcal{C}})^P(p) \wedge (\beta_{\mathcal{C}})^P(q)\} \\
(\gamma_{\mathcal{H}})^P(pq) &\leq \{(\gamma_{\mathcal{C}})^P(p) \vee (\gamma_{\mathcal{C}})^P(q)\} \\
(\delta_{\mathcal{H}})^P(pq) &\leq \{(\delta_{\mathcal{C}})^P(p) \vee (\delta_{\mathcal{C}})^P(q)\}, \quad \forall \quad pq \in \mathbb{D},
\end{aligned}
$$

$$
\begin{aligned}
(\alpha_{\mathcal{H}})^N(pq) &\geq \{(\alpha_{\mathcal{C}})^N(p) \vee (\alpha_{\mathcal{C}})^N(q)\} \\
(\beta_{\mathcal{H}})^N(pq) &\geq \{(\beta_{\mathcal{C}})^N(p) \vee (\beta_{\mathcal{C}})^N(q)\} \\
(\gamma_{\mathcal{H}})^N(pq) &\geq \{(\gamma_{\mathcal{C}})^N(p) \wedge (\gamma_{\mathcal{C}})^N(q)\} \\
(\delta_{\mathcal{H}})^N(pq) &\geq \{(\delta_{\mathcal{C}})^N(p) \wedge (\delta_{\mathcal{C}})^N(q)\}, \quad \forall \quad pq \in \mathbb{D}.
\end{aligned}
$$

Here $\mathcal{C}$ is the QBN vertex set of $\mathtt{G}$ and $\mathcal{H}$ is the QBN edge set of $\mathtt{G}$.

3. Quadripartitioned Bipolar Neutrosophic Competition Graphs (QBNCG)

Definition 3.1. A QBND on a non-empty set ϱ is a pair $\mathtt{G} = (\Psi, \overrightarrow{\Upsilon})$, where Ψ is a QBNS on ϱ and Υ is a QBN relation on Λ, such that

$$
\begin{aligned}
&\alpha^p_{\Upsilon}(\overrightarrow{p,q}) \leq \alpha^p_{\Psi}(p) \wedge \alpha^p_{\Psi}(q), \quad \beta^p_{\Upsilon}(\overrightarrow{p,q}) \leq \beta^p_{\Psi}(p) \wedge \beta^p_{\Psi}(q), \\
&\gamma^p_{\Upsilon}(\overrightarrow{p,q}) \leq \gamma^p_{\Psi}(p) \vee \gamma^p_{\Psi}(q), \quad \delta^p_{\Upsilon}(\overrightarrow{p,q}) \leq \delta^p_{\Psi}(p) \vee \delta^p_{\Psi}(q), \\
&\alpha^n_{\Upsilon}(\overrightarrow{p,q}) \geq \alpha^n_{\Psi}(p) \vee \alpha^n_{\Psi}(q), \quad \beta^n_{\Upsilon}(\overrightarrow{p,q}) \geq \beta^n_{\Psi}(p) \vee \beta^n_{\Psi}(q), \\
&\gamma^n_{\Upsilon}(\overrightarrow{p,q}) \geq \gamma^n_{\Psi}(p) \wedge \gamma^n_{\Psi}(q), \quad \delta^n_{\Upsilon}(\overrightarrow{p,q}) \geq \delta^n_{\Psi}(p) \wedge \delta^n_{\Psi}(q), \forall \quad p, q \in \Lambda
\end{aligned}
$$

Example 3.2. Consider QBND $\mathtt{G} = (\Psi, \overrightarrow{\Upsilon})$ on $\Lambda = (p, q, r)$ as shown in figure 2.

Definition 3.3. Let $\overrightarrow{\mathtt{G}}$ be a QBND the QBN out neighbourhood of a vertex q is a QBNS $\mathtt{Q}^+(p) = (\Lambda_a^+, \alpha_p^{(p)^+}, \beta_p^{(p)^+}, \gamma_p^{(p)^+}, \delta_p^{(p)^+}, \alpha_p^{(n)^+}, \beta_p^{(n)^+}, \gamma_p^{(n)^+}, \delta_p^{(n)^+})$, where
$\Lambda_p^+ = \{q | \Upsilon_1^p\overrightarrow{(p,q)} > 0, \Upsilon_2^p\overrightarrow{(p,q)} > 0, \Upsilon_3^p\overrightarrow{(p,q)} > 0, \Upsilon_4^p\overrightarrow{(p,q)} > 0, \Upsilon_1^n\overrightarrow{(p,q)} < 0, \Upsilon_2^n\overrightarrow{(p,q)} < 0, \Upsilon_3^n\overrightarrow{(p,q)} < 0, \Upsilon_4^n\overrightarrow{(p,q)} < 0, \}$
such that $\alpha_p^{(p)^+} : \Lambda_p^+ \to [0, 1]$ represent by $\alpha_p^{(p)^+}(q) = \Upsilon_1^p\overrightarrow{(p,q)}$, $\beta_p^{(p)^+} : \Lambda_p^+ \to [0, 1]$ represent by $\beta_p^{(p)^+}(q) = \Upsilon_2^p\overrightarrow{(p,q)}$, $\gamma_p^{(p)^+} : \varrho_p^+ \to [0, 1]$ represent by $\gamma_p^{(p)^+}(q) = \Upsilon_3^p\overrightarrow{(p,q)}$, $\delta_p^{(p)^+} : \Lambda_p^+ \to [0, 1]$ represent by $\delta_p^{(p)^+}(q) = \Upsilon_4^p\overrightarrow{(p,q)}$, $\alpha_p^{(n)^+} : \Lambda_p^+ \to [-1, 0]$ represent by $\alpha_p^{(n)^+}(q) = \Upsilon_1^n\overrightarrow{(p,q)}$, $\beta_p^{(n)^+} : \Lambda_p^+ \to [-1, 0]$ represent by $\beta_p^{(n)^+}(q) = \Upsilon_2^n\overrightarrow{(p,q)}$, $\gamma_p^{(n)^+} : \Lambda_p^+ \to [-1, 0]$ represent by $\gamma_p^{(n)^+}(q) = \Upsilon_3^n\overrightarrow{(p,q)}$, $\delta_p^{(n)^+} : \Lambda_p^+ \to [-1, 0]$ represent by $\delta_p^{(n)^+}(q) = \Upsilon_4^n\overrightarrow{(p,q)}$,

Definition 3.4. Let $\overrightarrow{\mathtt{G}}$ be a QBND the QBN in-neighbourhood of a vertex p is a QBNS $\mathtt{Q}^-(p) = (\Lambda_q^-, \alpha_p^{(p)^-}, \beta_p^{(p)^-}, \gamma_p^{(p)^-}, \delta_p^{(p)^-}, \alpha_p^{(n)^-}, \beta_p^{(n)^-}, \gamma_p^{(n)^-}, \delta_p^{(n)^-})$, where

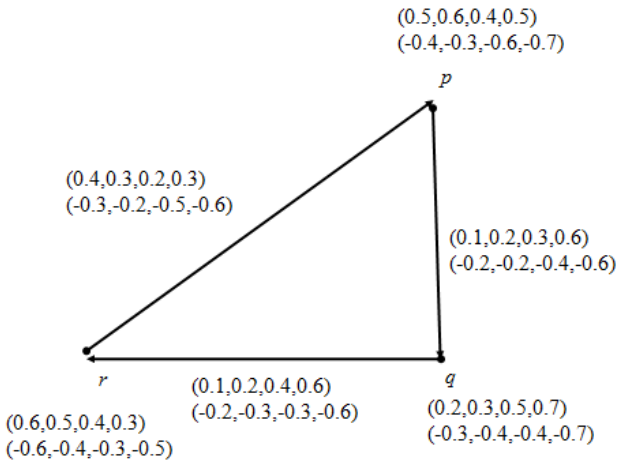


FIGURE 2. Quadripartitioned Bipolar Neutrosophic Digraph (QBND)

$\Lambda_p^- = \{p|\Upsilon_1^p(\overrightarrow{p,q}) > 0, \Upsilon_2^p(\overrightarrow{p,q}) > 0, \Upsilon_3^p(\overrightarrow{p,q}) > 0, \Upsilon_4^p(\overrightarrow{p,q}) > 0, \Upsilon_1^n(\overrightarrow{p,q}) < 0, \Upsilon_2^n(\overrightarrow{p,q}) < 0, \Upsilon_3^n(\overrightarrow{p,q}) < 0, \Upsilon_4^n(\overrightarrow{p,q}) < 0, \}$
such that $\alpha_p^{(p)^-} : \Lambda_p^- \to [0,1]$ represent by $\alpha_p^{(p)^-}(q) = \Upsilon_1^p(\overrightarrow{p,q})$, $\beta_p^{(p)^-} : \Lambda_q^- \to [0,1]$ represent by $\beta_p^{(p)^-}(q) = \Upsilon_2^p(\overrightarrow{p,q})$, $\gamma_p^{(p)^-} : \Lambda_p^- \to [0,1]$ represent by $\gamma_p^{(p)^-}(q) = \Upsilon_3^p(\overrightarrow{p,q})$, $\delta_p^{(p)^-} : \Lambda_p^- \to [0,1]$ represent by $\delta_p^{(p)^-}(q) = \Upsilon_4^p(\overrightarrow{p,q})$, $\alpha_p^{(n)^-} : \Lambda_p^- \to [-1,0]$ represent by $\alpha_p^{(n)^-}(q) = \Upsilon_1^n(\overrightarrow{p,q})$, $\beta_p^{(n)^-} : \Lambda_p^- \to [-1,0]$ represent by $\beta_p^{(n)^-}(q) = \Upsilon_2^n(\overrightarrow{p,q})$, $\gamma_p^{(n)^-} : \Lambda_q^- \to [-1,0]$ represent by $\gamma_p^{(n)^-}(q) = \Upsilon_3^n(\overrightarrow{p,q})$, $\delta_p^{(n)^-} : \Lambda_p^- \to [-1,0]$ represent by $\delta_p^{(n)^-}(q) = \Upsilon_4^n(\overrightarrow{p,q})$,

Example 3.5. Let an QBND $\mathtt{G} = (\Psi, \overrightarrow{\Upsilon})$ on $\Lambda = (p,q,r)$ as explained in figure 2. Tables 1 and 2 show the QBN in and out neighbourhoods, respectively.

QBN Out-neighbourhoods

Table 1.

ϱ	$\mathtt{Q}^+(\varrho)$
p	$\{q(0.1,0.2,0.3,0.6,-0.2,-0.2,-0.4,-0.6), r(0.4,0.3,0.2,0.3,-0.3,-0.2,-0.5,-0.6)\}$
q	$\emptyset$
r	$\{q(0.1,0.2,0.4,0.6,-0.2,-0.3,-0.3,-0.6)\}$

Table 2. QBN in-neighbourhoods

ϱ	$\mathtt{Q}^-(\varrho)$
p	$\emptyset$
q	$\{p(0.1,0.2,0.3,0.6,-0.2,-0.2,-0.4,-0.6), r(0.1,0.2,0.4,0.6,-0.2,-0.3,-0.3,-0.6)\}$
r	$\{p(0.4,0.3,0.2,0.3,-0.3,-0.2,-0.5,-0.6)\}$

Definition 3.6. The height of a QBNS $h(\mathtt{Q}) = (\Lambda, \alpha^p, \beta^p, \gamma^p, \delta^p, \alpha^n, \beta^n, \gamma^n, \delta^n,)$ and

$$h(\mathtt{Q}) = (h_1(\mathtt{Q}), h_2(\mathtt{Q}), h_3(\mathtt{Q}), h_4(\mathtt{Q}), h_5(\mathtt{Q}), h_6(\mathtt{Q}), h_7(\mathtt{Q}), h_8(\mathtt{Q}))$$

$=(\sup_{r\in\mathbb{X}} \alpha^p_{\mathtt{Q}(r)}, \sup_{r\in\mathbb{X}} \beta^p_{\mathtt{Q}(r)}, \inf_{r\in\mathbb{X}} \gamma^p_{\mathtt{Q}(r)}, \inf_{r\in\mathbb{X}} \delta^p_{\mathtt{Q}(r)}, \sup_{r\in\mathbb{X}} \alpha^p_{\mathtt{Q}(r)}, \sup_{r\in\mathbb{X}} \beta^p_{\mathtt{Q}(r)} \inf_{r\in\mathbb{X}} \gamma^p_{\mathtt{Q}(r)}, \inf_{r\in\mathbb{X}} \delta^p_{\mathtt{Q}(r)})$.

Example 3.7. The height of a QBNS

$\Psi = \{(p, (0.1, 0.2, 0.3, 0.6, -0.2, -0.2, -0.4, -0.6)), (q, (0.4, 0.3, 0.4, 0.7, -0.1, -0.2, -0.1, -0.6)),$

$(r, (0.2, 0.2, 0.4, 0.5, -0.2, -0.3, -0.3, -0.6))\}$ in $\varrho = \{p, q, r\}$ is $h(\Psi) = \{(0.4, 0.3, 0.3, 0.5, 0.4, 0.3, 0.3, 0.5)\}$

Definition 3.8. A QBNCG of a QBNG $\overrightarrow{\mathtt{G}} = (\Psi, \Upsilon)$ is an undirected graph QBNG $\beta(\overrightarrow{\mathtt{G}}) = (\Psi, \mathtt{Q})$ which has the same vertex set as in $\overrightarrow{\mathtt{G}}$ and there is an edge between two vertices p and q if and only if $\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)$ is non-empty. The positive TMF, CMF, IMF, FMF and negative TMF, CMF, IMF, FMF value of the edge (p, q) are defined as,

$$\alpha^p_{\mathtt{Q}}(p,q) = (\alpha^p_\Psi(p) \wedge \alpha^p_\Psi(q))h_1(\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)), \quad \beta^p_{\mathtt{Q}}(p,q) = (\beta^p_\Psi(p) \wedge \beta^p_\Psi(q))h_2(\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)),$$
$$\gamma^p_{\mathtt{Q}}(p,q) = (\gamma^p_\Psi(p) \vee \gamma^p_\Psi(q))h_3(\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)), \quad \delta^p_{\mathtt{Q}}(p,q) = (\delta^p_\Psi(p) \vee \beta^p_\Psi(q))h_4(\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)),$$
$$\alpha^n_{\mathtt{Q}}(p,q) = (\alpha^n_\Psi(p) \vee \alpha^n_\Psi(q))h_5(\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)), \quad \beta^n_{\mathtt{Q}}(p,q) = (\beta^n_\Psi(p) \vee \beta^n_\Psi(q))h_6(\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)),$$
$$\gamma^n_{\mathtt{Q}}(p,q) = (\gamma^n_\Psi(p) \wedge \gamma^n_\Psi(q))h_7(\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)), \quad \delta^n_{\mathtt{Q}}(p,q) = (\delta^n_\Psi(p) \wedge \beta^n_\Psi(q))h_8(\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)), \quad \forall p, q \in \Lambda$$

Example 3.9. Let $\mathtt{G} = (\Psi, \Upsilon)$ is a QBND on $\Lambda = (p, q, r)$, such that $\Psi = \{(p, 0.5, 0.6, 0.4, 0.5, -0.4, -0.3, -0.6, -0.7),$
$(q, 0.2, 0.3, 0.5, 0.7, -0.3, -0.4, -0.4, -0.7), (r, 0.6, 0.5, 0.4, 0.3, -0.6, -0.4, -0.3, -0.5)\}$, and $\Upsilon =$
$\{(\overrightarrow{pq}, 0.1, 0.2, 0.3, 0.6, -0.2, -0.2, -0.4, -0.6), (\overrightarrow{pr}, 0.4, 0.3, 0.2, 0.2, -0.3, -0.2, -0.5, -0.6),$
$(\overrightarrow{rq}, 0.1, 0.2, 0.4, 0.6, -0.2, -0.3, -0.3, -0.6)\}$ as shown in figure 3. By direct calculations Tables 3 and 4 show the out and in-neighborhoods of QBN, respectively.

Table 3. QBN Out-neighbourhoods

ϱ	$\mathtt{Q}^+(\varrho)$
p	$\{q(0.1, 0.2, 0.3, 0.6, -0.2, -0.2, -0.4, -0.6), r(0.4, 0.3, 0.2, 0.3, -0.3, -0.2, -0.5, -0.6)\}$
q	$\emptyset$
r	$\{q(0.1, 0.2, 0.4, 0.6, -0.2, -0.3, -0.3, -0.6)\}$

Table 4. QBN in-neighbourhoods

ϱ	$\mathtt{Q}^-(\varrho)$
p	$\emptyset$
q	$\{p(0.1, 0.2, 0.3, 0.6, -0.2, -0.2, -0.4, -0.6), r(0.1, 0.2, 0.4, 0.6, -0.2, -0.3, -0.3, -0.6)\}$
r	$\{p(0.4, 0.3, 0.2, 0.3, -0.3, -0.2, -0.5, -0.6)\}$

The QBNCG of Figure 3 is shown in Figure 4

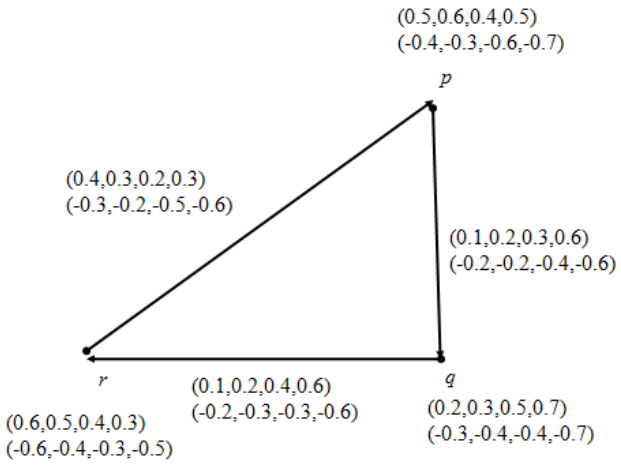


FIGURE 3. QBND

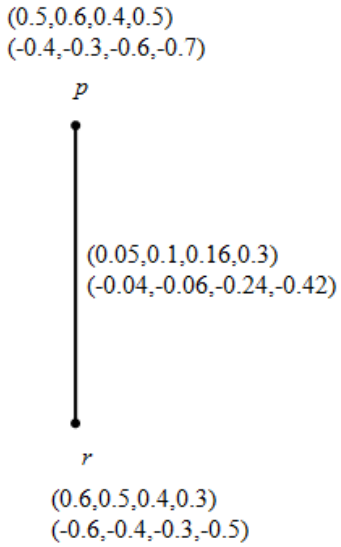


FIGURE 4. QBNCG

Definition 3.10. A QBN $\mathtt{G} = (\Psi, \Upsilon)$, where
$\Psi = (\Psi_1^p, \Psi_2^p, \Psi_3^p, \Psi_4^p, \Psi_1^n, \Psi_2^n, \Psi_3^n, \Psi_4^n)$, and $\Upsilon = (\Upsilon_1^p, \Upsilon_2^p, \Upsilon_3^p, \Upsilon_4^p, \Upsilon_1^n, \Upsilon_2^n, \Upsilon_3^n, \Upsilon_4^n)$ then, an edge (p, q),$p, q \in \Lambda$ is called independent strong if

$$\frac{1}{2}[\Psi_1^p(p) \wedge \Psi_1^p(q)] < \Upsilon_1^p(p,q), \quad \frac{1}{2}[\Psi_1^n(p) \vee \Psi_1^n(q)] > \Upsilon_1^n(p,q),$$
$$\frac{1}{2}[\Psi_2^p(p) \wedge \Psi_2^p(q)] < \Upsilon_2^p(p,q), \quad \frac{1}{2}[\Psi_2^n(p) \vee \Psi_2^n(q)] > \Upsilon_1^n(p,q),$$
$$\frac{1}{2}[\Psi_3^p(p) \vee \Psi_3^p(q)] > \Upsilon_3^p(p,q), \quad \frac{1}{2}[\Psi_3^n(p) \wedge \Psi_3^n(q)] < \Upsilon_3^n(p,q),$$
$$\frac{1}{2}[\Psi_4^p(p) \vee \Psi_4^p(q)] > \Upsilon_4^p(p,q), \quad \frac{1}{2}[\Psi_4^n(p) \wedge \Psi_4^n(q)] < \Upsilon_4^n(p,q).$$

In otherwords, one can say it is weak.

Theorem 3.11. *Let Λ be a QBND. If the intersection $\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)$ contains only one element of $\overrightarrow{\mathtt{G}}$, then the edge (p,q) of $\mathbb{C}(\overrightarrow{\mathtt{G}})$ is independently strong if and only if*

$$|[\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)]|_{\alpha^p} > 0.50 \quad |[\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)]|_{\alpha^n} < 0.50$$
$$|[\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)]|_{\beta^p} > 0.50 \quad |[\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)]|_{\beta^n} < 0.50$$
$$|[\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)]|_{\gamma^p} < 0.50 \quad |[\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)]|_{\gamma^n} < 0.50$$
$$|[\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)]|_{\delta^p} < 0.50 \quad |[\mathtt{Q}^+(p) \cap \mathtt{Q}^+(q)]|_{\delta^n} < 0.50$$

Proof.
Consider $\overrightarrow{\mathcal{X}}$ is a QBND. let $\varrho, \zeta \in \Lambda$.Suppose $\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta) = (p, p^p, q^p, r^p, s^p, p^n, q^n, r^n, s^n)$, where,$(p^p, q^p, r^p, s^p, p^n, q^n, r^n, s^n)$ are the positive TMF, CMF, IMF and FMF and negative TMF, CMF, IMF and FMF values of either the lines (ϱ, p) or the lines (ζ, p), respectively. Here,

$$|\,[\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)]\,|_{\alpha^p} = p^p = h_1(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta))$$
$$|\,[\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)]\,|_{\beta^p} = q^p = h_2(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta))$$
$$|\,[\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)]\,|_{\gamma^p} = r^p = h_3(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta))$$
$$|\,[\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)]\,|_{\delta^p} = s^p = h_4(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta))$$
$$|\,[\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)]\,|_{\alpha^n} = p^n = h_5(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta))$$
$$|\,[\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)]\,|_{\beta^n} = q^n = h_6(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta))$$
$$|\,[\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)]\,|_{\gamma^n} = r^n = h_7(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta))$$
$$|\,[\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)]\,|_{\delta^n} = s^n = h_8(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta))$$

Then,

$$\Upsilon_1^p(\varrho,\zeta) = p^p \times [\Psi_1^p(\varrho) \wedge \Psi_1^p(\zeta)], \quad \Upsilon_1^n(\varrho,\zeta) = p^n \times [\Psi_1^n(\varrho) \vee \Psi_1^n(\zeta)],$$
$$\Upsilon_2^p(\varrho,\zeta) = q^p \times [\Psi_2^p(\varrho) \wedge \Psi_2^p(\zeta)], \quad \Upsilon_2^n(\varrho,\zeta) = q^n \times [\Psi_2^n(\varrho) \vee \Psi_2^n(\zeta)],$$
$$\Upsilon_3^p(\varrho,\zeta) = r^p \times [\Psi_3^p(\varrho) \vee \Psi_3^p(\zeta)], \quad \Upsilon_3^n(\varrho,\zeta) = r^n \times [\Psi_3^n(\varrho) \wedge \Psi_3^n(\zeta)],$$
$$\Upsilon_4^p(\varrho,\zeta) = s^p \times [\Psi_4^p(\varrho) \vee \Psi_4^p(\zeta)], \quad \Upsilon_4^n(\varrho,\zeta) = s^n \times [\Psi_4^n(\varrho) \wedge \Psi_4^n(\zeta)].$$

Hence, the line (ϱ,ζ) in $C(\overrightarrow{\mathtt{G}})$ is independent strong if and only if $p^p > 0.50, q^p > 0.50, r^p < 0.50, s^p < 0.50$ $p^n < 0.50, q^n < 0.50, r^n > 0.50, s^n > 0.50$ Therefore, the line (ϱ,ζ) of $C(\overrightarrow{G})$ is independent strong if and only if

$$| [\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)] |_{\alpha^p} > 0.50 \quad | [\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)] |_{\alpha^n} < 0.50$$
$$| [\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)] |_{\beta^p} > 0.50 \quad | [\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)] |_{\beta^n} < 0.50$$
$$| [\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)] |_{\gamma^p} < 0.50 \quad | [\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)] |_{\gamma^n} < 0.50$$
$$| [\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)] |_{\delta^p} < 0.50 \quad | [\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)] |_{\delta^n} < 0.50$$

We visualize the theorem with an instance as verified in as shown in Figure 5 □

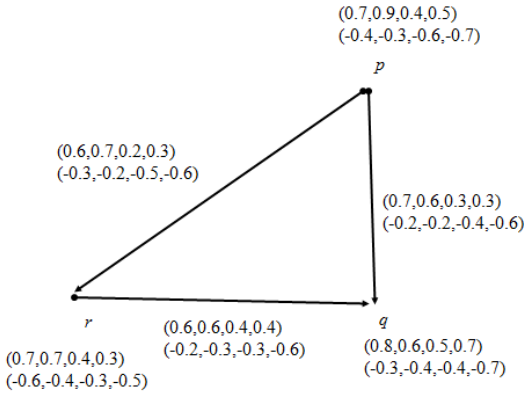
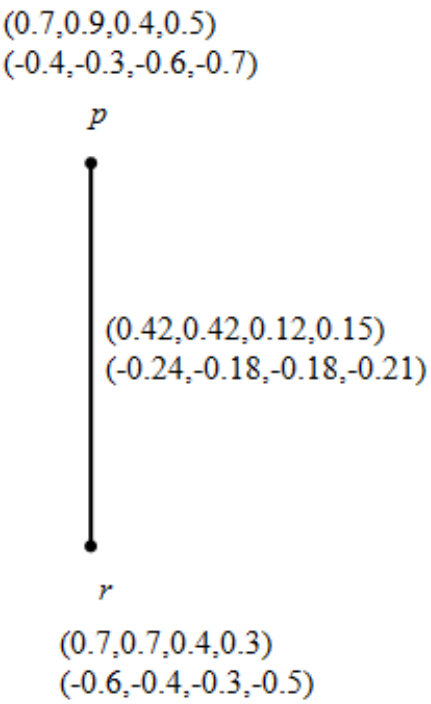


FIGURE 5. (5a)



(b)

FIGURE 6. (5b)
QBNCG (1) QBNCD (2)

Theorem 3.12. *If all the lines of QBND* $\overrightarrow{\mathtt{G}}$ *are independent strong, then*

$$\frac{\Upsilon_1^p(\varrho,\zeta)}{(\Psi_1^p(\varrho)\wedge\Psi_1^p(\zeta))^2} > 0.50 \quad \frac{\Upsilon_1^n(\varrho,\zeta)}{(\Psi_1^n(\varrho)\vee\Psi_1^n(\zeta))^2} < 0.50$$
$$\frac{\Upsilon_2^p(\varrho,\zeta)}{(\Psi_2^p(\varrho)\wedge\Psi_2^p(\zeta))^2} > 0.50 \quad \frac{\Upsilon_2^n(\varrho,\zeta)}{(\Psi_2^n(\varrho)\vee\Psi_2^n(\zeta))^2} < 0.50$$
$$\frac{\Upsilon_3^p(\varrho,\zeta)}{(\Psi_3^p(\varrho)\vee\Psi_3^p(\zeta))^2} < 0.50 \quad \frac{\Upsilon_3^n(\varrho,\zeta)}{(\Psi_3^n(\varrho)\wedge\Psi_3^n(\zeta))^2} < 0.50$$
$$\frac{\Upsilon_4^p(\varrho,\zeta)}{(\Psi_4^p(\varrho)\vee\Psi_4^p(\zeta))^2} < 0.50 \quad \frac{\Upsilon_4^n(\varrho,\zeta)}{(\Psi_4^n(\varrho)\wedge\Psi_4^n(\zeta))^2} < 0.50$$

for all edges (ϱ,ζ) *in* $C(\overrightarrow{\mathtt{G}})$.

Proof. Assume that QBND $\overrightarrow{\mathtt{G}}$ has independent, strong edges all around it.

$$\frac{1}{2}[\Psi_1^p(\varrho) \wedge \Psi_1^p(\zeta)] < \Upsilon_1^p(\varrho,\zeta), \quad \frac{1}{2}[\Psi_1^n(\varrho) \vee \Psi_1^n(\zeta)] > \Upsilon_1^n(\varrho,\zeta),$$
$$\frac{1}{2}[\Psi_2^p(\varrho) \wedge \Psi_2^p(\zeta)] < \Upsilon_2^p(\varrho,\zeta), \quad \frac{1}{2}[\Psi_2^n(\varrho) \vee \Psi_2^n(\zeta)] > \Upsilon_2^n(\varrho,\zeta),$$
$$\frac{1}{2}[\Psi_3^p(\varrho) \vee \Psi_3^p(\zeta)] > \Upsilon_3^p(\varrho,\zeta), \quad \frac{1}{2}[\Psi_3^n(\varrho) \wedge \Psi_3^n(\zeta)] < \Upsilon_3^n(\varrho,\zeta),$$
$$\frac{1}{2}[\Psi_4^p(\varrho) \vee \Psi_4^p(\zeta)] > \Upsilon_4^p(\varrho,\zeta), \quad \frac{1}{2}[\Psi_4^n(\varrho) \wedge \Psi_4^n(\zeta)] < \Upsilon_4^n(\varrho,\zeta).$$

for every lines (ϱ,ζ) in $\overrightarrow{\mathtt{G}}$. Let the corresponding QBNCG be $C(\overrightarrow{\mathtt{G}})$.

Then their arises two cases:

Case (1):

when $\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta) = \emptyset$ for all $\varrho, \zeta \in \Lambda$. Then there exists no edge in $C(\overrightarrow{\mathtt{G}})$ between ϱ and ζ. Thus, we have nothing to claim this case.

Case (2):

when $\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta) \neq \emptyset$. Let $\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta) = (a_1, m_1^p, r_1^p, p_1^p, q_1^p, m_1^n, r_1^n, p_1^n, q_1^n)$, $(a_2, m_2^p, r_2^p, p_2^p, q_2^p, m_2^n, r_2^n, p_2^n, q_2^n)$, $(a_3, m_3^p, r_3^p, p_3^p, q_3^p, m_3^n, r_3^n, p_3^n, q_3^n)$,...., $(a_l, m_l^p, r_l^p, p_l^p, q_l^p, m_l^n, r_l^n, p_l^n, q_l^n)$, where $(m_i^p, r_i^p, p_i^p, q_i^p, m_i^n, r_i^n, p_i^n, q_i^n)$ are the positive and negative TMF, CMF, IMF and FMF membership values of either $(\overrightarrow{\varrho, a_i})$ or $(\overrightarrow{\zeta, a_i})$ for $i = 1, 2, ..., l$, respectively, Thus,

$$m_i^p = [\Upsilon_1^p(\overrightarrow{\varrho, a_i}) \wedge \Upsilon_1^p(\overrightarrow{\zeta, a_i})], \quad m_i^n = [\Upsilon_1^n(\overrightarrow{\varrho, a_i}) \vee \Upsilon_1^n(\overrightarrow{\zeta, a_i})]$$
$$r_i^p = [\Upsilon_2^p(\overrightarrow{\varrho, a_i}) \wedge \Upsilon_2^p(\overrightarrow{\zeta, a_i})], \quad r_i^n = [\Upsilon_2^n(\overrightarrow{\varrho, a_i}) \vee \Upsilon_2^n(\overrightarrow{\zeta, a_i})]$$
$$p_i^p = [\Upsilon_3^p(\overrightarrow{\varrho, a_i}) \vee \Upsilon_3^p(\overrightarrow{\zeta, a_i})], \quad p_i^n = [\Upsilon_3^n(\overrightarrow{\varrho, a_i}) \wedge \Upsilon_3^n(\overrightarrow{\zeta, a_i})]$$
$$q_i^p = [\Upsilon_4^p(\overrightarrow{\varrho, a_i}) \vee \Upsilon_4^p(\overrightarrow{\zeta, a_i})], \quad q_i^n = [\Upsilon_4^n(\overrightarrow{\varrho, a_i}) \wedge \Upsilon_4^n(\overrightarrow{\zeta, a_i})]$$

for all $i = 1, 2, ..., l$.

Consider,

$$h_1(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)) = \max\{m_i^p, \quad i = 1, 2, .., l\} = m_{\max}^p$$
$$h_2(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)) = \max\{r_i^p, \quad i = 1, 2, .., l\} = r_{\max}^p$$
$$h_3(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)) = \min\{r_i^p, \quad i = 1, 2, .., l\} = r_{\min}^p$$
$$h_4(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)) = \min\{q_i^p, \quad i = 1, 2, .., l\} = q_{\min}^p$$
$$h_5(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)) = \min\{m_i^n, \quad i = 1, 2, .., l\} = m_{\max}^n$$
$$h_6(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)) = \min\{r_i^n, \quad i = 1, 2, .., l\} = r_{\max}^n$$
$$h_7(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)) = \min\{r_i^n, \quad i = 1, 2, .., l\} = r_{\min}^n$$
$$h_8(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)) = \min\{q_i^n, \quad i = 1, 2, .., l\} = q_{\min}^n$$

Obviously, $m^p_{\max} > \Upsilon^p_1(\overrightarrow{\varrho,\zeta})$, $r^p_{\max} > \Upsilon^p_2(\overrightarrow{\varrho,\zeta})$, $p^p_{\min} < \Upsilon^p_3(\overrightarrow{\varrho,\zeta})$, $q^p_{\min} < \Upsilon^p_4(\overrightarrow{\varrho,\zeta})$, $m^n_{\max} < \Upsilon^n_1(\overrightarrow{\varrho,\zeta})$, $r^n_{\max} < \Upsilon^n_2(\overrightarrow{\varrho,\zeta})$, $p^p_{\min} < \Upsilon^n_3(\overrightarrow{\varrho,\zeta})$, $q^p_{\min} < \Upsilon^p_4(\overrightarrow{\varrho,\zeta})$, for all edges $(\overrightarrow{\varrho,\zeta})$ shows that,

$$\frac{m^p_{\max}}{\Psi^p_1(\varrho)\wedge\Psi^p_1(\zeta)} > \frac{\Upsilon^p_1(\overrightarrow{\varrho,\zeta})}{\Psi^p_1(\varrho)\wedge\Psi^p_1(\zeta)} > 0.50, \quad \frac{m^n_{\max}}{\Psi^n_1(\varrho)\vee\Psi^n_1(\zeta)} < \frac{\Upsilon^n_1(\overrightarrow{\varrho,\zeta})}{\Psi^n_1(\varrho)\vee\Psi^n_1(\zeta)} < 0.50$$
$$\frac{r^p_{\max}}{\Psi^p_2(\varrho)\wedge\Psi^p_2(\zeta)} > \frac{\Upsilon^p_2(\overrightarrow{\varrho,\zeta})}{\Psi^p_2(\varrho)\wedge\Psi^p_2(\zeta)} > 0.50, \quad \frac{r^n_{\max}}{\Psi^n_2(\varrho)\vee\Psi^n_2(\zeta)} < \frac{\Upsilon^n_2(\overrightarrow{\varrho,\zeta})}{\Psi^n_2(\varrho)\vee\Psi^n_2(\zeta)} < 0.50$$
$$\frac{p^p_{\min}}{\Psi^p_3(\varrho)\vee\Psi^p_3(\zeta)} < \frac{\Upsilon^p_3(\overrightarrow{\varrho,\zeta})}{\Psi^p_3(\varrho)\vee\Psi^p_3(\zeta)} < 0.50, \frac{p^n_{\min}}{\Psi^n_3(\varrho)\wedge\Psi^n_3(\zeta)} < \frac{\Upsilon^n_3(\overrightarrow{\varrho,\zeta})}{\Psi^n_3(\varrho)\wedge\Psi^n_3(\zeta)} < 0.50$$
$$\frac{q^p_{\min}}{\Psi^p_4(\varrho)\vee\Psi^p_4(\zeta)} < \frac{\Upsilon^p_4(\overrightarrow{\varrho,\zeta})}{\Psi^p_4(\varrho)\vee\Psi^p_4(\zeta)} < 0.50, \quad \frac{q^n_{\min}}{\Psi^n_4(\varrho)\wedge\Psi^n_4(\zeta)} < \frac{\Upsilon^n_4(\overrightarrow{\varrho,\zeta})}{\Psi^n_4(\varrho)\wedge\Psi^n_4(\zeta)} < 0.50$$

therefore,

$$\Upsilon^p_1(\varrho,\zeta) = (\Psi^p_1(\varrho)\wedge\Psi^p_1(\zeta))h_1(\mathtt{Q}^+(\varrho)\cap\mathtt{Q}^+(\zeta)), \text{or} \quad \Upsilon^p_1(\varrho,\zeta) = [\Psi^p_1(\varrho)\wedge\Psi^p_1(\zeta)]\times m^p_{\max},$$
$$\text{or} \quad \frac{\Upsilon^p_1(\varrho,\zeta)}{(\Psi^p_1(\varrho)\wedge\Psi^p_1(\zeta))} = m^p_{\max}, \text{or} \quad \frac{\Upsilon^p_1(\varrho,\zeta)}{(\Psi^p_1(\varrho)\wedge\Psi^p_1(\zeta))^2} = \frac{m^p_{\max}}{(\Psi^p_1(\varrho)\wedge\Psi^p_1(\zeta))} > 0.50$$
$$\Upsilon^p_2(\varrho,\zeta) = (\Psi^p_2(\varrho)\wedge\Psi^p_2(\zeta))h_2(\mathtt{Q}^+(\varrho)\cap\mathtt{Q}^+(\zeta)), \text{or} \quad \Upsilon^p_2(\varrho,\zeta) = [\Psi^p_2(\varrho)\wedge\Psi^p_2(\zeta)]\times r^p_{\max},$$
$$\text{or} \quad \frac{\Upsilon^p_2(\varrho,\zeta)}{(\Psi^p_2(\varrho)\wedge\Psi^p_2(\zeta))} = r^p_{\max}, \text{or} \quad \frac{\Upsilon^p_2(\varrho,\zeta)}{(\Psi^p_2(\varrho)\wedge\Psi^p_2(\zeta))^2} = \frac{r^p_{\max}}{(\Psi^p_2(\varrho)\wedge\Psi^p_2(\zeta))} > 0.50$$
$$\Upsilon^p_3(\varrho,\zeta) = (\Psi^p_3(\varrho)\vee\Psi^p_3(\zeta))h_3(\mathtt{Q}^+(\varrho)\cap\mathtt{Q}^+(\zeta)), \text{or} \quad \Upsilon^p_3(\varrho,\zeta) = [\Psi^p_3(\varrho)\vee\Psi^p_3(\zeta)]\times p^p_{\min},$$
$$\text{or} \quad \frac{\Upsilon^p_3(\varrho,\zeta)}{(\Psi^p_3(\varrho)\wedge\Psi^p_3(\zeta))} = p^p_{\min}, \text{or} \quad \frac{\Upsilon^p_3(\varrho,\zeta)}{(\Psi^p_3(\varrho)\vee\Psi^p_3(\zeta))^2} = \frac{p^p_{\min}}{(\Psi^p_3(\varrho)\vee\Psi^p_3(\zeta))} < 0.50$$
$$\Upsilon^p_4(\varrho,\zeta) = (\Psi^p_4(\varrho)\vee\Psi^p_4(\zeta))h_4(\mathtt{Q}^+(\varrho)\cap\mathtt{Q}^+(\zeta)), \text{or} \quad \Upsilon^p_4(\varrho,\zeta) = [\Psi^p_4(\varrho)\vee\Psi^p_4(\zeta)]\times q^p_{\min},$$
$$\text{or} \quad \frac{\Upsilon^p_4(\varrho,\zeta)}{(\Psi^p_4(\varrho)\wedge\Psi^p_4(\zeta))} = q^p_{\min}, \text{or} \quad \frac{\Upsilon^p_4(\varrho,\zeta)}{(\Psi^p_4(\varrho)\vee\Psi^p_4(\zeta))^2} = \frac{q^p_{\min}}{(\Psi^p_4(\varrho)\vee\Psi^p_4(\zeta))} < 0.50$$

$$\Upsilon_1^n(\varrho,\zeta) = (\Psi_1^n(\varrho) \vee \Psi_1^n(\zeta))h_1(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)), \text{or} \quad \Upsilon_1^n(\varrho,\zeta) = [\Psi_1^n(\varrho) \vee \Psi_1^n(\zeta)] \times m_{\max}^n,$$
$$\text{or} \quad \frac{\Upsilon_1^n(\varrho,\zeta)}{(\Psi_1^n(\varrho) \vee \Psi_1^n(\zeta))} = m_{\max}^n, \text{or} \quad \frac{\Upsilon_1^n(\varrho,\zeta)}{(\Psi_1^n(\varrho) \vee \Psi_1^n(\zeta))^2} = \frac{m_{\max}^n}{(\Psi_1^n(\varrho) \vee \Psi_1^n(\zeta))} < 0.50$$
$$\Upsilon_2^n(\varrho,\zeta) = (\Psi_2^n(\varrho) \vee \Psi_2^n(\zeta))h_2(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)), \text{or} \quad \Upsilon_2^n(\varrho,\zeta) = [\Psi_2^n(\varrho) \vee \Psi_2^n(\zeta)] \times r_{\max}^n,$$
$$\text{or} \quad \frac{\Upsilon_2^n(\varrho,\zeta)}{(\Psi_2^n(\varrho) \vee \Psi_2^n(\zeta))} = r_{\max}^n, \text{or} \quad \frac{\Upsilon_2^n(\varrho,\zeta)}{(\Psi_2^n(\varrho) \vee \Psi_2^n(\zeta))^2} = \frac{r_{\max}^n}{(\Psi_2^n(\varrho) \vee \Psi_2^n(\zeta))} < 0.50$$
$$\Upsilon_3^n(\varrho,\zeta) = (\Psi_3^n(\varrho) \wedge \Psi_3^n(\zeta))h_3(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)), \text{or} \quad \Upsilon_3^n(\varrho,\zeta) = [\Psi_3^n(\varrho) \wedge \Psi_3^n(\zeta)] \times p_{\min}^n,$$
$$\text{or} \quad \frac{\Upsilon_3^n(\varrho,\zeta)}{(\Psi_3^n(\varrho) \wedge \Psi_3^n(\zeta))} = p_{\min}^n, \text{or} \quad \frac{\Upsilon_3^n(\varrho,\zeta)}{(\Psi_3^n(\varrho) \wedge \Psi_3^n(\zeta))^2} = \frac{p_{\min}^n}{(\Psi_3^n(\varrho) \wedge \Psi_3^n(\zeta))} < 0.50$$
$$\Upsilon_4^n(\varrho,\zeta) = (\Psi_4^n(\varrho) \wedge \Psi_4^n(\zeta))h_4(\mathtt{Q}^+(\varrho) \cap \mathtt{Q}^+(\zeta)), \text{or} \quad \Upsilon_4^n(\varrho,\zeta) = [\Psi_4^n(\varrho) \wedge \Psi_4^n(\zeta)] \times q_{\min}^n,$$
$$\text{or} \quad \frac{\Upsilon_4^n(\varrho,\zeta)}{(\Psi_4^n(\varrho) \wedge \Psi_4^n(\zeta))} = q_{\min}^n, \text{or} \quad \frac{\Upsilon_4^n(\varrho,\zeta)}{(\Psi_4^n(\varrho) \wedge \Psi_4^n(\zeta))^2} = \frac{q_{\min}^n}{(\Psi_4^n(\varrho) \wedge \Psi_4^n(\zeta))} < 0.50$$

Hence,

$$\frac{\Upsilon_1^p(\varrho,\zeta)}{(\Psi_1^p(\varrho) \wedge \Psi_1^p(\zeta))^2} > 0.50 \quad \frac{\Upsilon_1^n(\varrho,\zeta)}{(\Psi_1^n(\varrho) \vee \Psi_1^n(\zeta))^2} < 0.50$$
$$\frac{\Upsilon_2^p(\varrho,\zeta)}{(\Psi_2^p(\varrho) \wedge \Psi_2^p(\zeta))^2} > 0.50 \quad \frac{\Upsilon_2^n(\varrho,\zeta)}{(\Psi_2^n(\varrho) \vee \Psi_2^n(\zeta))^2} < 0.50$$
$$\frac{\Upsilon_3^p(\varrho,\zeta)}{(\Psi_3^p(\varrho) \vee \Psi_3^p(\zeta))^2} < 0.50 \quad \frac{\Upsilon_3^n(\varrho,\zeta)}{(\Psi_3^n(\varrho) \wedge \Psi_3^n(\zeta))^2} < 0.50$$
$$\frac{\Upsilon_4^p(\varrho,\zeta)}{(\Psi_4^p(\varrho) \vee \Psi_4^p(\zeta))^2} < 0.50 \quad \frac{\Upsilon_4^n(\varrho,\zeta)}{(\Psi_4^n(\varrho) \wedge \Psi_4^n(\zeta))^2} < 0.50,$$

for every lines (ϱ,ζ) in $C(\overrightarrow{\mathtt{G}})$. □

Definition 3.13. The QBN open-neighborhood of a vertex ϱ of a QBNG $\mathtt{G} = (\Psi, \Upsilon)$ is QBNS $\mathtt{Q}(\varrho) = (\Lambda, \alpha_\varrho^p, \beta_\varrho^p, \gamma_\varrho^p, \delta_\varrho^p, \alpha_\varrho^n, \beta_\varrho^n, \gamma_\varrho^n, \delta_\varrho^n)$, where, $\Lambda_\varrho = (\zeta | \Upsilon_1^p(\varrho,\zeta) > 0, \Upsilon_2^p(\varrho,\zeta) > 0, \Upsilon_3^p(\varrho,\zeta) > 0, \Upsilon_4^p(\varrho,\zeta)) > 0, \Upsilon_1^n(\varrho,\zeta) < 0, \Upsilon_2^n(\varrho,\zeta) < 0, \Upsilon_3^n(\varrho,\zeta) < 0, \Upsilon_4^p(\varrho,\zeta)) < 0$, and $\alpha_\varrho^p : \Lambda_\varrho \to [0,1]$ represent by $\alpha_\varrho^p = \Upsilon_1^p(\varrho,\zeta)$, $\beta_\varrho^p : \Lambda_\varrho \to [0,1]$ represent by $\beta_\varrho^p = \Upsilon_2^p(\varrho,\zeta)$, $\gamma_\varrho^p : \Lambda_\varrho \to [0,1]$ represent by $\gamma_\varrho^p = \Upsilon_3^p(\varrho,\zeta)$, $\delta_\varrho^p : \Lambda_\varrho \to [0,1]$ represent by $\delta_\varrho^p = \Upsilon_4^p(\varrho,\zeta)$, $\alpha_\varrho^n : \Lambda_\varrho \to [-1,0]$ represent by $\alpha_\varrho^n = \Upsilon_1^n(\varrho,\zeta)$, $\beta_\varrho^n : \Lambda_\varrho \to [-1,0]$ represent by $\beta_\varrho^n = \Upsilon_2^n(\varrho,\zeta)$, $\gamma_\varrho^n : \Lambda_\varrho \to [-1,0]$ represent by $\gamma_\varrho^n = \Upsilon_3^n(\varrho,\zeta)$, $\delta_\varrho^n : \Lambda_\varrho \to [-1,0]$ represent by $\delta_\varrho^n = \Upsilon_4^n(\varrho,\zeta)$. For every vertex $\varrho \in \Lambda$, the QBN singleton set, $\Psi_\varrho^| = (\varrho, \Psi_1^{p'}, \Psi_2^{p'}, \Psi_3^{p'}, \Psi_4^{p'}, \Psi_1^{n'}, \Psi_2^{n'}, \Psi_3^{n'}, \Psi_4^{n'})$ such that $\Psi_1^{p'} : \varrho \to [0,1], \Psi_2^{p'} : \varrho \to [0,1], \Psi_3^{p'} : \varrho \to [0,1], \Psi_4^{p'} : \varrho \to [0,1], \Psi_1^{n'} : \varrho \to [-1,0], \Psi_2^{n'} : \varrho \to [-1,0], \Psi_3^{n'} : \varrho \to [-1,0], \Psi_4^{n'} : \varrho \to [-1,0]$, represent by $\Psi_1^{p'} = \Psi_1^p(\varrho), \Psi_2^{p'} = \Psi_2^p(\varrho), \Psi_3^{p'} = \Psi_3^p(\varrho), \Psi_4^{p'} = \Psi_4^p(\varrho)$ and $\Psi_1^{n'} = \Psi_1^n(\varrho), \Psi_2^{n'} = \Psi_2^n(\varrho), \Psi_3^{n'} = \Psi_3^n(\varrho), \Psi_4^{n'} = \Psi_4^n(\varrho)$, respectively. The QBN closed neighbourhood of a vertex ϱ is $\mathtt{Q}[\varrho] = \mathtt{Q}(\varrho) \cup \Psi_\varrho$

Definition 3.14. Consider $\mathtt{G} = (\Psi, \Upsilon)$ is a QBNG. QBN open neighbourhood graph of $\mathtt{G}$ is a QBNG $\mathtt{Q}(\mathtt{G}) = (\Psi, \Upsilon')$ which has the same QBNS of a vertex in $\mathtt{G}$ and has a QBN edge between two vertices $\varrho, \zeta \in \Lambda$ in $(\mathtt{Q})$ if and only if $\mathtt{Q}(\varrho) \cap \mathtt{Q}(\zeta)$ is a non-empty QBNS in $\mathtt{G}$. The positive TMF, CMF, IMF and FMF, negative TMF, CMF, IMF and FMF value of the edge (ϱ, ζ) are given by:

$$\Upsilon_1^{p'}(\varrho,\zeta) = [\Psi_1^p(\varrho) \wedge \Psi_1^p(\zeta)]h_1(\mathtt{Q}(\varrho) \cap \mathtt{Q}(\zeta)), \quad \Upsilon_1^{n'}(\varrho,\zeta) = [\Psi_1^n(\varrho) \vee \Psi_1^n(\zeta)]h_1(\mathtt{Q}(\varrho) \cap \mathtt{Q}(\zeta)),$$
$$\Upsilon_2^{p'}(\varrho,\zeta) = [\Psi_2^p(\varrho) \wedge \Psi_2^p(\zeta)]h_2(\mathtt{Q}(\varrho) \cap \mathtt{Q}(\zeta)), \quad \Upsilon_2^{n'}(\varrho,\zeta) = [\Psi_2^n(\varrho) \vee \Psi_2^n(\zeta)]h_2(\mathtt{Q}(\varrho) \cap \mathtt{Q}(\zeta)),$$
$$\Upsilon_3^{p'}(\varrho,\zeta) = [\Psi_3^p(\varrho) \vee \Psi_3^p(\zeta)]h_3(\mathtt{Q}(\varrho) \cap \mathtt{Q}(\zeta)), \quad \Upsilon_3^{n'}(\varrho,\zeta) = [\Psi_3^n(\varrho) \wedge \Psi_3^n(\zeta)]h_3(\mathtt{Q}(\varrho) \cap \mathtt{Q}(\zeta)),$$
$$\Upsilon_4^{p'}(\varrho,\zeta) = [\Psi_4^p(\varrho) \vee \Psi_4^p(\zeta)]h_4(\mathtt{Q}(\varrho) \cap \mathtt{Q}(\zeta)), \quad \Upsilon_4^{n'}(\varrho,\zeta) = [\Psi_4^n(\varrho) \wedge \Psi_4^n(\zeta)]h_4(\mathtt{Q}(\varrho) \cap \mathtt{Q}(\zeta)).$$

Definition 3.15. Consider $\mathtt{G} = (\Psi, \Upsilon)$ is a QBNG. QBN open neighbourhood graph of $\mathtt{G}$ is a QBNG $\mathtt{Q}[\mathtt{G}] = (\Psi, \Upsilon')$ which has the same QBNS of a vertex in $\mathtt{G}$ and has a QBN edge between two vertices $\varrho, \zeta \in \Lambda$ in $\mathtt{Q}[\mathtt{G}]$ if and only if $\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]$ is a non-empty QBNS in $\mathtt{G}$. The positive TMF, CMF, IMF and FMF, negative TMF, CMF, IMF and FMF value of the edge (ϱ, ζ) are given by:

$$\Upsilon_1^{p'}(\varrho,\zeta) = [\Psi_1^p(\varrho) \wedge \Psi_1^p(\zeta)]h_1(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]), \quad \Upsilon_1^{n'}(\varrho,\zeta) = [\Psi_1^n(\varrho) \vee \Psi_1^n(\zeta)]h_1(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]),$$
$$\Upsilon_2^{p'}(\varrho,\zeta) = [\Psi_2^p(\varrho) \wedge \Psi_2^p(\zeta)]h_2(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]), \quad \Upsilon_2^{n'}(\varrho,\zeta) = [\Psi_2^n(\varrho) \vee \Psi_2^n(\zeta)]h_2(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]),$$
$$\Upsilon_3^{p'}(\varrho,\zeta) = [\Psi_3^p(\varrho) \vee \Psi_3^p(\zeta)]h_3(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]), \quad \Upsilon_3^{n'}(\varrho,\zeta) = [\Psi_3^n(\varrho) \wedge \Psi_3^n(\zeta)]h_3(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]),$$
$$\Upsilon_4^{p'}(\varrho,\zeta) = [\Psi_4^p(\varrho) \vee \Psi_4^p(\zeta)]h_4(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]), \quad \Upsilon_4^{n'}(\varrho,\zeta) = [\Psi_4^n(\varrho) \wedge \Psi_4^n(\zeta)]h_4(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]).$$

Example 3.16. Let $\mathtt{G} = (\Psi, \Upsilon)$ is a QBND, such that, $\Lambda = \{p, q, r, s\}$, $\Psi = \{(p, 0.3, 0.4, 0.5, 0.6, -0.4, -0.6, -0.7, -0.4),$
$(q, 0.6, 0.7, 0.6, 0.5, -0.6, -0.7, -0.5, -0.4), (r, 0.4, 0.3, 0.4, 0.4, -0.6, -0.7, -0.6, -0.5),$
$(s, 0.4, 0.6, 0.7, 0.6, -0.4, -0.3, -0.4, -0.3)\}$, and
$\Upsilon = \{(pq, 0.2, 0.3, 0.5, 0.5, -0.3, -0.5, -0.6, -0.3),$
$(qr, 0.3, 0.2, 0.5, 0.4, -0.5, -0.6, -0.4, -0.3), (sr, 0.3, 0.2, 0.6, 0.5, -0.3, -0.2, -0.4, -0.4),$
$(ps, 0.2, 0.3, 0.5, 0.5, -0.3, -0.2, -0.6, -0.3)\}$, according to figure 6.
Figures 7 and 8 show the corresponding QBN open and closed neighbourhood graphs.

Theorem 3.17. *For each edge of a QBNG* $\mathtt{G}$*, there exist an edge in* $\mathtt{Q}[\mathtt{G}]$.

Proof. If (ϱ, ζ) is an line of a QBNG $\mathtt{G} = (\Psi, \Upsilon)$. Suppose $\mathtt{Q}[\mathtt{G}] = (\Psi, \Upsilon')$ is the corresponding closed neighbourhood of a QBNG. Suppose $\varrho, \zeta \in \mathtt{Q}[\varrho]$ and $\varrho, \zeta \in \mathtt{Q}[\zeta]$. Then $\varrho, \zeta \in \mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]$. Hence,

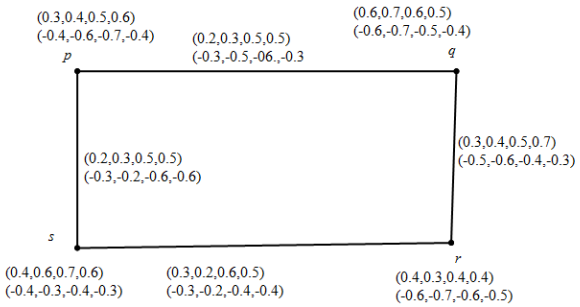


FIGURE 7. QBNG

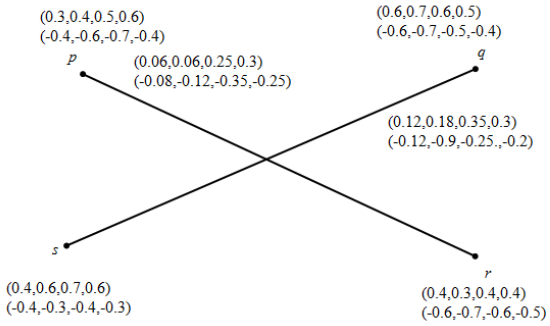


FIGURE 8. Q(G)

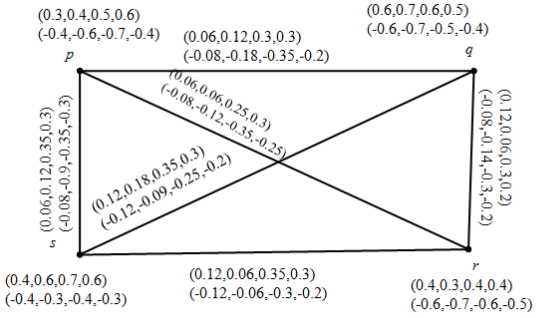


FIGURE 9. Q[G]

$$h_1(\mathbb{Q}[\varrho] \cap \mathbb{Q}[\zeta]) \neq 0 \quad h_2(\mathbb{Q}[\varrho] \cap \mathbb{Q}[\zeta]) \neq 0$$
$$h_3(\mathbb{Q}[\varrho] \cap \mathbb{Q}[\zeta]) \neq 0 \quad h_4(\mathbb{Q}[\varrho] \cap \mathbb{Q}[\zeta]) \neq 0$$
$$h_5(\mathbb{Q}[\varrho] \cap \mathbb{Q}[\zeta]) \neq 0 \quad h_6(\mathbb{Q}[\varrho] \cap \mathbb{Q}[\zeta]) \neq 0.$$

Then,

$$\Upsilon_1^{p'}(\varrho,\zeta) = [\Psi_1^p(\varrho) \wedge \Psi_1^p(\zeta)]h_1(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]) \neq 0.$$
$$\Upsilon_2^{p'}(\varrho,\zeta) = [\Psi_2^p(\varrho) \wedge \Psi_2^p(\zeta)]h_2(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]) \neq 0.$$
$$\Upsilon_3^{p'}(\varrho,\zeta) = [\Psi_3^p(\varrho) \vee \Psi_3^p(\zeta)]h_3(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]) \neq 0.$$
$$\Upsilon_4^{p'}(\varrho,\zeta) = [\Psi_4^p(\varrho) \vee \Psi_4^p(\zeta)]h_4(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]) \neq 0.$$
$$\Upsilon_1^{n'}(\varrho,\zeta) = [\Psi_1^n(\varrho) \vee \Psi_1^n(\zeta)]h_1(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]) \neq 0.$$
$$\Upsilon_2^{n'}(\varrho,\zeta) = [\Psi_2^n(\varrho) \wedge \Psi_2^n(\zeta)]h_2(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]) \neq 0.$$
$$\Upsilon_3^{n'}(\varrho,\zeta) = [\Psi_3^n(\varrho) \vee \Psi_3^n(\zeta)]h_3(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]) \neq 0.$$
$$\Upsilon_4^{n'}(\varrho,\zeta) = [\Psi_4^n(\varrho) \vee \Psi_4^n(\zeta)]h_4(\mathtt{Q}[\varrho] \cap \mathtt{Q}[\zeta]) \neq 0.$$

Thus, for each edge (ϱ,ζ) in QBNG $\mathtt{G}$, there exists an edge (ϱ,ζ) in $\mathtt{Q}[\mathtt{G}]$. □

Theorem 3.18. *Let $C(\overrightarrow{\mathtt{G}_1}) = (\Psi_1, \Upsilon_1)$ and $C(\overrightarrow{\mathtt{G}_2}) = (\Psi_1, \Upsilon_2)$ be two QBNCG of QBND $\overrightarrow{\mathtt{G}_1} = (\Psi_1, l_1)$ and $\overrightarrow{\mathtt{G}_2} = (\Psi_1, l_2)$, respectively. Then $C(\overrightarrow{\mathtt{G}_1}\Box\overrightarrow{\mathtt{G}_2}) = \mathtt{G}_{C(\overrightarrow{\mathtt{G}_1})^*\Box C(\overrightarrow{\mathtt{G}_2})^*} \cup \mathtt{G}^{\Box}$, where, $\mathtt{G}_{C(\overrightarrow{\mathtt{G}_1})^*\Box C(\overrightarrow{\mathtt{G}_2})^*}$ is a QBNG on crisp graph $(\varrho_1 \times \varrho_2, E^*_{C(\overrightarrow{\mathtt{G}_1})}\Box E^*_{C(\overrightarrow{\mathtt{G}_2})})$, $C(\overrightarrow{\mathtt{G}_1})^*$ and $C(\overrightarrow{\mathtt{G}_2})^*$ are the crisp competition graphs of $\overrightarrow{\mathtt{G}_1}$ and $\overrightarrow{\mathtt{G}_2}$, respectively. $\mathtt{G}^{\Box}$ is a QBNG on $(Y_1 \times Y_2, E^{\Box})$ such that:*

(1.) $E^{\Box} = \{(\varrho_1,\varrho_2)(\zeta_1,\zeta_2) : \zeta_1 \in \mathtt{Q}^-(\varrho_1^*), \zeta_2 \in \mathtt{Q}^+(\varrho_2^*)\}$

$$E^*_{C(\overrightarrow{\mathtt{G}_1})}\Box E^*_{C(\overrightarrow{\mathtt{G}_2})} = \{(\varrho_1,\varrho_2)(\varrho_1,\zeta_2) : \varrho_1 \in Y_1, \varrho_2\zeta_2 \in E^*_{C(\overrightarrow{\mathtt{G}_2})}\} \cup \{(\varrho_1,\varrho_2)(\zeta_1,\varrho_2) : \varrho_2 \in Y_2, \varrho_1\zeta_1 \in E^*_{C(\overrightarrow{\mathtt{G}_1})}\}$$

(2.)
$$\alpha^p_{\Psi_1\Box\Psi_2} = \alpha^p_{\Psi_1}(\varrho_1) \wedge \alpha^p_{\Psi_2}(\varrho_2) \quad \beta^p_{\Psi_1\Box\Psi_2} = \beta^p_{\Psi_1}(\varrho_1) \wedge \beta^p_{\Psi_2}(\varrho_2)$$
$$\gamma^p_{\Psi_1\Box\Psi_2} = \gamma^p_{\Psi_1}(\varrho_1) \vee \gamma^p_{\Psi_2}(\varrho_2) \quad \delta^p_{\Psi_1\Box\Psi_2} = \delta^p_{\Psi_1}(\varrho_1) \vee \delta^p_{\Psi_2}(\varrho_2)$$
$$\alpha^n_{\Psi_1\Box\Psi_2} = \alpha^n_{\Psi_1}(\varrho_1) \vee \alpha^n_{\Psi_2}(\varrho_2) \quad \beta^n_{\Psi_1\Box\Psi_2} = \beta^n_{\Psi_1}(\varrho_1) \vee \beta^n_{\Psi_2}(\varrho_2)$$
$$\gamma^n_{\Psi_1\Box\Psi_2} = \gamma^n_{\Psi_1}(\varrho_1) \wedge \gamma^n_{\Psi_2}(\varrho_2) \quad \delta^n_{\Psi_1\Box\Psi_2} = \delta^p_{\Psi_1}(\varrho_1) \wedge \delta^n_{\Psi_2}(\varrho_2)$$

(3) $\alpha^p_\Upsilon((\varrho_1,\varrho_2),(\varrho_1,\zeta_2)) = [\alpha^p_{\Psi_1}(\varrho_1) \wedge \alpha^p_{\Psi_2}(\varrho_2) \wedge \alpha^p_{\Psi_2}(\zeta_2)] \times \vee_{a_2}\{\alpha^p_{\Psi_1}(\varrho_1) \wedge \alpha^p_{\overrightarrow{l_2}}(\varrho_2 a_2) \wedge \alpha^p_{\overrightarrow{l_2}}(\zeta_2 a_2)\}$
$(\varrho_1,\varrho_2)(\varrho_1,\zeta_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_2 \in (\mathbb{Q}^+(\varrho_2) \cap \mathbb{Q}^+(\zeta_2))^*$

(4) $\beta^p_\Upsilon((\varrho_1,\varrho_2),(\varrho_1,\zeta_2)) = [\beta^p_{\Psi_1}(\varrho_1) \wedge \beta^p_{\Psi_2}(\varrho_2) \wedge \beta^p_{\Psi_2}(\zeta_2)] \times \vee_{a_2}\{\beta^p_{\Psi_1}(\varrho_1) \wedge \beta^p_{\overrightarrow{l_2}}(\varrho_2 a_2) \wedge \beta^p_{\overrightarrow{l_2}}(\zeta_2 a_2)\}$
$(\varrho_1,\varrho_2)(\varrho_1,\zeta_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_2 \in (\mathbb{Q}^+(\varrho_2) \cap \mathbb{Q}^+(\zeta_2))^*$

(5) $\gamma^p_\Upsilon((\varrho_1,\varrho_2),(\varrho_1,\zeta_2)) = [\gamma^p_{\Psi_1}(\varrho_1) \vee \gamma^p_{\Psi_2}(\varrho_2) \wedge \gamma^p_{\Psi_2}(\zeta_2)] \times \vee_{a_2}\{\gamma^p_{\Psi_1}(\varrho_1) \vee \gamma^p_{\overrightarrow{l_2}}(\varrho_2 a_2) \vee \gamma^p_{\overrightarrow{l_2}}(\zeta_2 a_2)\}$
$(\varrho_1,\varrho_2)(\varrho_1,\zeta_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_2 \in (\mathbb{Q}^+(\varrho_2) \cap \mathbb{Q}^+(\zeta_2))^*$

(6) $\delta^p_\Upsilon((\varrho_1,\varrho_2),(\varrho_1,\zeta_2)) = [\delta^p_{\Psi_1}(\varrho_1) \vee \delta^p_{\Psi_2}(\varrho_2) \wedge \delta^p_{\Psi_2}(\zeta_2)] \times \vee_{a_2}\{\delta^p_{\Psi_1}(\varrho_1) \vee \delta^p_{\overrightarrow{l_2}}(\varrho_2 a_2) \vee \delta^p_{\overrightarrow{l_2}}(\zeta_2 a_2)\}$
$(\varrho_1,\varrho_2)(\varrho_1,\zeta_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_2 \in (\mathbb{Q}^+(\varrho_2) \cap \mathbb{Q}^+(\zeta_2))^*$

(7) $\alpha^n_\Upsilon((\varrho_1,\varrho_2),(\varrho_1,\zeta_2)) = [\alpha^n_{\Psi_1}(\varrho_1) \vee \alpha^n_{\Psi_2}(\varrho_2) \vee \alpha^n_{\Psi_2}(\zeta_2)] \times \vee_{a_2}\{\alpha^n_{\Psi_1}(\varrho_1) \vee \alpha^n_{\overrightarrow{l_2}}(\varrho_2 a_2) \vee \alpha^n_{\overrightarrow{l_2}}(\zeta_2 a_2)\}$
$(\varrho_1,\varrho_2)(\varrho_1,\zeta_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_2 \in (\mathbb{Q}^+(\varrho_2) \cap \mathbb{Q}^+(\zeta_2))^*$

(8) $\beta^n_\Upsilon((\varrho_1,\varrho_2),(\varrho_1,\zeta_2)) = [\beta^n_{\Psi_1}(\varrho_1) \vee \beta^n_{\Psi_2}(\varrho_2) \vee \beta^n_{\Psi_2}(\zeta_2)] \times \vee_{a_2}\{\beta^n_{\Psi_1}(\varrho_1) \vee \beta^n_{\overrightarrow{l_2}}(\varrho_2 a_2) \vee \beta^n_{\overrightarrow{l_2}}(\zeta_2 a_2)\}$
$(\varrho_1,\varrho_2)(\varrho_1,\zeta_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_2 \in (\mathbb{Q}^+(\varrho_2) \cap \mathbb{Q}^+(\zeta_2))^*$

(9) $\gamma^n_\Upsilon((\varrho_1,\varrho_2),(\varrho_1,\zeta_2)) = [\gamma^n_{\Psi_1}(\varrho_1) \wedge \gamma^n_{\Psi_2}(\varrho_2) \wedge \gamma^n_{\Psi_2}(\zeta_2)] \times \vee_{a_2}\{\gamma^n_{\Psi_1}(\varrho_1) \wedge \gamma^n_{\overrightarrow{l_2}}(\varrho_2 a_2) \wedge \gamma^n_{\overrightarrow{l_2}}(\zeta_2 a_2)\}$
$(\varrho_1,\varrho_2)(\varrho_1,\zeta_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_2 \in (\mathbb{Q}^+(\varrho_2) \cap \mathbb{Q}^+(\zeta_2))^*$

(10) $\delta^n_\Upsilon((\varrho_1,\varrho_2),(\varrho_1,\zeta_2)) = [\delta^n_{\Psi_1}(\varrho_1) \wedge \delta^n_{\Psi_2}(\varrho_2) \wedge \delta^n_{\Psi_2}(\zeta_2)] \times \vee_{a_2}\{\delta^n_{\Psi_1}(\varrho_1) \wedge \delta^n_{\overrightarrow{l_2}}(\varrho_2 a_2) \wedge \delta^n_{\overrightarrow{l_2}}(\zeta_2 a_2)\}$
$(\varrho_1,\varrho_2)(\varrho_1,\zeta_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_2 \in (\mathbb{Q}^+(\varrho_2) \cap \mathbb{Q}^+(\zeta_2))^*$

(11) $\alpha^p_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\varrho_2)) = [\alpha^p_{\Psi_1}(\varrho_1) \wedge \alpha^p_{\Psi_1}(\zeta_1) \wedge \alpha^p_{\Psi_2}(\varrho_2)] \times \vee_{a_1}\{\alpha^p_{\Psi_2}(\varrho_2) \wedge \alpha^p_{\overrightarrow{l_1}}(\varrho_1 a_1) \wedge \alpha^p_{\overrightarrow{l_1}}(\zeta_1 a_1)\}$
$(\varrho_1,\varrho_2)(\zeta_1,\varrho_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_1 \in (\mathbb{Q}^+(\varrho_1 \cap \mathbb{Q}^+(\zeta_1))^*$

(12) $\beta^p_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\varrho_2)) = [\beta^p_{\Psi_1}(\varrho_1) \wedge \beta^p_{\Psi_1}(\zeta_1) \wedge \beta^p_{\Psi_2}(\varrho_2)] \times \vee_{a_1}\{\beta^p_{\Psi_2}(\varrho_2) \wedge \beta^p_{\overrightarrow{l_1}}(\varrho_1 a_1) \wedge \beta^p_{\overrightarrow{l_1}}(\zeta_1 a_1)\}$
$(\varrho_1,\varrho_2)(\zeta_1,\varrho_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_1 \in (\mathbb{Q}^+(\varrho_1) \cap \mathbb{Q}^+(\zeta_1))^*$

(13) $\gamma^p_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\varrho_2)) = [\gamma^p_{\Psi_1}(\varrho_1) \vee \gamma^p_{\Psi_1}(\zeta_1) \vee \gamma^p_{\Psi_2}(\varrho_2)] \times \vee_{a_1}\{\gamma^p_{\Psi_2}(\varrho_2) \vee \gamma^p_{\overrightarrow{l_1}}(\varrho_1 a_1) \vee \gamma^p_{\overrightarrow{l_1}}(\zeta_1 a_1)\}$
$(\varrho_1,\varrho_2)(\zeta_1,\varrho_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_1 \in (\mathbb{Q}^+(\varrho_1) \cap \mathbb{Q}^+(\zeta_1))^*$

(14) $\delta^p_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\varrho_2)) = [\delta^p_{\Psi_1}(\varrho_1) \vee \delta^p_{\Psi_1}(\zeta_1) \vee \delta^p_{\Psi_2}(\varrho_2)] \times \vee_{a_1}\{\delta^p_{\Psi_2}(\varrho_2) \vee \delta^p_{\overrightarrow{l_1}}(\varrho_1 a_1) \vee \delta^p_{\overrightarrow{l_1}}(\zeta_1 a_1)\}$
$(\varrho_1,\varrho_2)(\zeta_1,\varrho_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_1 \in (\mathbb{Q}^+(\varrho_1) \cap \mathbb{Q}^+(\zeta_1))^*$

(15) $\alpha^n_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\varrho_2)) = [\alpha^n_{\Psi_1}(\varrho_1) \vee \alpha^n_{\Psi_1}(\zeta_1) \vee \alpha^n_{\Psi_2}(\varrho_2)] \times \vee_{a_1}\{\alpha^n_{\Psi_2}(\varrho_2) \vee \alpha^n_{\overrightarrow{l_1}}(\varrho_1 a_1) \vee \alpha^n_{\overrightarrow{l_1}}(\zeta_1 a_1)\}$
$(\varrho_1,\varrho_2)(\zeta_1,\varrho_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_1 \in (\mathbb{Q}^+(\varrho_1) \cap \mathbb{Q}^+(\zeta_1))^*$

(16) $\beta^n_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\varrho_2)) = [\beta^n_{\Psi_1}(\varrho_1) \vee \beta^n_{\Psi_1}(\zeta_1) \vee \beta^n_{\Psi_2}(\varrho_2)] \times \vee_{a_1}\{\beta^n_{\Psi_2}(\varrho_2) \vee \beta^n_{\overrightarrow{l_1}}(\varrho_1 a_1) \vee \beta^n_{\overrightarrow{l_1}}(\zeta_1 a_1)\}$
$(\varrho_1,\varrho_2)(\zeta_1,\varrho_2) \in E_{C(\overrightarrow{\mathbb{G}_1^*})} \Box E_{C(\overrightarrow{\mathbb{G}_2^*})}, a_1 \in (\mathbb{Q}^+(\varrho_1) \cap \mathbb{Q}^+(\zeta_1))^*$

(17) $\gamma^n_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\varrho_2)) = [\gamma^n_{\Psi_1}(\varrho_1) \wedge \gamma^n_{\Psi_1}(\zeta_1) \wedge \gamma^n_{\Psi_2}(\varrho_2)] \times \vee_{a_1}\{\gamma^n_{\Psi_2}(\varrho_2) \wedge \gamma^n_{\overrightarrow{l_1}}(\varrho_1 a_1) \wedge \gamma^n_{\overrightarrow{l_1}}(\zeta_1 a_1)\}$
$(\varrho_1,\varrho_2)(\zeta_1,\varrho_2) \in E_{C(\overrightarrow{\mathtt{G}_1^*})} \square E_{C(\overrightarrow{\mathtt{G}_2^*})}, a_1 \in (\mathtt{Q}^+(\varrho_1) \cap \mathtt{Q}^+(\zeta_1))^*$

(18) $\delta^n_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\varrho_2)) = [\delta^n_{\Psi_1}(\varrho_1) \wedge \delta^n_{\Psi_1}(\zeta_1) \wedge \delta^n_{\Psi_2}(\varrho_2)] \times \vee_{a_1}\{\delta^n_{\Psi_2}(\varrho_2) \wedge \delta^n_{\overrightarrow{l_1}}(\varrho_1 a_1) \wedge \delta^n_{\overrightarrow{l_1}}(\zeta_1 a_1)\}$
$(\varrho_1,\varrho_2)(\zeta_1,\varrho_2) \in E_{C(\overrightarrow{\mathtt{G}_1^*})} \square E_{C(\overrightarrow{\mathtt{G}_2^*})}, a_1 \in (\mathtt{Q}^+(\varrho_1) \cap \mathtt{Q}^+(\zeta_1))^*,$

(19) $\alpha^p_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\zeta_2)) = [\alpha^p_{\Psi_1}(\varrho_1) \wedge \alpha^p_{\Psi_1}(\zeta_1) \wedge \alpha^p_{\Psi_2}(\varrho_2) \wedge \alpha^p_{\Psi_2}(\zeta_2)] \times \{\alpha^p_{\Psi_1}(\varrho_1) \wedge \alpha^p_{\overrightarrow{l_1}}(\zeta_1\varrho_1) \wedge \alpha^p_{\Psi_2}(\zeta_2) \wedge \alpha^p_{\overrightarrow{l_2}}(\varrho_2\zeta_2)\}$
$(\varrho_1,\zeta_1)(\varrho_2,\zeta_2) \in E^\square$

(20) $\beta^p_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\zeta_2)) = [\beta^p_{\Psi_1}(\varrho_1) \wedge \beta^p_{\Psi_1}(\zeta_1) \wedge \beta^p_{\Psi_2}(\varrho_2) \wedge \beta^p_{\Psi_2}(\zeta_2)] \times \{\beta^p_{\Psi_1}(\varrho_1) \wedge \beta^p_{\overrightarrow{l_1}}(\zeta_1\varrho_1) \wedge \beta^p_{\Psi_2}(\zeta_2) \wedge \beta^p_{\overrightarrow{l_2}}(\varrho_2\zeta_2)\}$
$(\varrho_1,\zeta_1)(\varrho_2,\zeta_2) \in E^\square$

(21) $\gamma^p_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\zeta_2)) = [\gamma^p_{\Psi_1}(\varrho_1) \vee \gamma^p_{\Psi_1}(\zeta_1) \vee \gamma^p_{\Psi_2}(\varrho_2) \vee \gamma^p_{\Psi_2}(\zeta_2)] \times \{\gamma^p_{\Psi_1}(\varrho_1) \vee \gamma^p_{\overrightarrow{l_1}}(\zeta_1\varrho_1) \vee \gamma^p_{\Psi_2}(\zeta_2) \vee \gamma^p_{\overrightarrow{l_2}}(\varrho_2\zeta_2)\}$
$(\varrho_1,\zeta_1)(\varrho_2,\zeta_2) \in E^\square$

(22) $\delta^p_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\zeta_2)) = [\delta^p_{\Psi_1}(\varrho_1) \vee \delta^p_{\Psi_1}(\zeta_1) \vee \delta^p_{\Psi_2}(\varrho_2) \vee \delta^p_{\Psi_2}(\zeta_2)] \times \{\delta^p_{\Psi_1}(\varrho_1) \vee \delta^p_{\overrightarrow{l_1}}(\zeta_1\varrho_1) \vee \delta^p_{\Psi_2}(\zeta_2) \vee \delta^p_{\overrightarrow{l_2}}(\varrho_2\zeta_2)\}$
$(\varrho_1,\zeta_1)(\varrho_2,\zeta_2) \in E^\square$

(23) $\alpha^n_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\zeta_2)) = [\alpha^n_{\Psi_1}(\varrho_1) \vee \alpha^n_{\Psi_1}(\zeta_1) \vee \alpha^n_{\Psi_2}(\varrho_2) \vee \alpha^n_{\Psi_2}(\zeta_2)] \times \{\alpha^n_{\Psi_1}(\varrho_1) \vee \alpha^n_{\overrightarrow{l_1}}(\zeta_1\varrho_1) \vee \alpha^n_{\Psi_2}(\zeta_2) \vee \alpha^n_{\overrightarrow{l_2}}(\varrho_2\zeta_2)\}$
$(\varrho_1,\zeta_1)(\varrho_2,\zeta_2) \in E^\square$

(24) $\beta^n_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\zeta_2)) = [\beta^n_{\Psi_1}(\varrho_1) \vee \beta^n_{\Psi_1}(\zeta_1) \vee \beta^n_{\Psi_2}(\varrho_2) \vee \beta^n_{\Psi_2}(\zeta_2)] \times \{\beta^n_{\Psi_1}(\varrho_1) \vee \beta^n_{\overrightarrow{l_1}}(\zeta_1\varrho_1) \vee \beta^n_{\Psi_2}(\zeta_2) \vee \beta^n_{\overrightarrow{l_2}}(\varrho_2\zeta_2)\}$
$(\varrho_1,\zeta_1)(\varrho_2,\zeta_2) \in E^\square$

(25) $\gamma^n_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\zeta_2)) = [\gamma^n_{\Psi_1}(\varrho_1) \wedge \gamma^n_{\Psi_1}(\zeta_1) \wedge \gamma^n_{\Psi_2}(\varrho_2) \wedge \gamma^n_{\Psi_2}(\zeta_2)] \times \{\gamma^n_{\Psi_1}(\varrho_1) \wedge \gamma^n_{\overrightarrow{l_1}}(\zeta_1\varrho_1) \wedge \gamma^n_{\Psi_2}(\zeta_2) \wedge \gamma^n_{\overrightarrow{l_2}}(\varrho_2\zeta_2)\}$
$(\varrho_1,\zeta_1)(\varrho_2,\zeta_2) \in E^\square$

(26) $\delta^n_\Upsilon((\varrho_1,\varrho_2),(\zeta_1,\zeta_2)) = [\delta^n_{\Psi_1}(\varrho_1) \wedge \delta^n_{\Psi_1}(\zeta_1) \wedge \delta^n_{\Psi_2}(\varrho_2) \wedge \delta^n_{\Psi_2}(\zeta_2)] \times \{\delta^n_{\Psi_1}(\varrho_1) \wedge \delta^n_{\overrightarrow{l_1}}(\zeta_1\varrho_1) \wedge \delta^n_{\Psi_2}(\zeta_2) \wedge \delta^n_{\overrightarrow{l_2}}(\varrho_2\zeta_2)\}$
$(\varrho_1,\zeta_1)(\varrho_2,\zeta_2) \in E^\square.$

Example 3.19. Consider $\overrightarrow{\mathtt{G}_1} = (\Psi_1, l_1)$ and $\overrightarrow{\mathtt{G}_2} = (\Psi_2, l_2)$ be two QBND, as shown in figure 9. The QBN out and in-neighbourhood of $\overrightarrow{\mathtt{G}_1}$ and $\overrightarrow{\mathtt{G}_2}$ are given in tables 5 and 6. The QBNC graphs $C(\overrightarrow{\mathtt{G}_1})$ and $C(\overrightarrow{\mathtt{G}_2})$ are given in Figure 10.

Table 5. QBN Out neighbourhoods of $\overrightarrow{\mathtt{G}_1}$

ϱ	$\mathtt{Q}^+(\varrho)$
p_1	$\{q_1(0.4, 0.3, 0.2, 0.1, -0.2, -0.1, -0.2, -0.3)\}$
q_1	$\emptyset$
r_1	$\{q_1(0.5, 0.3, 0.6, 0.5, -0.2, -0.1, -0.4, -0.5)\}$
s_1	$\{r_1(0.4, 0.4, 0.6, 0.5, -0.3, -0.2, -0.4, -0.5)\}$

Table 6. QBN Out neighbourhoods of $\overrightarrow{\mathtt{G}_2}$

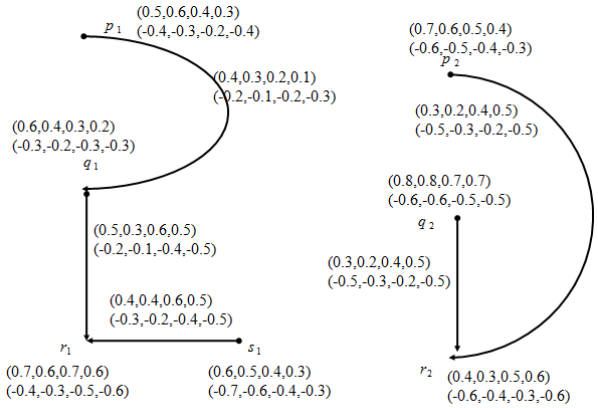


FIGURE 10. QBND

ϱ	$\mathtt{Q}^{+}(\varrho)$
p_2	$\{r_2(0.3, 0.2, 0.4, 0.5, -0.5, -0.3, -0.2, -0.2)\}$
q_2	$\{r_2(0.3, 0.2, 0.4, 0.5, -0.5, -0.3, -0.2, -0.5)$
r_2	$\{q_1(0.4, 0.3, 0.5, 0.6, -0.6, -0.4, -0.3, -0.6)\}$

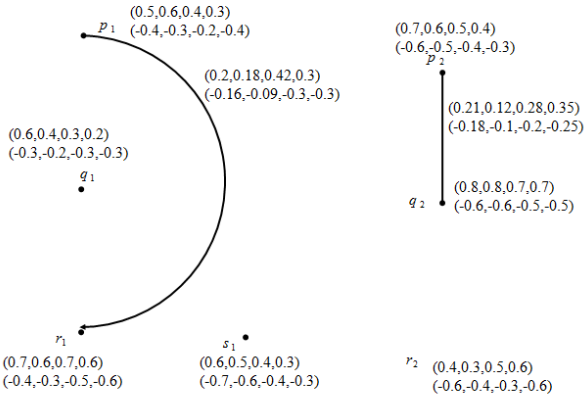


FIGURE 11. QBNCG

We now construct the QBNC graph $\mathtt{G}_{C(\overrightarrow{\mathtt{G}_1^*})\square C(\overrightarrow{\mathtt{G}_2^*})} \cup \mathtt{G}^{\square} = (W, B)$, where, $W = (\alpha_W^p, \beta_W^p, \gamma_W^p, \delta_W^p, \alpha_W^n, \beta_W^n, \gamma_W^n, \delta_W^n)$ and $B = (\alpha_B^p, \beta_B^p, \gamma_B^p, \delta_B^p, \alpha_B^n, \beta_B^n, \gamma_B^n, \delta_B^n)$ from $C(\overrightarrow{\mathtt{G}_1^*})$ and

$C(\overrightarrow{\mathtt{G}_2}^*)$ using Theorem(3.13). We obtain the following set of edges by using condition (1).

$$\begin{aligned} E_{C(\overrightarrow{\mathtt{G}_1}^*)\Box C(\overrightarrow{\mathtt{G}_2^*})} =&\{(p_1,p_2)(p_1,q_2)(q_1,p_2)(q_1,q_2),(r_1,p_2),(r_1,q_2),\\ &(s_1,p_2),(s_1,q_2),(p_1,p_2)(r_1,p_2),\\ &(p_1,q_2)(p_1,r_2),(p_1,r_2)(r_1,r_2)\},\\ E^{\Box} =&\{(q_1,p_2)(p_1,r_2),(q_1,p_2)(r_1r_2),(q_1,q_2)(p_1,r_2),\\ (q_1,q_2)(r_1,r_2),&(r_1,p_2)(s_1,r_2),(r_1,q_2)(s_1,r_2)\} \end{aligned}$$

According to condition (3) to (28), the degree of positive TMF, CMF, IMF and FMF, negative TMF, CMF, IMF and FMF of the adjacent edges of $\mathtt{G}_{C(\overrightarrow{\mathtt{G}_1}^*)\Box C(\overrightarrow{\mathtt{G}_2^*})}$ and $\mathtt{G}^{\Box}$ are given in Table 7.

Table 7. Adjacent edges of $\mathtt{G}_{C(\overrightarrow{\mathtt{G}_1}^*)\Box C(\overrightarrow{\mathtt{G}_2^*})} \cup \mathtt{G}^{\Box} = (W, B)$

$(\varrho_1,\varrho_2)(\zeta_1,\zeta_2)$	$\Upsilon(\varrho_1,\varrho_2)(\zeta_1,\zeta_2)$
$(p_1,p_2)(p_1,q_2)$	$\{(0.15,0.12,0.28,0.35,-0.12,-0.06,-0.2,-0.25)\}$
$(q_1,p_2)(q_1q_2)$	$\{(0.18,0.08,0.28,0.35,-0.09,-0.04,-0.2,-0.25)\}$
$(r_1,p_2)(r_1q_2)$	$\{(0.21,0.12,0.49,0.42,-0.12,-0.06,-0.35,-0.36)\}$
$(s_1,p_2)(s_1,q_2)$	$\{(0.18,0.1,0.28,0.35,-0.18,-0.1,-0.20,-0.25)\}$
$(p_1,p_2)(r_1,p_2)$	$\{(0.2,0.18,0.42,0.3,-0.16,-0.09,-0.24,-0.20)\}$
$(p_1,q_2)(r_1,q_2)$	$\{(0.2,0.18,0.49,0.49,-0.16,-0.09,-0.35,-0.42)\}$
$(p_1,r_2)(r_1,r_2)$	$\{(0.16,0.09,0.42,0.36,-0.16,-0.09,-0.30,-0.36)\}$
$(q_1,p_2)(p_1,r_2)$	$\{(0.12,0.06,0.2,0.36,-0.09,-0.04,-0.2,-0.24)\}$
$(q_1,p_2)(r_1,r_2)$	$\{(0.12,0.06,0.42,0.42,-0.09,-0.04,-0.3,-0.36)\}$
$(q_1,q_2)(p_1,r_2)$	$\{(0.12,0.06,0.35,0.42,-0.09,-0.04,-0.25,-0.30)\}$
$(q_1,q_2)(r_1,r_2)$	$\{(0.12,0.06,0.42,0.42,-0.09,-0.3,-0.25,-0.36)\}$
$(r_1,p_2)(s_1,r_2)$	$\{(0.12,0.06,0.49,0.36,-0.12,-0.06,-0.35,-0.36)\}$
$(r_1,q_2)(s_1,r_2)$	$\{(0.12,0.06,0.49,0.42,-0.12,-0.06,-0.35,-0.36)\}$

The QBNCG obtained using this model is given in Figure 11. where, the solid line indicate the part of QBNCG obtained from $\mathtt{G}_{C(\overrightarrow{\mathtt{G}_1}^*)\Box C(\overrightarrow{\mathtt{G}_2^*})}$, the dotted lines represent the part $\mathtt{G}^{\Box}$. The Cartesian product $\overrightarrow{\mathtt{G}_1}\Box\overrightarrow{\mathtt{G}_2}$ of QBND $\overrightarrow{\mathtt{G}_1}$ and $\overrightarrow{\mathtt{G}_2}$ is shown in figure 12. The QBN out-neighborhood of $\overrightarrow{\mathtt{G}_1}\Box\overrightarrow{\mathtt{G}_2}$ are calculate in Table 8. The QBNCG of $\overrightarrow{\mathtt{G}_1}\Box\overrightarrow{\mathtt{G}_2}$ is shown in figure 13.

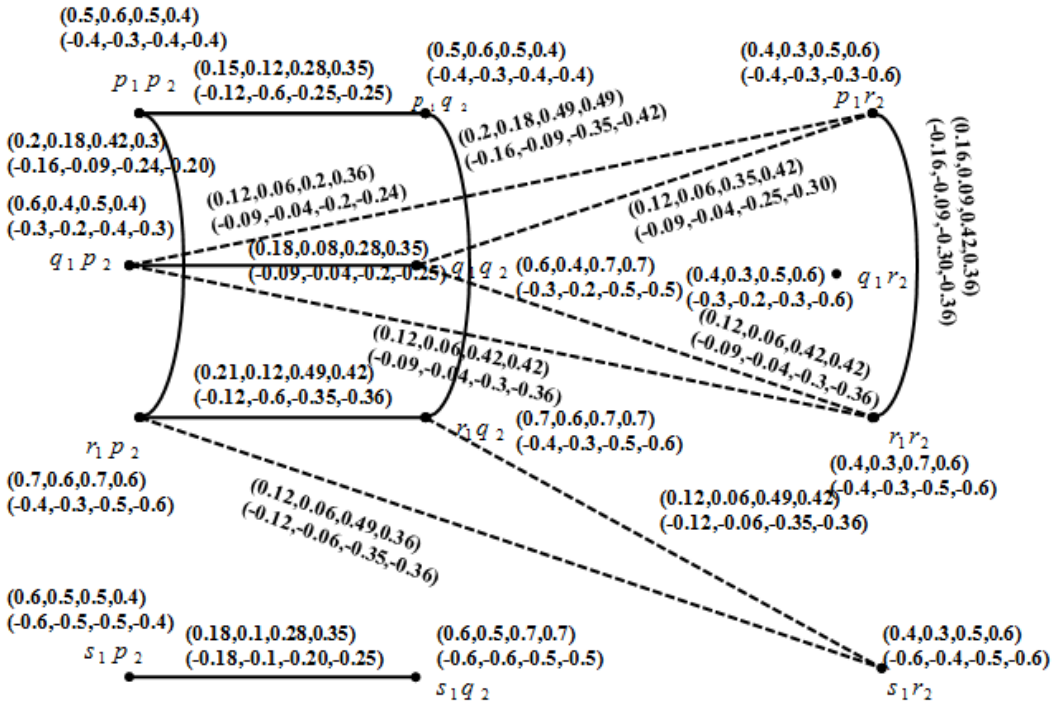


FIGURE 12

$\mathtt{G}_{C(\overrightarrow{\mathtt{G}_1^*})\Box C(\overrightarrow{\mathtt{G}_2^*})} \cup \mathtt{G}^{\Box}$

Table 8. QBN out-neighborhood of $\overrightarrow{\mathtt{G}_1}\Box\overrightarrow{\mathtt{G}_2}$

(ϱ,ζ)	$\mathtt{Q}^+(\varrho,\zeta)$
(p_1,p_2)	$\{(p_1,r_2)(0.3,0.2,0.4,0.5,-0.4,-0.3,-0.2,-0.4),(\mathtt{Q}_1,p_2)(0.4,0.3,0.5,0.4,-0.2,-0.1,-0.4,-0.3)\}$
(p_1,q_2)	$\{(p_1,r_2)(0.3,0.2,0.4,0.5,-0.4,-0.3,-0.2,-0.5),(\mathtt{Q}_1,q_2)(0.4,0.3,0.7,0.7,-0.2,-0.1,-0.5,-0.5)\}$
(p_1,r_2)	$\{(q_1r_2)(0.4,0.3,0.5,0.6,-0.2,-0.1,-0.3,-0.6)\}$
(q_1,p_2)	$\{(q_1,r_2)(0.3,0.2,0.4,0.5,-0.3,-0.2,-0.3,-0.3)\}$
(q_1,q_2)	$\{(q_1r_2)(0.3,0.2,0.4,0.5,-0.3,-0.2,-0.3,-0.5)\}$
(r_1,p_2)	$\{(r_1r_2)(0.3,0.2,0.7,0.6,-0.4,-0.3,-0.5,-0.6)\}$
(r_1,q_2)	$\{(r_1,r_2)(0.3,0.2,0.7,0.6,-0.4,-0.3,-0.5,-0.6),(q_1,q_2)(0.5,0.3,0.7,0.7,-0.2,-0.1,-0.5,-0.5)\}$
(r_1,r_2)	$\{(q_1,r_2)(0.4,0.3,0.6,0.6,-0.2,-0.1,-0.4,-0.6)\}$
(q_1,r_2)	$\emptyset$
(s_1,p_2)	$\{(r_1p_2)(0.4,0.4,0.6,0.5,-0.3,-0.2,-0.4,-0.5),(s_1r_2)(0.3,0.2,0.4,0.5,-0.5,-0.3,-0.5,-0.4)\}$
(s_1,q_2)	$\{(r_1,q_2)(0.4,0.4,0.7,0.7,-0.3,-0.2,-0.5,-0.5),(s_1,r_2)(0.3,0.2,0.4,0.5,-0.5,-0.3,-0.5,-0.4)\}$
(s_1,r_2)	$\{(r_1r_2)(0.4,0.3,0.6,0.6,-0.3,-0.2,-0.4,-0.6)\}$

It is clear from figure 11 and figure 13 that $\mathtt{G}_{C(\overrightarrow{\mathtt{G}_1^*})\Box C(\overrightarrow{\mathtt{G}_2^*})} \cup \mathtt{G}^{\Box} \cong C(\overrightarrow{\mathtt{G}_1}\Box\overrightarrow{\mathtt{G}_2})$.

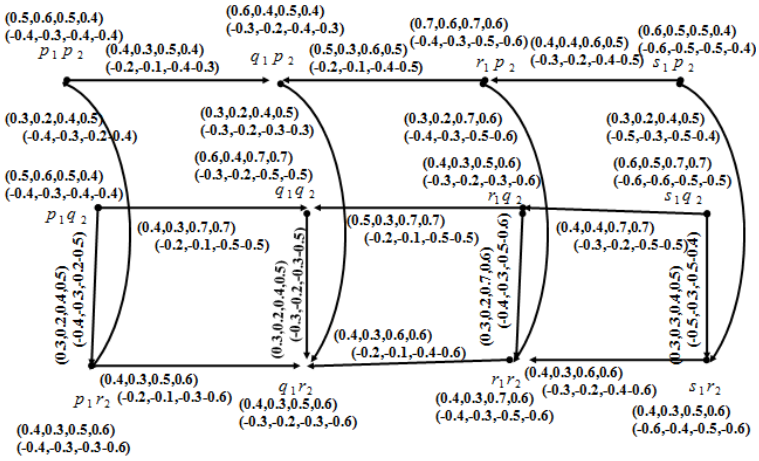


FIGURE 13
$\overrightarrow{G_1}\square\overrightarrow{G_2}$

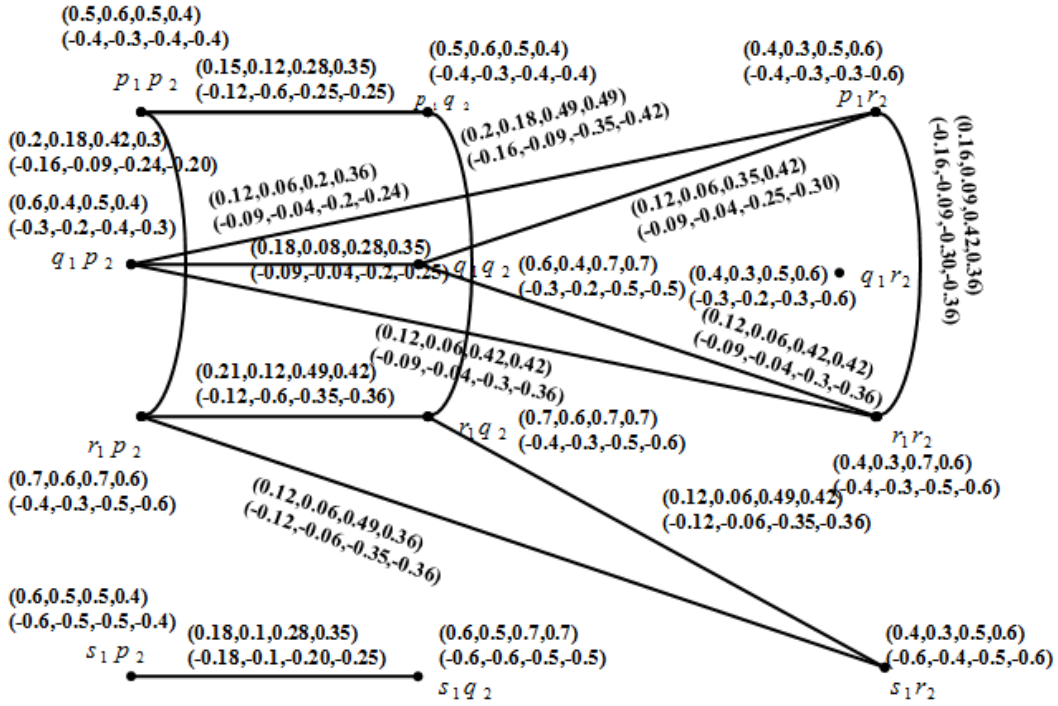


FIGURE 14
$C(\overrightarrow{G_1}\square\overrightarrow{G_2})$

4. **Application**

4.1. *Job competition on QBNCG*

Let the bipolar quadripartitioned neutrosophic competition digraph in figure 14 represent the competition amongst job seekers in a company. Let (Hussain, Aslam, Mohammed, Satham, Durga) be the set of applicants for the jobs Vice Chancellor (VC), Pro Vice Chancellor (PVC), Academic Dean (AD), Professor (PR). The positive degree of truth membership $\alpha(\varrho)$ of each applicant denote the percentage of academic performance, contradiction membership degree $\beta(\varrho)$ of each applicant represent the percentage of background knowledge, $\gamma(\varrho)$ and $\delta(\varrho)$ represent the research and teaching aptitude percentage. The negative degree of membership $\alpha(\varrho)$ represents the percentage that the applicant in not effective in order to fulfil the goals of the organization, etc. The bipolar quadripartitioned neutrosophic competition graph can be utilized in order to find the suitable jobs of the applicants.

Table 9. QBN out-neighborhood of Applicant and Jobs

$\varrho \in Y$	$\mathbb{Q}^+(\varrho, \zeta)$
Hussain	{(VC)(0.4,0.3,0.3,0.3,-0.3,-0.2,-0.4,-0.4), (PVC)(0.4,0.3,0.3,0.3,-0.3,-0.2,-0.4,-0.6), (AD)(0.3,0.3,0.3,0.3,-0.2,-0.1,-0.5,-0.5)}
Durga	{(PR)(0.2,0.2,0.2,0.2,-0.1,-0.1,-0.5,-0.5), (PVC)(0.6,0.6,0.6,0.6,-0.2,-0.2,-0.2,-0.2)}
Aslam	{(VC)(0.3,0.4,0.2,0.5,-0.5,-0.3,-0.3,-0.3), (PVC)(0.2,0.2,0.2,0.2,-0.1,-0.1,-0.1,-0.1), (AD)(0.3,0.2,0.5,0.5,-0.2,-0.2,-0.2,-0.2)}
Satham	{(AD)(0.2,0.1,0.3,0.3,-0.1,-0.1,-0.1,-0.1), (PR)(0.2,0.2,0.2,0.2,-0.1,-0.1,-0.1,-0.1)}
Mohammed	{(PR)(0.3,0.3,0.3,0.3,-0.3,-0.2,-0.2,-0.2)}

Here, $\mathbb{Q}^+(Hussain) \cap \mathbb{Q}^+(Mohammed) = \emptyset$, $\mathbb{Q}^+(Aslam) \cap \mathbb{Q}^+(Mohammed) = \emptyset$,
$\mathbb{Q}^+(Hussain) \cap \mathbb{Q}^+(Durga) = (PVC, 0.4, 0.3, 0.6, 0.6, -0.2, -0.2, -0.4, -0.6)$, $\mathbb{Q}^+(Hussain) \cap \mathbb{Q}^+(Satham) = (AD, 0.3, 0.2, 0.5, 0.5, -0.1, -0.1, -0.5, -0.5)$
$\mathbb{Q}^+(Mohammed) \cap \mathbb{Q}^+(Durga) = (PR, 0.2, 0.2, 0.3, 0.3, -0.1, -0.1, -0.5, -0.5)$,
$\mathbb{Q}^+(Mohammed) \cap \mathbb{Q}^+(Satham) = (PR, 0.2, 0.2, 0.3, 0.3, -0.1, -0.1, -0.2, -0.2)$
$\mathbb{Q}^+(Satham) \cap \mathbb{Q}^+(Durga) = (PR, 0.2, 0.2, 0.2, 0.2, -0.1, -0.1, -0.5, -0.5)$, $\mathbb{Q}^+(Aslam) \cap \mathbb{Q}^+(Durga) = (PVC, 0.2, 0.2, 0.6, 0.6, -0.1, -0.1, -0.2, -0.2)$
$\mathbb{Q}^+(Satham) \cap \mathbb{Q}^+(Aslam) = (AD, 0.2, 0.1, 0.5, 0.5, -0.1, -0.1, -0.2, -0.2)$,
$\mathbb{Q}^+(Hussain) \cap \mathbb{Q}^+(Aslam) = (PVC, 0.2, 0.2, 0.3, 0.3, -0.1, -0.1, -0.4, -0.6)$,
$(VC, 0.3, 0.3, 0.3, 0.5, -0.2, -0.1, -0.5, -0.5)$, $(AD, 0.3, 0.2, 0.5, 0.5, -0.2, -0.1, -0.5, -0.5)$.
Figure 15 depicts a bipolar quadripartitioned neutrosophic competition graph. The dotted lines reflect the applicant competing for the specific employment, and the solid

lines show how fiercely two applicants are competing. For instance, Hussain and Mohammed both are competing for the jobs, AD and strength of competition between them $(0.12, 0.18, 0.21, 0.24, -0.12, -0.09, -0.21, -0.24)$.

Here table 10, $\alpha(p, q)$ indicates the value of strength of competition of applicant p for Job q with respect to academic performance in order fulfill the goals of that jobs. In a similar fashion, the table 10 is constructed as follows:

Table 10. QBN out-neighborhood of Applicant and Jobs

Applicant, Job	Competition	α(Applicant, Job)	Sr(Applicant, Job)
Hussain,VC	Aslam	(0.12,0.18,0.21,0.24,-0.12,-0.09,-0.21,-0.24)	2.09
Aslam,VC	Hussain	(0.12,0.18,0.21,0.24,-0.12,-0.09,-0.21,-0.24)	2.09
Hussain,AD	Aslam, Satham	(0.105,0.15,0.28,0.32,-0.105,-0.065,-0.28,-0.32)	2.085
Aslam,AD	Hussain, Satham	(0.09,0.125,0.18,0.33,-0.09,-0.055,-0.28,-0.36)	2.2
Satham,AD	Hussain, Aslam	(0.075,0.095,0.25,0.41,-0.075,-0.055,-0.35,-0.44)	2.17
Hussain,PVC	Aslam, Durga	(0.16,0.18,0.345,0.36,-0.1,-0.075,-0.31,-0.36)	2.13
Aslam,PVC	Hussain, Durga	(0.1,0.17,0.345,0.36,-0.08,-0.065,-0.315,-0.36)	2.095
Durga,PVC	Hussain, Aslam	(0.14,0.17,0.48,0.48,-0.06,-0.05,-0.42,-0.48)	2.14
Mohammed,PR	Satham, Durga	(0.06,0.08,0.18,0.225,-0.05,-0.04,-0.12,-0.12)	1.885
Durga,PR	Mohammed, Satham	(0.06,0.12,0.20,0.20,-0.04,-0.04,-0.08,-0.08)	1.86
Satham,PR	Mohammed, Durga	(0.06,0.12,0.14,0.185,-0.05,-0.04,-0.08,-0.08)	1.925

Algorithm

(i) For a set of r applicants, enter the values for truth positive membership, contradiction membership, ignorance membership and false membership..

(ii) If for every two points p_i and p_j, $\alpha^+(r_ir_j) > 0.$ $\beta^+(r_ir_j) > 0.$, $\gamma^+(r_ir_j) > 0.$ $\delta^+(r_ir_j) > 0.$ and $\alpha^-(r_ir_j) < 0.$, $\beta^-(r_ir_j) < 0.$, $\gamma^-(r_ir_j) < 0.$ $\delta^-(r_ir_j) < 0.$, then $(r_j, \alpha^+(r_ir_j), \beta^+(r_ir_j), \gamma^+(r_ir_j), \delta^+(r_ir_j),$
$\alpha^-(r_ir_j), \beta^-(r_ir_j), \gamma^+(r_ir_j), \delta^-(r_ir_j) \in \mathtt{Q}^+(r_i))$

(iii) Repeat the step (ii) for all points r_i and r_j to calculate bipolar quadripartitioned neutrosophic out-neighbourhoods $\mathtt{Q}^+(r_i)$.

(iv) Calculate $\mathtt{Q}^+(r_i) \cap \mathtt{Q}^+(r_j)$ for every pair of distinct points r_i and r_j.

(v) Calculate $h[\mathtt{Q}^+(r_i) \cap \mathtt{Q}^+(r_j)]$

(vi) If $\mathtt{Q}^+(r_i) \cap \mathtt{Q}^+(r_j) \neq \emptyset$ then draw an edge r_ir_j

(vii) Repeat step (vi) for every pair of distinct points.

(viii) Provide each edge membership values. With the conditions r_ir_j

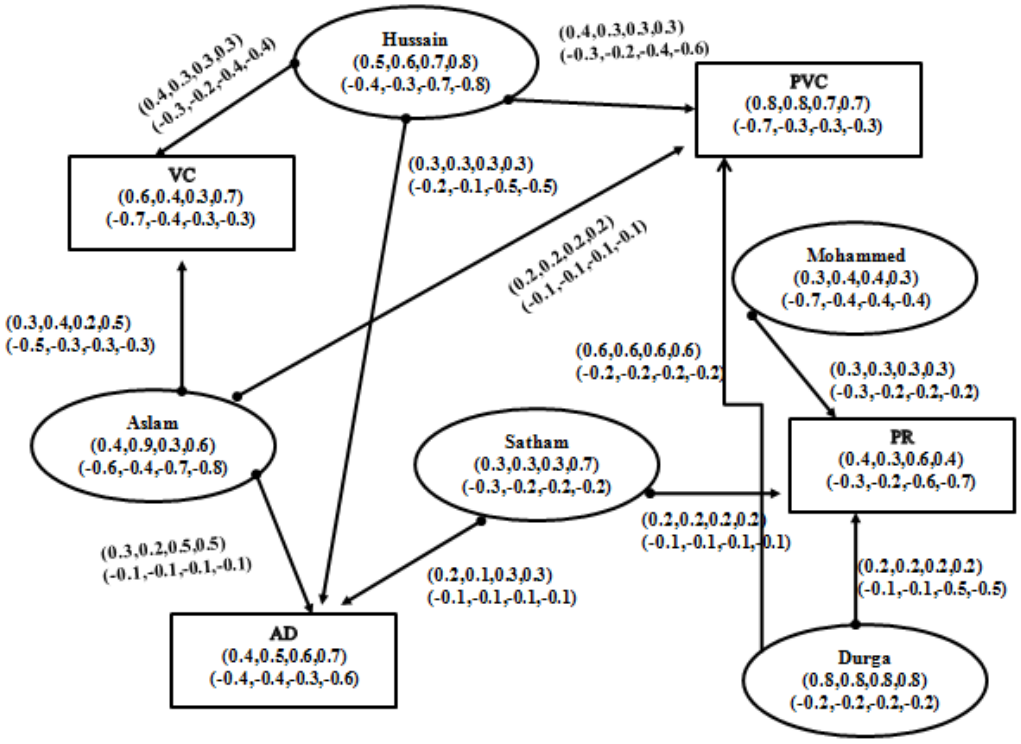


FIGURE 15. Bipolar Quadripartitioned Neutrosophic Digraph

$$\alpha^+(r_ir_j) = (r_i \wedge r_j)h_1[\mathbb{Q}^+(r_i) \cap \mathbb{Q}^+(r_j)], \alpha^-(r_ir_j) = (r_i \vee r_j)h_1[\mathbb{Q}^+(r_i) \cap \mathbb{Q}^+(r_j)]$$
$$\beta^+(r_ir_j) = (r_i \wedge r_j)h_1[\mathbb{Q}^+(r_i) \cap \mathbb{Q}^+(r_j)], \beta^-(r_ir_j) = (r_i \vee r_j)h_1[\mathbb{Q}^+(r_i) \cap \mathbb{Q}^+(r_j)]$$
$$\gamma^+(r_ir_j) = (r_i \vee r_j)h_1[\mathbb{Q}^+(r_i) \cap \mathbb{Q}^+(r_j)], \gamma^-(r_ir_j) = (r_i \wedge r_j)h_1[\mathbb{Q}^+(r_i) \cap \mathbb{Q}^+(r_j)]$$
$$\delta^+(r_ir_j) = (r_i \vee r_j)h_1[\mathbb{Q}^+(r_i) \cap \mathbb{Q}^+(r_j)], \delta^-(r_ir_j) = (r_i \wedge r_j)h_1[\mathbb{Q}^+(r_i) \cap \mathbb{Q}^+(r_j)].$$

(ix) If $r, s_1, s_2, s_3, ..., s_p$ are the applicants computing for designation t, then strength of competition $\alpha(r,t) = (\alpha^+(r,t), \beta^+(r,t), \gamma^+(r,t), \delta^+(r,t), \alpha^-(r,t), \beta^-(r,t), \gamma^-(r,t), \delta^-(r,t))$ of each applicant r for the designation t is

$$\alpha(r,t) = \frac{\alpha^+(rs_1) + ... + \alpha^+(ys_1), \beta^+(ts_1) + ... + \beta^+(ys_1), \gamma^+(rs_1) + ... + \gamma^+(ys_1), \delta^+(rs_1) + ... + \delta^+(ys_1)}{s},$$

$$\frac{\alpha^-(rs_1) + ... + \alpha^-(ys_1), \beta^-(rs_1) + ... + \beta^-(ys_1), \gamma^-(rs_1) + ... + \gamma^-(ys_1), \delta^-(rs_1) + ... + \delta^-(ys_1)}{s}$$

(x) Calculate $Sr(r,t)$, the strength of competition of each applicant r and t.

$$Sr(r,t) = \alpha^+(r,t) + \beta^+(r,t) - \gamma^+(r,t) - \delta^+(r,t) + 2 + \alpha^-(r,t) + \beta^-(r,t) - (\gamma^+(r,t) + \delta^+(r,t)).$$

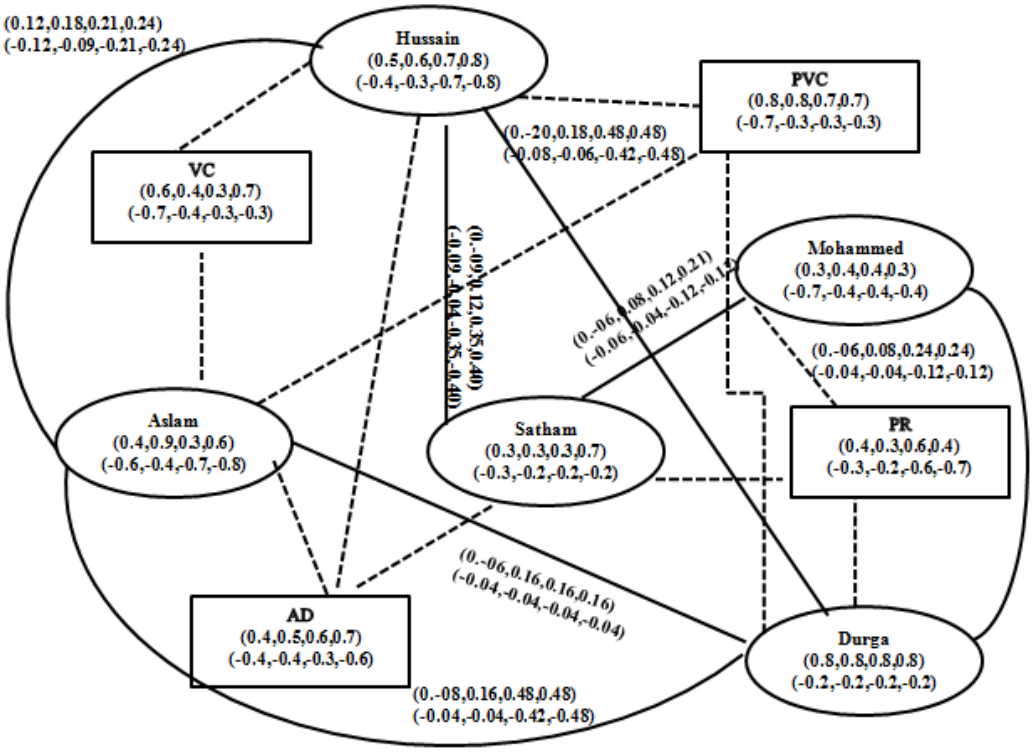


FIGURE 16. Bipolar Quadripartitioned Neutrosophic Competition Graph

5. **Results and Discussion**

In this study, quadripartitioned bipolar neutrosophic competition graphs are introduced as the generalisation of bipolar neutrosophic competition graphs. The sum of positive and negative truth, contradiction, ignorance and falsity membership functions are lies between -4 and 4. Finding a suitable job by the applicant and selecting a suitable applicant is the major issue. The proposed work is applied to find the suitable designation for the applicants with algorithm. We established the job competition model in quadripartitioned bipolar neutrosophic competition graphs to represent competition between the applicants along with algorithm. It generalises the existing works in the literature [19, 27]. A quadripartitioned neutrosophic pythagorean set is a powerful general format framework that generalizes the concept of quadripartitioned neutrosophic sets and neutrosophic pythagorean sets [32]. One can interpret the proposed method which yields the new concept in quadripartitioned bipolar neutrosophic pythagorean graphs. Pentapartitioned neutrosophic sets are a generalization of the single-valued and quadri-partitioned single-valued neutrosophic sets, and utilizes five symbol-valued neutrosophic logic. Pentapartitioned neutrosophic graphs and applications are established in [33]. Using the generalised score function, one can derive the application of pentapartitioned bipolar neutrosophic competition graphs. A new ranking function [34, 35]

can be defined to apply the proposed concept in quadripartitioned triangular neutrosophic graphs. Also the proposed concepts are extended to quadripartitioned neutrosophic statistical models [36].

Conclusion and Future Direction

This manuscript dealt with the new concept of quadripartitioned bipolar neutrosophic competition graphs and the operations like a Cartesian product and direct product of quadripartitioned neutrosophic bipolar competition graphs with their properties have been discussed. The proposed concepts are illustrated with examples and application. The positive membership function denotes the applicant's eligibility and the negative membership function represents the ineligible percentage of applicants. The significant advantage of the proposed work is to find accuracy of suitable jobs by the applicants can be done by employing the quadripartitioned bipolar neutrosophic competition graphs. One can apply the proposed work in various quadripartitioned neutrosophic fields. The concept of considered graphs provided sufficient feasibility to suppress the related concepts on imprecise information. In future, we can extend the proposed graphs based on the properties of quadripartitioned neutrosophic rough competition graphs, interval valued quadripartitioned neutrosophic competition graphs, quadripartitioned neutrosophic trees and applications. Also, one can extend the developed concepts into isomorphic properties and regularity properties in the proposed graph structures. Furthermore, authors planned to use the ideas mentioned in the articles [37, 38] and extend as Quadripartitioned trapezoidal bipolar neutrosophic competition graphs and application in decision making problems.

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Compliance with ethical standards

Conflict of interest: The authors declare that they have no conflict of interest.

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Enhancing Strategies for Improving Safety Risk Management in Fabricated Residential Construction: A Neutrosophic Approach to Minimize Hazards and Ensure Efficiency

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Abstract: Safety risk management in fabricated residential construction persists as an important yet challenging task because of the inherent uncertainties, ambiguities, and complexities associated with construction projects. Traditional risk management approaches often struggle to adequately capture these uncertainties, leading to suboptimal decision-making. In response to this gap, this paper proposes a novel Interval-Valued Fermatean Neutrosophic Set (IVFNS)-based ELECTRE II method for improving safety risk management. The proposed method integrates the strengths of IVFNS, which effectively models uncertainty, indeterminacy, and inconsistency, with the ELECTRE II outranking approach, which provides a structured framework for ranking alternatives based on pairwise comparisons. A public case study from a real construction company is introduced, involving five fabricated residential construction projects, to demonstrate and analyze the applicability of different safety techniques under different choice factors. The quantitative and qualitative analysis demonstrate that our methodology can offer a new solution to advance risk management in fabricated residential construction.

Keywords: Neutrosophic theory, Integrated circuits, Teaching Quality, Decision making

1. Introduction

The construction industry, particularly the residential sector, is one of the most hazardous industries worldwide, with safety risks pose significant challenges to project success and worker well-being [1]. Despite advancements in construction technologis and safety protocols, the dynamic and complex nature of fabricated residential construction projects often leads to uncertainties, ambiguities, and incomplete information in risk assessment and management [2]. Conventional risk management approaches, which rely on deterministic mechanisms, usually falls short in addressing these uncertainties, which can result in insufficient safety procedure and increasing vulnerability to accidents [3].

In recent years, the adoption of prefabricated as well as modular construction technique had been gaining momentum becuase of potential to improved efficiency, reducing waste, and enhancd quality control. However, these methods also present unique safety risks, such as those related to the transportation, assembly, and integration of prefabricated components [2], [3], [4]. These risks are often influenced by a combination of human, technical, and environmental factors, making them difficult to quantify using conventional risk management frameworks [4].

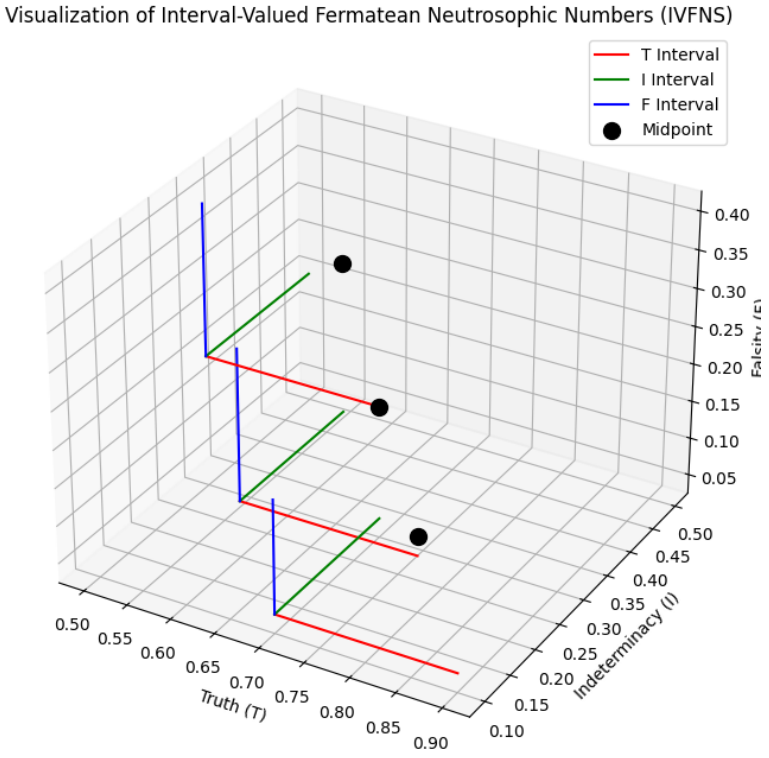

Figure 1. Visualization of Interval-Valued Fermatean Neutrosophic Numbers

For modeling uncertainty in such dynamic environment, Smarandache [5] presented an extension to standard theory of fuzzy logic by developing neutrosophic sets (NSs), which is a theoretical approach for representing information with three membership components: truth (𝕥), indeterminacy (𝕚), and falsity (𝕗). This way, NSs offer an enlightening description uncertainty by simultaneously capturing degrees of truth, falsehood, and ambiguity, making it particularly effective in addressing complex and inconsistent data. According to that, the application of NSs validate valued advantages in broad-range of arena thropugh building a powerful mechanism to factually analyzes and mediates uncertainty in dynamic environments [6]. To further enhance its practicality, interval-valued neutrosophic set (IVNS) [7] was proposed to extend the traditional NSs, where exact values of truth, falsity, and indeterminacy are difficult to determine. Instead of relying on careful values, IVNS represent each component with an interval within the range [0,1].

To that end, this research aims to develop a robust neutrosophic-based framework for identifying, assessing, and mitigating safety risks in fabricated residential construction projects. We propose to integrate expert knowledge, ancient data, and real-time project information into our framework to improve the accuracy and reliability of risk valuations, thereby improve safety outcomes and reduce the likelihood of coincidences. To sum up, the contribution of this research is pointed as follows:

- A comprehensive neutrosophic framework that integrates interval-valued Fermatean neutrosophic sets (IVFNS) with an extended multi-criteria decision-making technique for safety risk management in fabricated residential construction.
- Practical insights into the prioritization of safety measures, enabling construction practitioners to allocate resources more effectively and reduce the likelihood of accidents. A quantitative case study is presented, on which proof-of-concept analysis is performed to assess and evaluate the results of the proposed approach, comparing its performance against previous approaches.

2. Fundamentals & Definitions

Definition 1. IVNS is defined as an extension of standard with three interval-based membership components $T_A(x) = [infT_A(x), supT_A(x)]$, $I_A(x) = [infI_A(x), supI_A(x)]$, and $F_A(x) = [infF_A(x), supF_A(x)] \subseteq [0,1]$.

$$A = \left\{ \left\langle x, \begin{matrix} [infT_A(x), supT_A(x)], [infI_A(x), supI_A(x)], \\ [infF_A(x), supF_A(x)] \end{matrix} \right\rangle \mid x \in X \right\} \quad (1)$$

where

$$0 \leq supT\_A\ (x) + supI\_A\ (x) + supF\_A\ (x) \leq 3 \quad (2)$$

Definition 2. Given two IVNs $A = \{\langle [infT_A(x), supT_A(x)], [infI_A(x), supI_A(x)], [infF_A(x), supF_A(x)] \rangle\}$, and $A = \{\langle [infT_B(x), supT_B(x)], [infI_B(x), supI_B(x)], [infF_B(x), supF_B(x)] \rangle\}$, you can apply the following mathematical operations:

- Complement

$$A^c = \langle [infT_A(x), supT_A(x)], [1 - infI_A(x), 1 - supI_A(x)], [infF_A(x), supF_A(x)] \rangle \quad (3)$$

- Addition

(4)

$$A \oplus B = \langle [infT_A(x) + infT_B(x) - infT_A(x) \cdot infT_B(x), supT_A(x) + supT_B(x) - supT_A(x) \cdot supT_B(x)], [infI_A(x) \cdot infI_B(x), supI_A(x) \cdot supI_B(x)], [infF_A(x) \cdot infF_B(x), supF_A(x) \cdot supF_B(x)] \rangle$$

- Multiplication

$$A \otimes B = \langle [infT_A(x) \cdot infT_B(x), supT_A(x) \cdot supT_B(x)], [infI_A(x) + infI_B(x) - infI_A(x) \cdot infI_B(x), supI_A(x) \cdot supI_B(x) - supI_A(x) \cdot supI_B(x)], [infF_A(x) + infF_B(x) - infF_A(x) \cdot infF_B(x), supF_A(x) + supF_B(x) - supF_A(x) \cdot supF_B(x)] \rangle \quad (5)$$

Definition 3. Given two IVNS Given two IVNs $A = \{\langle [infT_A(x), supT_A(x)], [infI_A(x), supI_A(x)], [infF_A(x), supF_A(x)] \rangle\}$, and $B = \{\langle [infT_B(x), supT_B(x)], [infI_B(x), supI_B(x)], [infF_B(x), supF_B(x)] \rangle\}$, the following relations apply as follows:

$$A \subseteq B \quad iff \quad infT_A(x) \leq infT_B(x), supT_A(x) \leq supT_B(x); infI_A(x) \geq infI_B(x), supI_A(x) \geq supI_B(x);\ infF_A(x) \geq infF_B(x), supF_A(x) \geq supF_B(x) \quad (6)$$

$$A = B\ iff\ A \subseteq B\ and\ B \subseteq A. \quad (7)$$

Definition 4. Given two IVNS Given two IVNs $A = \{\langle [infT_A(x), supT_A(x)], [infI_A(x), supI_A(x)], [infF_A(x), supF_A(x)] \rangle\}$, and $B = \{\langle [infT_B(x), supT_B(x)], [infI_B(x), supI_B(x)], [infF_B(x), supF_B(x)] \rangle\}$, and $C = \{\langle [infT_C(x), supT_C(x)], [infI_C(x), supI_C(x)], [infF_C(x), supF_C(x)] \rangle\}$, the following relations apply as follows:

$$\begin{aligned}
&(1) A + B = B + A,\\
&(2) A \cdot B = B \cdot A,\\
&(3) \lambda(A + B) = \lambda A + \lambda B, \lambda > 0,\\
&(4) (A \cdot B)^{\lambda} = A^{\lambda} + B^{\lambda}, \lambda > 0,\\
&(5) \lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A, \lambda_1 > 0, \lambda_2 > 0\\
&(6) A^{\lambda_1} \cdot A^{\lambda_2} = A^{(\lambda_1 + \lambda_2)}, \lambda_1 > 0, \lambda_2 > 0,\\
&(7) (A + B) + C = A + (B + C),\\
&(8) (A \cdot B) \cdot C = A \cdot (B \cdot C)
\end{aligned} \quad (8)$$

Definition 5. The interval-valued fermatean neutrosophic set (IVFNs) is defined as the extension of standard IVNSs with three interval-based membership components $T_A(x) = [infT_A(x), supT_A(x)]$, $I_A(x) = [infI_A(x), supI_A(x)]$, and $F_A(x) = [infF_A(x), supF_A(x)] \subseteq [0,1]$.

$$A = \left\{ \left\langle x, \begin{matrix} [infT_A(x), supT_A(x)], [infI_A(x), supI_A(x)], \\ [infF_A(x), supF_A(x)] \end{matrix} \right\rangle \mid x \in X \right\} \quad (9)$$

where

$$0 \le supT_A(x) + supI_A(x) + supF_A(x) \le 3 \quad (10)$$

$$0 \le [infT_A(x)]^3 + [infF_A]^3 \le 1 \; and \; 0 \le [infI_A(x)]^3 \le 1 \quad (11)$$

$$0 \le [infT_A(x)]^3 + [infI_A(x)]^3 + [infF_A(x)]^3 \le 2 \quad (12)$$

$$0 \le [supT_A(x)]^3 + [supI_A(x)]^3 + [supF_A(x)]^3 \le 2 \quad (13)$$

Given two IVFNs $A = \{\langle [infT_A(x), supT_A(x)], [infI_A(x), supI_A(x)], [infF_A(x), supF_A(x)]\rangle\}$, and $A = \{\langle [infT_B(x), supT_B(x)], [infI_B(x), supI_B(x)], [infF_B(x), supF_B(x)]\rangle\}$, you can apply the following mathematical operations:

- Complement

$$A^c = \left[\sqrt[3]{1 - [1 - \{infT_A(x)\}^3]^{\mu}}, \sqrt[3]{1 - [1 - \{supT_A(x)\}\}^3]^{\mu}}\right], [\{infI_A(x)\}^3, \{supI_A(x)\}^3], [\{infF_A(x)\}^3, \{infF_A(x)\}^3] \quad (14)$$

- Addition

$$A \oplus B = \left\langle \left[\begin{matrix} \sqrt[3]{\{infT_A(x)\}^3 + \{infT_B(x)\}^3 - \{infT_A(x)\}^3 \cdot \{\{infT_B(x)\}^3}, \\ \sqrt[3]{\{supT_A(x)\}^3 + \{supT_B(x)\}^3 - \{supT_A(x)\}^3 \cdot \{supT_B(x)\}^3} \end{matrix}\right], [infI_A(x) \cdot infI_B(x), supI_A(x) \cdot supI_B(x)], [infF_A(x) \cdot infF_B(x), supF_A(x) \cdot supF_B(x)]\right\rangle \quad (15)$$

- Multiplication

$$A \oplus B = \left\langle \begin{matrix} [infT_A(x) \cdot infT_B(x), supT_A(x) \cdot supT_B(x)], \\ \left[\begin{matrix} \sqrt[3]{\{infI_A(x)\}^3 + \{infI_B(x)\}^3 - \{infI_A(x)\}^3 \cdot \{infI_B(x)\}^3}, \\ \sqrt[3]{\{supI_A(x)\}^3 + \{supI_B(x)\}^3 - \{supI_A(x)\}^3 \cdot \{supI_B(x)\}^3} \end{matrix}\right], \\ \left[\begin{matrix} \sqrt[3]{\{infF_A(x)\}^3 + \{infF_B(x)\}^3 - \{infF_A(x)\}^3 \cdot \{infF_B(x)\}^3}, \\ \sqrt[3]{\{supF_A(x)\}^3 + \{supF_B(x)\}^3 - \{supF_A(x)\}^3 \cdot \{supF_B(x)\}^3} \end{matrix}\right] \end{matrix} \right\rangle \quad (16)$$

Definition 6: The weighted averaging operator for IVNS is defined to aggregate a number of IVNS weighted by the weight vector $Y = (y_1, \dots, y_j, \dots, y_n)$ and $\sum_{j=1}^{n} y_j = 1$.

$$WA_{IVNNs}(x_1, \dots, x_j, \dots, x_n) = \sum_{j=1}^{n} y_j x_j = \left\langle \begin{matrix} \left[1 - \prod_{j=1}^{n}(1 - infT_A(x))^{y_j}, 1 - \prod_{j=1}^{n}(1 - supT_A(x))^{y_j}\right] \\ \left[\prod_{j=1}^{n}(infI_A(x))^{y_j}, \prod_{j=1}^{n}(supI_A(x))^{y_j}\right], \left[\prod_{j=1}^{n}(infF_A(x))^{y_j}, \prod_{j=1}^{n}(supF_A(x))^{y_j}\right] \end{matrix} \right\rangle \quad (17)$$

Definition 7. Given an IVNS $A = \{\langle [infT_A(x), supT_A(x)], [infI_A(x), supI_A(x)], [infF_A(x), supF_A(x)]\rangle\}$, the de-neutrosophication can be calculated as given below:

$$A_{den} = \left(\frac{\left(infT_A(x) + supT_A(x)\right)}{2} + \left(1 - \frac{\left(infI_A(x) + supI_A(x)\right)}{2}\right)\left(supI_A(x)\right) - \left(\frac{\left(infF_A(x) + supF_A(x)\right)}{2}\right)\left(1 - supF_A(x)\right)\right) \tag{18}$$

Definition 8: Given an IVFNS $A = \{\langle [infT_A(x), supT_A(x)], [infI_A(x), supI_A(x)], [infF_A(x), supF_A(x)]\rangle\}$, the score function of can be computed as follows:

$$S(A) = \frac{[infT_A(x)]^3 + [supT_A(x)]^3 + [infI_A(x)]^3 + [supI_A(x)]^3 + [infF_A(x)]^3 + [supF_A(x)]^3}{2} \tag{19}$$

3. Research Method

Herein, we propose a detailed explanation of the proposed IVNS decision-making approaches for evaluating the safety risks in fabricated residential construction, generating useful insights that enable the decision makers to appropriately and efficiently manage the construction sites. Our framework begin with integration of criteria weighting approach that take into account contrast intensity (reflected by standard deviation) as well as the collaboration among criteria [8]. This pass through the following steps.

Step 1, we create IVFNS decision matrix, where the alternatives are evaluated against the criteria using the triplet

$$.D_{IVFN} = \begin{bmatrix} \begin{Bmatrix} [infT_A(x), supT_A(x)], \\ [infI_A(x), supI_A(x)], \\ [infF_A(x), supF_A(x)] \end{Bmatrix}_{11} & \cdots & \begin{Bmatrix} [infT_A(x), supT_A(x)], [infI_A(x), supI_A(x)], \\ [infF_A(x), supF_A(x)] \end{Bmatrix}_{1m} \\ \vdots & \ddots & \vdots \\ \begin{Bmatrix} [infT_A(x), supT_A(x)], \\ [infI_A(x), supI_A(x)], \\ [infF_A(x), supF_A(x)] \end{Bmatrix}_{n1} & \cdots & \begin{Bmatrix} [infT_A(x), supT_A(x)], [infI_A(x), supI_A(x)], \\ [infF_A(x), supF_A(x)] \end{Bmatrix}_{nm} \end{bmatrix} \tag{20}$$

Step 2, we compute the normalized IVFNS decision matrix, D_{IVFN}^{norm}, as follows:

$$\tilde{T}_{ij} = \left[\frac{\inf T_{ij}}{\max(\sup T_j)}, \frac{\sup T_{ij}}{\max(\sup T_j)}\right], \tilde{I}_{ij} = \left[\frac{\inf I_{ij}}{\max(\sup I_j)}, \frac{\sup I_{ij}}{\max(\sup I_j)}\right]$$
$$\tilde{F}_{ij} = \left[\frac{\inf F_{ij}}{\max(\sup F_j)}, \frac{\sup F_{ij}}{\max(\sup F_j)}\right] \quad (21)$$

when it comes to cost criteria, we can use the complement of the IVFNS values:

$$\tilde{T}_{ij} = \left[1 - \frac{\sup T_{ij}}{\max(\sup T_j)}, 1 - \frac{\inf T_{ij}}{\max(\sup T_j)}\right], \tilde{I}_{ij} = \left[1 - \frac{\sup I_{ij}}{\max(\sup I_j)}, 1 - \frac{\inf I_{ij}}{\max(\sup I_j)}\right]$$
$$\tilde{F}_{ij} = \left[1 - \frac{\sup F_{ij}}{\max(\sup F_j)}, 1 - \frac{\inf F_{ij}}{\max(\sup F_j)}\right] \quad (22)$$

Step 3: we perform de-neutrosophication of each IVFNS number to obtain crisp decision matrix.

$$A_{den} = \left(\frac{(inf T_A(x) + sup T_A(x))}{2} + \left(1 - \frac{(inf I_A(x) + sup I_A(x))}{2}\right)(sup I_A(x)) - \left(\frac{(inf F_A(x) + sup F_A(x))}{2}\right)(1 - sup F_A(x))\right) \quad (23)$$

Step 4, we compute the STD associated with each criterion, S_j based on the following:

$$S_j = \sqrt{\frac{\sum_{i=1}^{m}([D_{IVFN}^{norm}]_{ij} - \overline{[D_{IVFN}^{norm}]_j})^2}{n-1}} \quad (24)$$

Step 5, the correlation between different criteria I calculated as follows:

$$r_{bj} = \frac{\sum_{b,j=1}^{n}([D_{IVFN}^{norm}]_{ib} - \overline{[D_{IVFN}^{norm}]_b})([D_{IVFN}^{norm}]_{ij} - \overline{[D_{IVFN}^{norm}]_j})}{\sqrt{\sum_{b=1}^{n}([D_{IVFN}^{norm}]_{ib} - [D_{IVFN}^{norm}]_b)^2 \sum_{j=1}^{n}([D_{IVFN}^{norm}]_{ij} - \overline{[D_{IVFN}^{norm}]_j})^2}} \quad (25)$$

Step 5, the contradictory variation foreach pair of criteria is computed as follows:

$$P_j \leftarrow \sum_{b=1}^{n}(1 - r_{bj}) \quad (26)$$

Step 5, the volume of information to be measured by $j-th$ criterion, C_j, and the corresponding weights are computed as follows:

$$C_j \leftarrow S_j P_j \quad (27)$$
$$w_j \leftarrow \frac{C_j}{\sum_{j=1}^{n} C_j} \quad (28)$$

Following, we apply IVFNS-ELECTRE-II algorithm to decide on the best construction alternatives that can provide better risk management (Refer to Algorithm 1).

Algorithm 1: pseudocode of the IVFNS-ELECTRE-II.

Input: The concordance threshold, discordance threshold, criteria weights w_j, and D_{IVFN}^{norm}

1 Make use of normalized Decision Matrix, D_{IVFN}^{norm}.

2 Apply weights to decision matrix, D_{IVFN}^{norm}: $[D_{IVFN}^{norm}]_w = [D_{IVFN}^{norm}]_{ij} \times w_j$.

3 Decide the Concordance Sets: $c_{kl} = \left\{j, {D_{IVFN}^{norm}}_{kj} \geq y_{lj}\right\}$ for J = 1,2,3, … , n.

4 Decide the Discordance Sets: $d_{kj} = \left\{j, {D_{IVFN}^{norm}}_{kj} < y_{lj}\right\}$ for J = 1,2,3, … , n

5 Build the Concordance Matrices $\mathcal{F} = \begin{pmatrix} - & \mathrm{cm}_{12} & \mathrm{cm}_{13} & \dots & \mathrm{cm}_{1n} \\ \mathrm{cm}_{21} & - & f_{23} & \dots & \mathrm{cm}_{2n} \\ \mathrm{cm}_{31} & \mathrm{cm}_{32} & - & \dots & \mathrm{cm}_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \mathrm{cm}_{n1} & \mathrm{cm}_{n2} & \mathrm{cm}_{n3} & \dots & - \end{pmatrix}$.

6 Build the Discordance Matrices $\mathcal{G} = \begin{pmatrix} - & dd_{12} & dd_{13} & \dots & dd_{1n} \\ dd_{21} & - & dd_{23} & \dots & dd_{2n} \\ dd_{31} & dd_{32} & - & \dots & dd_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ dd_{n1} & dd_{n2} & dd_{n3} & \dots & - \end{pmatrix}$.

Determine the concordance based dominance matrix:

7 $$H = \begin{pmatrix} - & h_{12} & h_{13} & \dots & h_{1n} \\ h_{21} & - & h_{23} & \dots & h_{2n} \\ h_{31} & h_{32} & - & \dots & h_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ h_{n1} & h_{n2} & h_{n3} & \dots & - \end{pmatrix}, where\ h_{pq} = \begin{cases} 1, & \mathrm{cm}_{pq} \geq \overline{\mathrm{cm}}, \\ 0, & \mathrm{cm}_{pq} < \overline{\mathrm{cm}}. \end{cases} such\ that\ \mathrm{cm}$$
$$= \frac{1}{n(n-1)} \sum_{\substack{p=1 \\ p \neq q}}^{n} \sum_{\substack{q=1 \\ q \neq p}}^{n} \mathrm{cm}_{pq}$$

Determine the discordance based dominance index matrix: $L =$

8 $\begin{pmatrix} - & l_{12} & l_{13} & \dots & l_{1n} \\ l_{21} & - & l_{23} & \dots & l_{2n} \\ l_{31} & l_{32} & - & \dots & l_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \dots & - \end{pmatrix}, where\ l_{pq} = \begin{cases} 1, & g_{pq} < \overline{dd}, \\ 0, & g_{pq} \geq \overline{dd}. \end{cases}$ such that $\overline{dd} =$
$\frac{1}{n(n-1)} \sum_{\substack{p=1 \\ p \neq q}}^{n} \sum_{\substack{q=1 \\ q \neq p}}^{n} dd_{pq}$

9 Construct the strong and weak outranking relations

4. Results and Discussions

This section introduces a quantitative case study for our work based on a construction company, which is planning to implement safety measures using either of five fabricated residential

construction projects (alternatives). In Project A, the company introduce a high-rise modular residential building in an urban area. In Project B, the company emphasize Low-rise prefabricated housing development in a suburban area. In project C, the company emphasize mixed-use residential and commercial modular complex in a semi-urban area. In project D, the company emphasize affordable housing projects using prefabricated components in a rural area. In project E, the company emphasize Luxury modular villas in a coastal area with challenging environmental conditions. As shown, the project varies in many characteristics that affect the safety risks and related costs. These criteria include Implementation Cost (C1), Operational Disruption (C2), Risk Reduction Effectiveness (C3), Worker Compliance and Acceptance (C4), and Long-Term Sustainability (C5).

In Table 1, we show the results of aggregating the IVFNS decision matrices from different experts, where each cell represents the IVFNS evaluation of an alternative A_i with respect to the criterion C_j:

Table 1. Aggregated IVFNS decision matrix from different domain experts.

Alternatives	**C_1**	**C_2**	**C_3**	**C_4**	**C_5**
A_1	⟨[0.6, 0.8], [0.2, 0.4], [0.1, 0.3]⟩	⟨[0.5, 0.7], [0.3, 0.5], [0.2, 0.4]⟩	⟨[0.7, 0.9], [0.1, 0.3], [0.05, 0.2]⟩	⟨[0.6, 0.8], [0.2, 0.4], [0.1, 0.3]⟩	⟨[0.5, 0.7], [0.3, 0.5], [0.2, 0.4]⟩
A_2	⟨[0.7, 0.9], [0.1, 0.3], [0.05, 0.2]⟩	⟨[0.6, 0.8], [0.2, 0.4], [0.1, 0.3]⟩	⟨[0.5, 0.7], [0.3, 0.5], [0.2, 0.4]⟩	⟨[0.7, 0.9], [0.1, 0.3], [0.05, 0.2]⟩	⟨[0.6, 0.8], [0.2, 0.4], [0.1, 0.3]⟩
A_3	⟨[0.5, 0.7], [0.3, 0.5], [0.2, 0.4]⟩	⟨[0.7, 0.9], [0.1, 0.3], [0.05, 0.2]⟩	⟨[0.6, 0.8], [0.2, 0.4], [0.1, 0.3]⟩	⟨[0.5, 0.7], [0.3, 0.5], [0.2, 0.4]⟩	⟨[0.7, 0.9], [0.1, 0.3], [0.05, 0.2]⟩
A_4	⟨[0.6, 0.8], [0.2, 0.4], [0.1, 0.3]⟩	⟨[0.5, 0.7], [0.3, 0.5], [0.2, 0.4]⟩	⟨[0.7, 0.9], [0.1, 0.3], [0.05, 0.2]⟩	⟨[0.6, 0.8], [0.2, 0.4], [0.1, 0.3]⟩	⟨[0.5, 0.7], [0.3, 0.5], [0.2, 0.4]⟩
A_5	⟨[0.7, 0.9], [0.1, 0.3], [0.05, 0.2]⟩	⟨[0.6, 0.8], [0.2, 0.4], [0.1, 0.3]⟩	⟨[0.5, 0.7], [0.3, 0.5], [0.2, 0.4]⟩	⟨[0.7, 0.9], [0.1, 0.3], [0.05, 0.2]⟩	⟨[0.6, 0.8], [0.2, 0.4], [0.1, 0.3]⟩
A_6	⟨[0.4,0.6],[0.3,0.5], [0.2,0.4]⟩	⟨[0.5,0.7],[0.2,0.4], [0.1,0.3]⟩	⟨[0.6,0.8],[0.1,0.3], [0.05,0.2]⟩	⟨[0.5,0.7],[0.3,0.5],[0.2,0.4]⟩	⟨[0.7,0.9],[0.1,0.3],[0.05,0.2]⟩

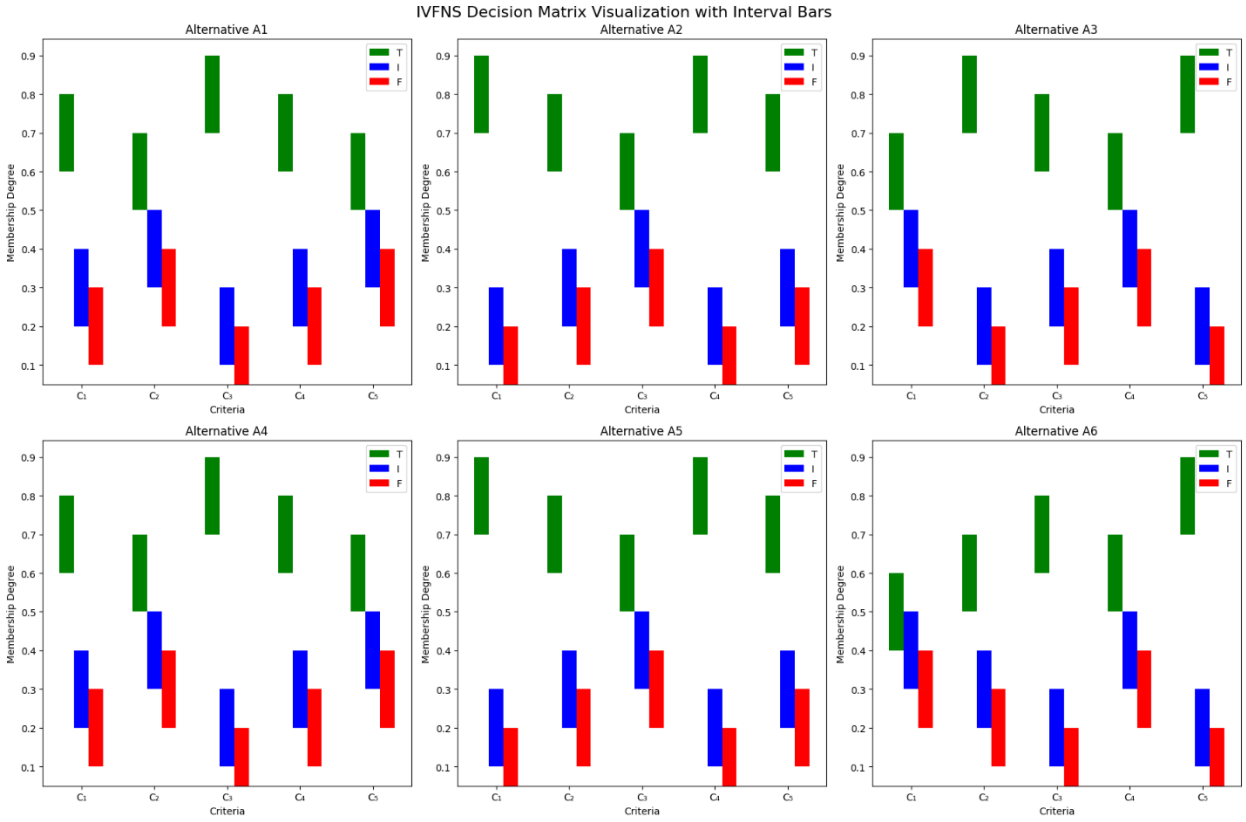

Figure 2. visual representation of IVFNS decision matrix.

Then, we convert the IVFNS decision matrix into a crisp matrix using a predefined de-neutrosophication method.

Table 2: Quantitative results of De-neutrosophication of the IVFNS Decision Matrix.

Criteria / Alternative	C1	C2	C3	C4	C5
$\mathbf{A_1}$	0.84	0.72	0.94	0.84	0.72
$\mathbf{A_2}$	0.94	0.84	0.72	0.94	0.84
$\mathbf{A_3}$	0.72	0.94	0.84	0.72	0.94
$\mathbf{A_4}$	0.84	0.72	0.94	0.84	0.72
$\mathbf{A_5}$	0.94	0.84	0.72	0.94	0.84
$\mathbf{A_6}$	0.62	0.74	0.84	0.72	0.94

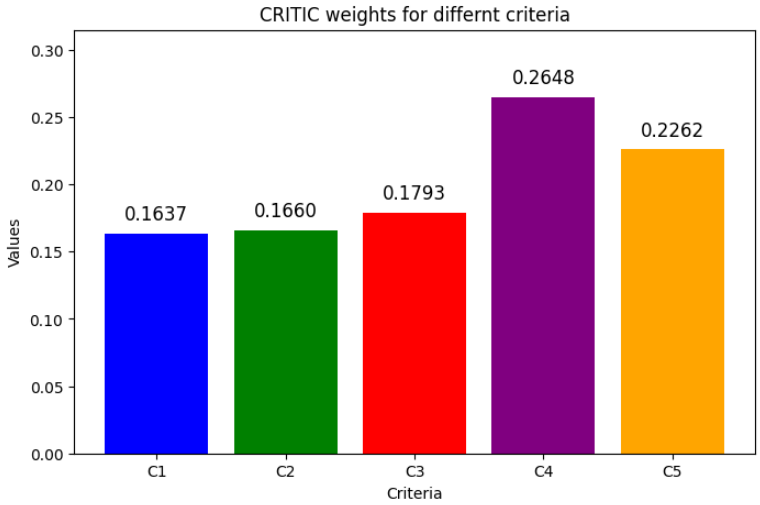

Figure 3. Qualitative representation of CRITIC weights.

Based on IVFNS-based CRITIC method, the weights of the criteria are presented in Figure 3, representing the relative importance of each construction criterion in the safety risk management

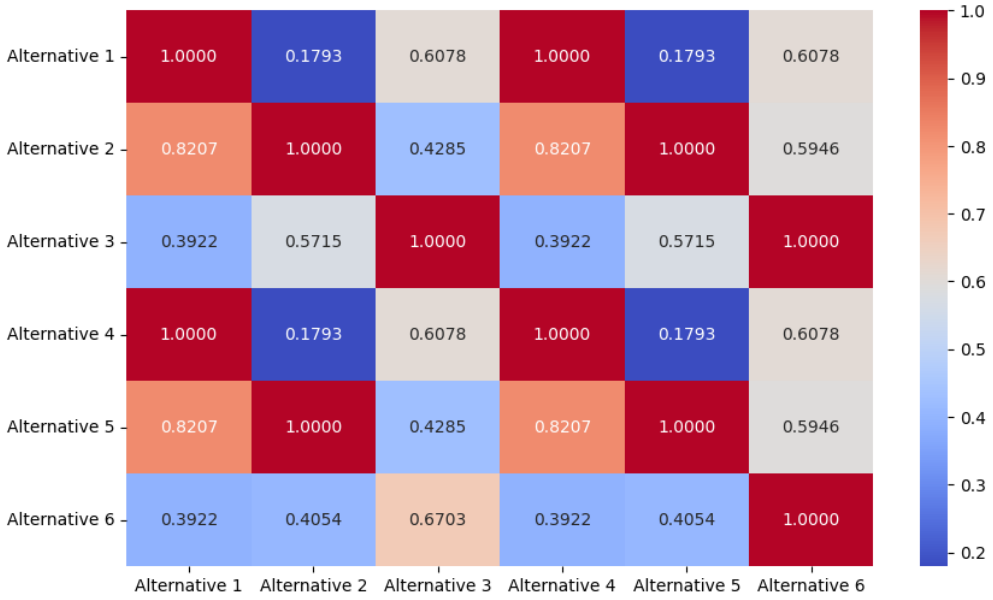

Figure 4. Concordance Matrix

process.

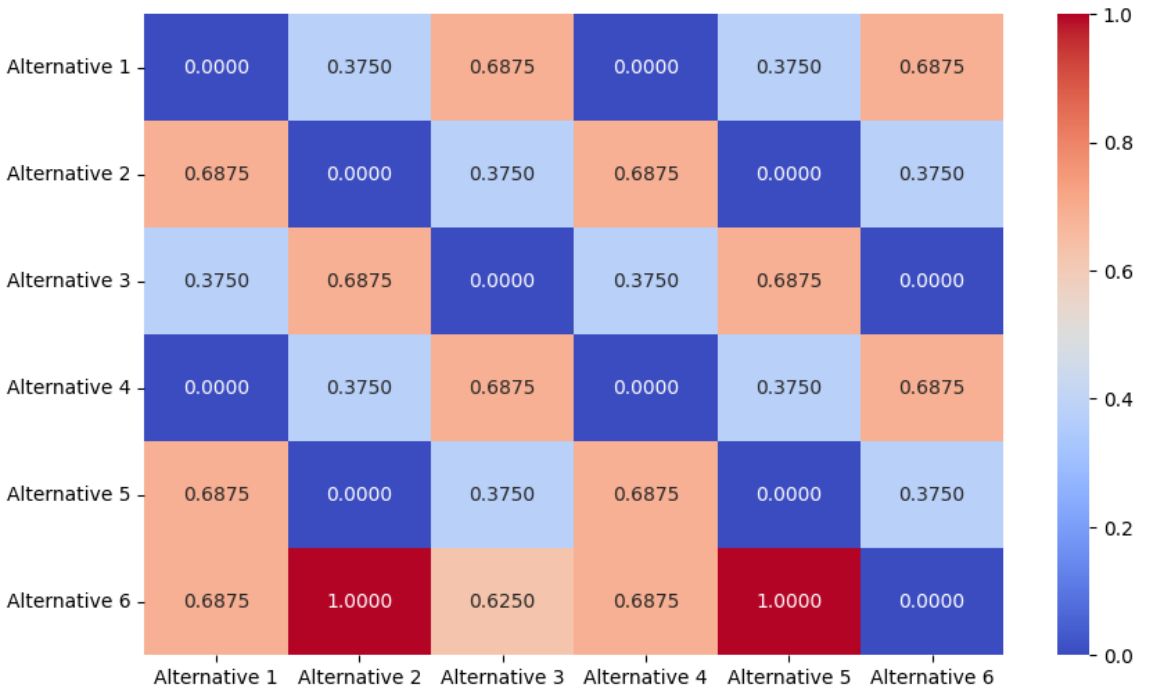

Figure 5. Discordance Matrix

To offer an informative summary of stages of arrangement between the alternatives, we present the concordance matrix are presented in Figure 4. The presented row values explain the degree to which a particular alternative being concordant with other alternatives based on the weighted upright of the different safe construction criteria. Moreover, to show the degree of divergence between alternatives, we present the discordance in Figure 5, which represent a corresponding matrix for the concordance matrix, where we could recognize the zones in which alternatives show a notable variance.

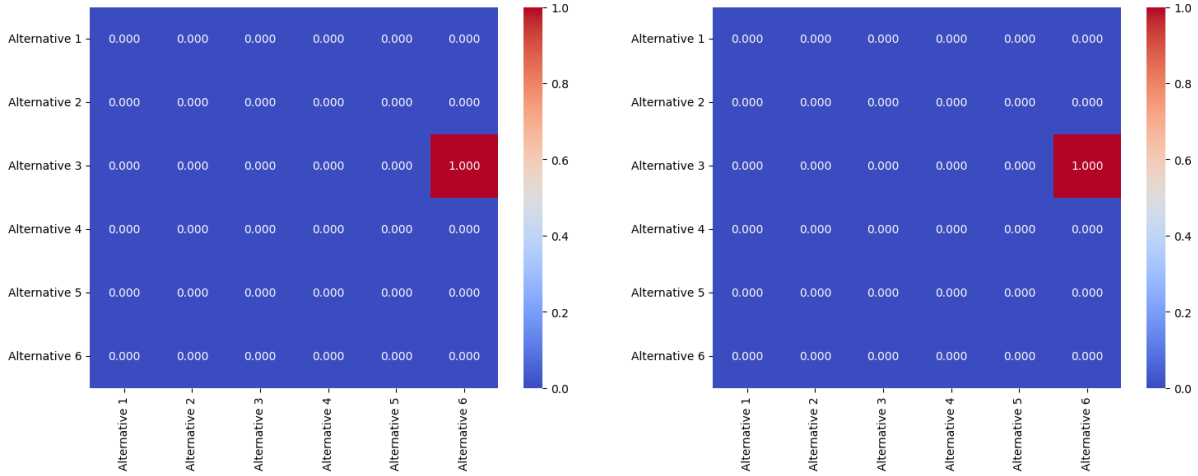

Figure 6. Strong vs weak dominance Matrix

To ascertain the dominant correlations among options, the dominance weak matrix and dominance strong matrix, displayed in Figure 6, combines the data from the concordance and discordance matrices. Depending on the combination assessment of concordance and discordance levels, every choice indicates whether one option is superior than others.

The ranking results obtained from the proposed IVFNS-based ELECTRE II method were compared with those from IVNS-ELECTRE-II, Combinative Distance-Based Assessment(CODAS) reported in [9], and TOPSIS to evaluate the consistency and robustness of our framework. As shown in Table 2, the rankings produced by all methods exhibit a high degree of agreement, particularly for the top-performing alternatives, which underscores the reliability of the proposed IVFNS-ELECTRE-II. However, minor discrepancies were observed in the middle and lower ranks, primarily because of contradictory mathematical basis. For instance, while TOPSIS relies on distance-based measures to identify the ideal and anti-ideal solutions, CODAS incorporates a threshold-based approach to assess the relative performance of alternatives. In contrast, our solution exploits outranking relations and IVFNS to capture the inherent uncertainty, in the safety decision-making process in construction sites.

Table 2. Comparative analysis

Rank	**IVFNS-ELECTRE-II**	**IVNS-ELECTRE-I**[10]	**CODAS** [9]	**TOPSIS**[11]
1	A6	A6	A6	A6
2	A1	A1	A5	A4
3	A2	A2	A2	A1
4	A4	A4	A4	A5
5	A5	A5	A1	A2
6	A3	A3	A3	A3

5. Conclusions

This study proposed a novel IVFNS-based ELECTRE-II method for improving safety risk management in fabricated residential construction. Our framework takes advantage of both IVFNS and the ELECTRE II outranking approach to build effective solution for addressing the inherent uncertainty in the task of evaluating and prioritizing safety measures. The application of our IVFNS-based ELECTRE-II method to a real-world case study demonstrated its practicality and effectiveness in ranking construction projects based on multiple cost and benefit criteria. The proof-of-concept of comparative analysis against other decision-making methods revealed a high degree of consistency in the ranking results, while also highlighting the unique advantages of the proposed approach in handling complex and ambiguous decision environments.

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The definite integrals of 2- refined neutrosophic functions and its properties

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Abstract: This article aims to study the definite integrals of 2- refined neutrosophic functions and its properties. The definite integrals of 2- refined neutrosophic functions are defined in this study, and a set of theories are presented, including the Fundamental theorem of 2- refined neutrosophic integral calculus, the mean-value theorem of 2- refined neutrosophic integral calculus with its two parts, in addition to the properties of the definite integrals of 2- refined neutrosophic functions. Also, we used the n-Refined AH-Isometry [11].

Keywords: definite integrals; 2- refined neutrosophic functions; properties; mean-value theorem.

1. Introduction and Preliminaries

As an alternative to the existing logics, Smarandache proposed the neutrosophic logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R \text{ or } C$ [1]. Agboola introduced the concept of refined neutrosophic algebraic structures [2]. In addition, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. Abobala presented the papers on some special substructures of refined neutrosophic rings and a study of ah-substructures in n-refined neutrosophic vector spaces [6-7].

Alhasan.Y and Abdulfatah. R also presented a study on the division of refined neutrosophic number [8].

There are papers presented in n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Neutrosophic Rings I [4-5] and studying the integral calculus according to the logic of neutrosophic by presenting a set of papers on that [9-10].

In addition, the AH-Isometry was extended to n-Refined AH-Isometry by Smarandache & Abobala in 2024 [11]

2. The definite integrals of 2- refined neutrosophic functions

Theorem 1 (Fundamental theorem of 2- refined neutrosophic integral calculus)

Let be $f(x, I_1, I_2)$ a continuous 2- refined neutrosophic function defined in $[a + a_1I_1 + a_2I_2, b + b_1I_1 + b_2I_2]$, and let $F(x, I_1, I_2)$ be the anti-derivative of $f(x, I_1, I_2)$, that is

$$\int f(x, I_1, I_2)dx = F(x, I_1, I_2)$$

then:

$$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = F(a + a_1I_1 + a_2I_2) - F(b + b_1I_1 + b_2I_2)$$

Where a, a_1, a_2, b, b_1, b_2 are real number, I_1, I_2 represent indeterminacy.

Example 1

$$(1)\ \int_{-2+I_1-I_2}^{5-I_1+I_2} (-3I_1 + 2x)dx = [-3I_1x + x^2]_{-2+I_1-I_2}^{5-I_1+I_2}$$

$$= [-3I_1(5 - I_1 + I_2) + (5 - I_1 + I_2)^2] - [3I_1(-2 + I_1 - I_2) + (-2 + I_1 - I_2)^2]$$

$$= -15I_1 + 25 - 11I_1 + 12I_2 - 6I_1 - 4 + 5I_1 - 5I_2$$

$$= 21 - 27I_1 + 7I_2$$

$$(2)\ \int_0^{\frac{\pi}{2}+\pi I_1+\pi I_2} cos\left(x - \pi I_1 + \frac{\pi}{4}I_2\right)dx = \left[sin\left(x - \pi I_1 + \frac{\pi}{4}I_2\right)\right]_0^{\frac{\pi}{2}+\pi I_1+\pi I_2}$$

$$= \left[sin\left(\frac{\pi}{2} + \pi I_1 + \pi I_2 - \pi I_1 + \frac{\pi}{4}I_2\right)\right] - \left[sin\left(-\pi I_1 + \frac{\pi}{4}I_2\right)\right]$$

$$= \left[sin\left(\frac{\pi}{2} + \frac{5\pi}{4}I_2\right)\right] - \left[sin\left(-\pi I_1 + \frac{\pi}{4}I_2\right)\right]$$

$$= sin\left(\frac{\pi}{2}\right) + I_1\left[sin\left(\frac{7\pi}{4}\right) - sin\left(\frac{7\pi}{4}\right)\right] + I_2\left[sin\left(\frac{7\pi}{4}\right) - sin\left(\frac{\pi}{2}\right)\right] - I_1\left[sin\left(-\frac{3\pi}{4}\right) - sin\left(\frac{\pi}{4}\right)\right] - I_2 sin\left(\frac{\pi}{4}\right)$$

$$= 1 + \sqrt{2}I_1 + \left(-\sqrt{2} - 1\right)I_2$$

$$(3)\ \int_{2I_2}^{-2+I_1-I_2} -2x(4 - 2I_1 + 5I_2 - x^2)^2dx = \left[\frac{(4 - 2I_1 + 5I_2 - x^2)^3}{3}\right]_{2I_2}^{-2+I_1-I_2}$$

$$= \left[\frac{(4 - 5I_1 + 4I_2 - (-2 + I_1 - I_2)^2)^3}{3}\right] - \left[\frac{(4 - 5I_1 + 4I_2 - (2I_2)^2)^3}{3}\right]$$

$$= \left[\frac{(4 - 5I_1 + 4I_2 - 4 + 5I_1 - 5I_2)^3}{3}\right] - \left[\frac{(4 - 5I_1 + 4I_2 - 4I_2)^3}{3}\right]$$

$$= \left[\frac{-1}{3}I_2\right] - \left[\frac{64}{3} - \frac{65}{3}I_1\right] = -\frac{64}{3} + \frac{65}{3}I_1 - \frac{1}{3}I_2$$

$$(4)\int_{5+29I_1+6I_2}^{2+26I_1+25I_2}\frac{1}{2\sqrt{x-1-2I_1-I_2}}dx=\left[\sqrt{x-1-2I_1-I_2}\right]_{5+29I_1+6I_2}^{2+26I_1+25I_2}$$

$$=\sqrt{2+26I_1+25I_2-1-2I_1-I_2}-\sqrt{5+29I_1+6I_2-1-2I_1-I_2}$$

$$=\sqrt{1+24I_1+24I_2}-\sqrt{4+27I_1+5I_2}$$

$$=1+2I_1+4I_2-2-3I_1-2I_2$$

$$=-1-I_1+2I_2$$

Theorem 2 (The mean- value theorem of 2- refined neutrosophic integral calculus_ part I)

If $f(x,I_1,I_2)$ is continuous on an interval, then we say that $f(x,I_1,I_2)$ has an anti- derivative on that interval. In specific, if $a+a_1I_1+a_2I_2$ is any point in the interval, then $f(x,I_1,I_2)$ defined by:

$$(1)\ \frac{d}{dx}\left[\int_{a+a_1I_1+a_2I_2}^{x}f(t,I_1,I_2)dt\right]=f(x,I_1,I_2)$$

$$(2)\ \frac{d}{dx}\left[\int_{x}^{a+a_1I_1+a_2I_2}f(t,I_1,I_2)dt\right]=-f(x,I_1,I_2)$$

Example 2

$$(1)\ \frac{d}{dx}\left[\int_{\pi+\frac{\pi}{3}I_1+\frac{\pi}{6}I_2}^{x}(tan^2t+2-3I_1+4I_2)dt\right]=tan^2x+2-3I_1+4I_2$$

$$(2)\ \frac{d}{dx}\left[\int_{9-5I_1+7I_2}^{x}\sqrt{5I_1t+4I_2}dt\right]=\sqrt{5I_1x+4I_2}$$

$$(3)\frac{d}{dx}\left[\int_{x}^{-6+I_1-I_2}(5I_1t^3+I_2t^2+t-2I_1+5I_2)dt\right]=-5I_1x^3-I_2x^2-x+2I_1-5I_2$$

Remarks 1

$$(1)\ \frac{d}{dx}\left[\int_{a+a_1I_1+a_2I_2}^{g(x,I_1,I_2)}f(t,I_1,I_2)dt\right]=f\big(g(x,I_1,I_2)\big)\acute{g}(x,I_1,I_2)$$

Proof:

$$\frac{d}{dx}\left[\int_{a+a_1I_1+a_2I_2}^{g(x,I_1,I_2)}f(t,I_1,I_2)dt\right]=\frac{d}{dx}\big[F\big(g(x,I_1,I_2)\big)\big]$$

$$=\acute{F}\big(g(x,I_1,I_2)\big)\acute{g}(x,I_1,I_2)$$

$$= f\big(g(x,I_1,I_2)\big)\acute{g}(x,I_1,I_2)$$

$$(2)\frac{d}{dx}\left[\int_{g(x,I_1,I_2)}^{a+a_1I_1+a_2I_2} f(t,I_1,I_2)dt\right] = -f\big(g(x,I_1,I_2)\big)\acute{g}(x,I_1,I_2)$$

Proof:

$$\frac{d}{dx}\left[\int_{g(x,I_1,I_2)}^{a+a_1I_1+a_2I_2} f(t,I_1,I_2)dt\right] = \frac{d}{dx}\left[-F\big(g(x,I_1,I_2)\big)\right]$$

$$= -\acute{F}\big(g(x,I_1,I_2)\big)\acute{g}(x,I_1,I_2)$$

$$= -f\big(g(x,I_1,I_2)\big)\acute{g}(x,I_1,I_2)$$

$$(3)\ \frac{d}{dx}\left[\int_{g_1(x,I_1,I_2)}^{g_2(x,I_1,I_2)} f(t,I_1,I_2)dt\right] = f\big(g_2(x,I_1,I_2)\big)\ \acute{g}_2(x,I_1,I_2) - f\big(g_1(x,I_1,I_2)\big)\ \acute{g}_1(x,I_1,I_2)$$

Proof:

$$\frac{d}{dx}\left[\int_{g_1(x,I_1,I_2)}^{g_2(x,I_1,I_2)} f(t,I_1,I_2)dt\right] = \frac{d}{dx}\left[\int_{g_1(x,I_1,I_2)}^{0+0I_1+0I_2} f(t,I_1,I_2)dt + \int_{0+0I_1+0I_2}^{g_2(x,I_1,I_2)} f(t,I_1,I_2)dt\right]$$

$$= f\big(g_2(x,I_1,I_2)\big)\ \acute{g}_2(x,I_1,I_2) - f\big(g_1(x,I_1,I_2)\big)\ \acute{g}_1(x,I_1,I_2)$$

Example 3

$$(1)\ \frac{d}{dx}\left[\int_{1+I_1+I_2}^{cos\ (x+\frac{\pi}{3}I_1+\pi I_2)} (t^3 - I_1 + 7I_2)dt\right] = -(cos^3(x+\frac{\pi}{3}I_1+\pi I_2) - I_1 + 7I_2)sin\ (x+\frac{\pi}{3}I_1+\pi I_2)$$

$$(2)\ \frac{d}{dx}\left[\int_{-9-I_1+6I_2}^{\sqrt{I_1+3I_2-5x}} (-8+I_2-t)dt\right] = (-8+I_2-\sqrt{I_1+3I_2-5x})\frac{-5}{2\sqrt{I_1+3I_2-5x}}$$

$$(3)\ \frac{d}{dx}\left[\int_{ln(x+4+I_1+I_2)}^{5+2I_1+I_2} tdt\right] = -\frac{1}{x+4+I_1+I_2}\ ln(x+4+I_1+I_2)$$

$$(4)\ \frac{d}{dx}\left[\int_{-7x+I_1+4I_2}^{6x^2-2I_1+I_2} \frac{2-I_1+I_2}{2t+I_1+3I_2}dt\right] = \frac{(2-I_1+I_2)x}{12x^2-4I_1+2I_2+I_1+3I_2} + \frac{14-7I_1+7I_2}{-14x+2I_1+8I_2+I_1+3I_2}$$

Theorem 3 (The mean- value theorem of 2- refined neutrosophic integral calculus_ part II)

If $f(x, I_1, I_2)$ is continuous on a closed 2- refined interval $[a + a_1I_1 + a_2I_2, b + b_1I_1 + b_2I_2]$, then there is at least one point $x^* = x_0 + x_1I_1 + x_2I_2$ in $[a + a_1I_1 + a_2I_2, b + b_1I_1 + b_2I_2]$ such that:

$$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = f(x^*, I_1, I_2)\big(b + b_1I_1 + b_2I_2 - (a + a_1I_1 + a_2I_2)\big)$$

Example 4
If $f(x, I_1, I_2) = 4x + 5I_1$ let's find x^* that satisfy the mean-value theorem of f on $[2 + I_1 + 3I_2, 4 + I_1 + 5I_2]$.

$$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = f(x^*, I_1, I_2)\big(b + b_1I_1 + b_2I_2 - (a + a_1I_1 + a_2I_2)\big)$$

$$\int_{2+I_1+3I_2}^{4+I_1+5I_2} (4x + 5I_1)dx = (4x^* + 5I_1)(2 + 2I_2)$$

$$[2x^2 + 5I_1x]_{2+I_1+3I_2}^{4+I_1+5I_2} = (4x^* + 5I_1)(2 + 2I_2)$$

$$[2(4 + I_1 + 5I_2)^2 + 5I_1(4 + I_1 + 5I_2)] - [2(2 + I_1 + 3I_2)^2 + 5I_1(2 + I_1 + 3I_2)] = (4x^* + 5I_1)(2 + 2I_2)$$

$$24 + 22I_1 - 6I_2 = (4x^* + 5I_1)(2 + 2I_2)$$

$$4x^* + 5I_1 = \frac{24 + 36I_1 + 88I_2}{2 + 2I_2}$$

$$4x^* + 5I_1 = \frac{12 + 18I_1 + 44I_2}{1 + 0 + I_2}$$

$$4x^* + 5I_1 = 12 + 9I_1 + 16I_2$$

$$4x^* = 12 + 4I_1 + 16I_2$$

$$x^* = 3 + I_1 + 4I_2 \in [2 + I_1 + 3I_2, 4 + I_1 + 5I_2]$$

Example 5
If $f(x, I_1, I_2) = 9\sqrt{x}$ let's find x^* that satisfy the mean-value theorem of f on $[0 + 0I_1 + 0I_2, 1 + 12I_1 + 3I_2]$.

$$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = f(x^*, I_1, I_2)\big(b + b_1I_1 + b_2I_2 - (a + a_1I_1 + a_2I_2)\big)$$

$$\int_{0+0I_1+0I_2}^{1+12I_1+3I_2} 9\sqrt{x}\, dx = 9(1 + 12I_1 + 3I_2)\sqrt{x^*}$$

$$\left[6x\sqrt{x}\right]_{0+0I_1+0I_2}^{1+12I_1+3I_2} = 9(1+12I_1+3I_2)\sqrt{x^*}$$

$$6(1+12I_1+3I_2)\sqrt{1+12I_1+3I_2} = 9(1+12I_1+3I_2)\sqrt{x^*}$$

$$\sqrt{x^*} = \frac{2\sqrt{1+12I_1+3I_2}}{3}$$

$$x^* = \frac{4(1+12I_1+3I_2)}{9} = \frac{4+48I_1+12I_2}{9}$$

$$x^* = \frac{4}{9} + \frac{16}{3}I_1 + \frac{4}{3}I_2 \ \in \ [0+0I_1+0I_2, 1+12I_1+3I_2]$$

3. Properties of the definite integrals of 2- refined neutrosophic functions

Let $f(x, I_1, I_2)$ and $g(x, I_1, I_2)$ then:

(1) $$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = \int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(t, I_1, I_2)dt$$

(2) $$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = \int_{a+a_1I_1+a_2I_2}^{c+c_1I_1+c_2I_2} f(x, I_1, I_2)dx + \int_{c+c_1I_1+c_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx$$

where: $a + a_1I_1 + a_2I_2 \leq c + c_1I_1 + c_2I_2 \leq b + b_1I_1 + b_2I_2$

(3) $$\int_{a+a_1I_1+a_2I_2}^{a+a_1I_1+a_2I_2} f(x, I_1, I_2)dx = 0$$

(4) $$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = -\int_{b+b_1I_1+b_2I_2}^{a+a_1I_1+a_2I_2} f(x, I_1, I_2)dx$$

(5) $$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} (c+c_1I_1+c_2I_2)f(x, I_1, I_2)dx = (c+c_1I_1+c_2I_2)\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx$$

(6) $$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} [f(x, I_1, I_2) \pm g(x, I_1, I_2)]dx$$
$$= \int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx \pm \int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} g(x, I_1, I_2)dx$$

$$(7)\quad \int_{-(a+a_1I_1+a_2I_2)}^{a+a_1I_1+a_2I_2} f(x,I_1,I_2)dx$$

$$= \begin{cases} 2\displaystyle\int_0^{a+a_1I_1+a_2I_2} f(x,I_1,I_2)dx & ;if\ f(x,I_1,I_2)\ is\ even\ function \\ 0 & ;if\ f(x,I_1,I_2)\ is\ odd\ function \end{cases}$$

Example 6

$$1)\quad \int_{9+4I_1-7I_2}^{9+4I_1-7I_2} (t^3+(1+I_1+3I_2)t^2-12I_1-9I_2)dx = 0$$

$$2)\quad \int_{-\pi-\frac{\pi}{12}I_1+\frac{\pi}{3}I_2}^{\pi+\frac{\pi}{12}I_1-\frac{\pi}{3}I_2} -7I_2\ sin^7x\ cos^6x\ dx$$

Put: $f(x,I_1,I_2) = -7I_2\ sin^7x\ cos^6x$, then:

$$f(-x,I_1,I_2) = -7I_2\ sin^7(-x)cos^6(-x)$$

$$= -7I_2(sin(-x))^7(\cos(-x))^6 = 7I_2\ sin^7x\ cos^6x$$

$$= -f(x,I_1,I_2)$$

Hence $f(x,I)$ is odd function, so by property 7, we get:

$$\int_{-\pi-\frac{\pi}{12}I_1+\frac{\pi}{3}I_2}^{\pi+\frac{\pi}{12}I_1-\frac{\pi}{3}I_2} -7I_2\ sin^7x\ cos^6x\ dx = 0$$

3. Conclusions

This paper introduced the concept of the definite integrals of 2- refined neutrosophic functions and its properties, by discussing a set of theorems related to the definite integrals of 2- refined neutrosophic functions, where these theorems were proven. Also, the properties of the definite integrals of 2- refined neutrosophic functions were studied, which facilitated finding the result of the integrals of 2- refined neutrosophic function faster and directly. In addition, the importance of this paper comes from the fact that it facilitates many studies in the field of the neutrosophic integrals calculate.

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An Effective and Practical Probabilistic Simplified Neutrosophic Approach for Accurate and Reliable Innovative Design Evaluation for Digital Media Arts

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Abstract: The evaluation and classification of digital artworks remain a challenging task due to the inherent uncertainty, subjectivity, and imprecision in artistic design. Traditional computational techniques are no longer effective and fail to capture the ambiguity and probabilistic behavior associated with artistic evaluation, gradually becoming untrusted for innovative design assessment. To that end, we propose a Probabilistic Simplified Neutrosophic Set (PSNS)-based convolutional art classifier, that integrates neutrosophic logic to foster the representational power of simple Convolutional Network classifier to be able to learn and make a correct decision about uncertain art evaluation scenarios. With the encapsulation of PSNS, we can introduce probabilistic truth, indeterminacy, and falsity values to efficiently model the intrinsic uncertainties in digital media art evaluation. To validate our claims, we carefully choose a public case of digital art classification to conduct experiments to analyze the performance of the proposed approach. The quantitative results prove that our approach not only accentuates the applicability but also can improve decision-making through evaluating and refining the quality of elderly care services.

Keywords: Neutrosophic sets, Probabilistic Simplified Neutrosophic Set (PSNS), Machine Learning, Digital Media Arts, Uncertainty Modeling.

I. INTRODUCTION

Digital media arts encompass a wide range of creative disciplines, which include graphic design, animation, visual properties, as well as digital storytelling. The process of evaluation of the quality and effectiveness of digital designs entail a systematic approach that take into account both structured (measurable) and unstructured (subjective) aspects [1]. The swift

evolution of digital media arts usually led to cumulative complexity in bright design evaluation. Out-of-date evaluation methods relies deeply on subjective human judgment, which is frequently unpredictable and disposed to bias. With the emergence of artificial intelligence (AI) and machine learning (ML), automated approaches have gained attention for their ability to provides objective and data-driven design assessment [2], [3]. However, prevailing AI-based evaluation approaches are unable to model the uncertainty, imprecision, and contradictory information inherent in artistic design [1].

To provide a solution to that challenges, Neutrosophic Logic (NL) [4], [5] emerged as a innovative mathematical framework to capture truth, indeterminacy, and falsity of data stream at the same time. Though scholars developed different Neutrosophic Set (NS) extensions, most of them suffer from computational complexity or lacks the aptitude to represent probabilistic differences in ambiguous info [6], [7]. This limitations had motivated the development of the Probabilistic Simplified Neutrosophic Set (PSNS), which introduces probability-based weighting to truth, indeterminacy, and falsity membership functions, offering a more flexible and computationally efficient representation of uncertain information [8]. PSNS had been presented to handle uncertainty, imprecision, and indeterminacy in a more organized and probabilistic way. Like NS, PSNS represent an element with three components: truth membership (T), indeterminacy membership (I), and falsity membership (F), nevertheless, it allocates each of these mechanisms with probabilistic values, which tolerates a more elastic representations of information by participating unpredictability as well as indeterminacy alongside [9], [10].

Assuming the dynamic natures of the industry of digital media arts, where design basics are prejudiced by both quantitative attributes (like color balance, contrast, as well as symmetry) and qualitative aesthetics (like artistic creativity, and emotional impact), a PSNS-based solution can offer the ideal framework for innovative design evaluation. In this paper, we propose a PSNS approach for smart design evaluation in digital media arts. The key contributions of this approach are:

- We develop a PSNS-based Convolution classifier to automatically learn inherent evaluation patterns and design features for digital art images.
- With PSNS integration, the representational power of our approach is fostered with probabilistic neutrosophic reasoning capability, handling the ambiguities and

differences in the human insight of design quality.

- Proof-of-concept analysis is made by testing our framework on real-world case study of digital artworks, and the results demonstrated the effectiveness of our framework to provide dependable and robust design evaluations.

The residual part of this research is systematized as follows. Sect. 2 deliberate definitions and critical thoughts. The proposed methodology is defined in Sect. 3. Following, we explain the case study in Sect 4. Sect. 5 debate the results and analysis. Sect. 5 concludes the main findings.

II. PRELIMINARIES

Definition 1. The SVNSs are formulated as:

$$N = \{(\theta, T(\theta), I(\theta), F(\theta))|\theta \in \Theta\} \quad (1)$$

where $T(\theta), I(\theta), F(\theta)$ is membership, indeterminacy-membership and falsity-membership,$T(\theta), I(\theta), F(\theta) \in [0,1],\ 0 \leq T(\theta) + I(\theta) + F(\theta) \leq 3$.

Definition 2. The PSNSs are formulated as:

$$N = \left\{\left(\theta, T(\theta)(\mathfrak{P}T(\theta)), I(\theta)(\mathfrak{P}I(\theta)), F(\theta)(\mathfrak{P}F(\theta))\right)|\theta \in \Theta\right\} \quad (2)$$

where $T(\theta), I(\theta), F(\theta)$ is truth-membership, indeterminacy-membership and falsity-membership, $T(\theta), I(\theta), F(\theta) \in [0,1]$, $0 \leq T(\theta) + I(\theta) + F(\theta) \leq 3$, $0 \leq \mathfrak{P}T(\theta), \mathfrak{P}I(\theta), \mathfrak{P}F(\theta) \leq 1$, the $\mathfrak{P}T(\theta), \mathfrak{P}I(\theta), \mathfrak{P}F(\theta)$is possibility values of $T(\theta), I(\theta), F(\theta)$. The PSNS is listed as$P = (T(\mathfrak{P}T), I(\mathfrak{P}I), F(\mathfrak{P}F))$.

Definition 3. Let $N_1 = (T_1(\mathfrak{P}T_1), I_1(\mathfrak{P}I_1), \mathrm{F}_1(\mathfrak{P}\mathrm{F}_1))$, $N_2 = (T_2(\mathfrak{P}T_2), I_2(\mathfrak{P}I_2), \mathrm{F}_2(\mathfrak{P}\mathrm{F}_2))$, the basic operations are put forward:

A) add

$$\mathrm{N}_1 \oplus \mathrm{N}_2 = \begin{pmatrix} \mathrm{T}_1 + \mathrm{T}_2 - \mathrm{T}_1 \cdot \mathrm{T}_2 \left(2! \frac{\mathfrak{P}\mathrm{T}_1 \cdot \mathfrak{P}\mathrm{T}_2}{\mathfrak{P}\mathrm{T}_1 + \mathfrak{P}\mathrm{T}_2}\right), \\ \mathrm{I}_1 \cdot \mathrm{I}_2 \left(2! \frac{\mathfrak{P}\mathrm{I}_1 \cdot \mathfrak{P}\mathrm{I}_2}{\mathfrak{P}\mathrm{I}_1 + \mathfrak{P}\mathrm{I}_2}\right), \mathrm{F}_1 \cdot \mathrm{F}_2 \left(2! \frac{\mathfrak{P}\mathrm{F}_1 \cdot \mathfrak{P}\mathrm{F}_2}{\mathfrak{P}\mathrm{F}_1 + \mathfrak{P}\mathrm{F}_2}\right) \end{pmatrix}; \quad (3)$$

B) Multiply

$$N_1 \otimes N_2 = \begin{pmatrix} T_1 \cdot T_2\left(2!\frac{\mathfrak{P}T_1 \cdot \mathfrak{P}T_2}{\mathfrak{P}T_1 + \mathfrak{P}T_2}\right), \\ I_1 + I_2 - I_1 \cdot I_2\left(2!\frac{\mathfrak{P}I_1 \cdot \mathfrak{P}I_2}{\mathfrak{P}I_1 + \mathfrak{P}I_2}\right), \\ \mathrm{F}_1 + \mathrm{F}_2 - \mathrm{F}_1 \cdot \mathrm{F}_2\left(2!\frac{\mathfrak{P}\mathrm{F}_1 \cdot \mathfrak{P}\mathrm{F}_2}{\mathfrak{P}\mathrm{F}_1 + \mathfrak{P}\mathrm{F}_2}\right) \end{pmatrix}; \quad (4)$$

C) Lamda

$$\lambda N = \left(1 - (1 - T)^{\lambda}(\mathfrak{P}T), (I)^{\lambda}(\mathfrak{P}I), (F)^{\lambda}(\mathfrak{P}F)\right), \lambda > 0; \quad (5)$$

D) Power

$$(N)^{\lambda} = \left((T)^{\lambda}(\mathfrak{P}T), 1 - (1 - I)^{\lambda}(\mathfrak{P}I), 1 - (1 - F)^{\lambda}(\mathfrak{P}F)\right), \lambda > 0. \quad (6)$$

Definition 4. Given two PSNSs $\mathrm{N}_1 = \left(\mathrm{T}_1(\mathfrak{P}\mathrm{T}_1), \mathrm{I}_1(\mathfrak{P}\mathrm{I}_1), \mathrm{F}_1(\mathfrak{P}\mathrm{F}_1)\right)$, $N_2 = \left(T_2(\mathfrak{P}T_2), I_2(\mathfrak{P}I_2), \mathrm{F}_2(\mathfrak{P}\mathrm{F}_2)\right)$, then , the following characterstics apply:

A) Subset

$$N_1 \subseteq N_2 \Leftrightarrow \mathrm{T}_1(x) \leq \mathrm{T}_2(x), \mathfrak{P}\mathrm{T}_1(x) \leq \mathfrak{P}\mathrm{T}_2(x); \mathrm{I}_1(x) \geq \mathrm{I}_2(x), \mathfrak{P}\mathrm{I}_1(x) \geq \mathfrak{P}\mathrm{I}_2(x); \mathrm{F}_1(x) \geq \mathrm{F}_2, \mathfrak{P}\mathrm{F}_1(x) \geq \mathfrak{P}\mathrm{F}_2(x)\ for\ x \in X \quad (7)$$

B) Complement

$$A^C = \left\{\left\langle x; F_A(x)\left(P_{F_A(x)}\right), \left(1 - I_A(x)\right)\left(1 - \mathfrak{P}\mathrm{T}_1(x)\right), \mathrm{T}_1(x)\left(\mathfrak{P}\mathrm{T}_1(x)\right)\right\rangle : x \in X\right\} \quad (8)$$

C) Intersection

$$A_1 \cap A_2 = \left\{ \begin{matrix} \{\langle x; \min\{\mathrm{T}_1(x), \mathrm{T}_2(x)\}(\min\{\mathfrak{P}\mathrm{T}_1(x), \mathfrak{P}\mathrm{T}_2(x)\}), \\ \max\{\mathrm{I}_1(x), \mathrm{I}_2(x)\}(\max\{\mathfrak{P}\mathrm{I}_1(x), \mathfrak{P}\mathrm{I}_2(x)\}), \\ \max\{\mathrm{F}_1(x), \mathrm{F}_2(x)\}(\max\{\mathfrak{P}\mathrm{F}_1(x), \mathfrak{P}\mathrm{F}_2(x)\})\rangle : x \in X \end{matrix} \right\}$$

(9)

D) Union

$$A_1 \cup A_2 = \left\{ \begin{matrix} \langle x; \max\{\mathrm{T}_1(x), \mathrm{T}_2(x)\}(\max\{\mathfrak{P}\mathrm{T}_1(x), \mathfrak{P}\mathrm{T}_2(x)\}), \\ \min\{\mathrm{I}_1(x), \mathrm{I}_1(x)\}(\min\{\mathfrak{P}\mathrm{I}_1(x), \mathfrak{P}\mathrm{I}_2(x)\}), \\ \min\{\mathrm{F}_1(x), \mathrm{F}_2(x)\}(\min\{\mathfrak{P}\mathrm{F}_1(x), \mathfrak{P}\mathrm{F}_2(x)\})\rangle : x \in X \end{matrix} \right\} \quad (10)$$

Definition 5. Given $N_1 = \left(T_1(\mathfrak{P}T_1), I_1(\mathfrak{P}I_1), \mathrm{F}_1(\mathfrak{P}\mathrm{F}_1)\right)$, $N_2 = \left(T_2(\mathfrak{P}T_2), I_2(\mathfrak{P}I_2), \mathrm{F}_2(\mathfrak{P}\mathrm{F}_2)\right)$, the Logarithmic distance ($LD_{\mathfrak{P}SNN}$) between N_1 and N_2 is constructed:

$$LD_{\mathfrak{P}SNN}(N_1, N_2) = \frac{1}{3}\left(\begin{array}{l}(T_1 \times \mathfrak{P}T_1)\, log \dfrac{(T_1 \times \mathfrak{P}T_1)}{\dfrac{(T_1 \times \mathfrak{P}T_1) + (T_2 \times \mathfrak{P}T_2)}{2}} \\ +(T_2 \times \mathfrak{P}T_2)\, log \dfrac{(T_2 \times \mathfrak{P}T_2)}{\dfrac{(T_1 \times \mathfrak{P}T_1) + (T_2 \times \mathfrak{P}T_2)}{2}} \\ +(I_1 \times \mathfrak{P}I_1)\, log \dfrac{(I_1 \times \mathfrak{P}I_1)}{\dfrac{(I_1 \times \mathfrak{P}I_1) + (I_2 \times \mathfrak{P}I_2)}{2}} \\ +(I_2 \times \mathfrak{P}I_2)\, log \dfrac{(I_2 \times \mathfrak{P}I_2)}{\dfrac{(I_1 \times \mathfrak{P}I_1) + (I_2 \times \mathfrak{P}I_2)}{2}} \\ +(\mathrm{F}_1 \times \mathfrak{P}\mathrm{F}_1)\, log \dfrac{(\mathrm{F}_1 \times \mathfrak{P}\mathrm{F}_1)}{\dfrac{(\mathrm{F}_1 \times \mathfrak{P}\mathrm{F}_1) + (\mathrm{F}_2 \times \mathfrak{P}\mathrm{F}_2)}{2}} \\ +(\mathrm{F}_2 \times \mathfrak{P}\mathrm{F}_2)\, log \dfrac{(\mathrm{F}_2 \times \mathfrak{P}\mathrm{F}_2)}{\dfrac{(\mathrm{F}_1 \times \mathfrak{P}\mathrm{F}_1) + (\mathrm{F}_2 \times \mathfrak{P}\mathrm{F}_2)}{2}}\end{array}\right) \tag{11}$$

III. Research Methodology

This part of our paper presents the systematic approach adopted to achieve the objectives of the study. Our research methodology demonstrates a new approach that integrates PSNS-based reasoning with representational learning of convolutional network to offer uncertainity aware model of visual patterns of digital arts.

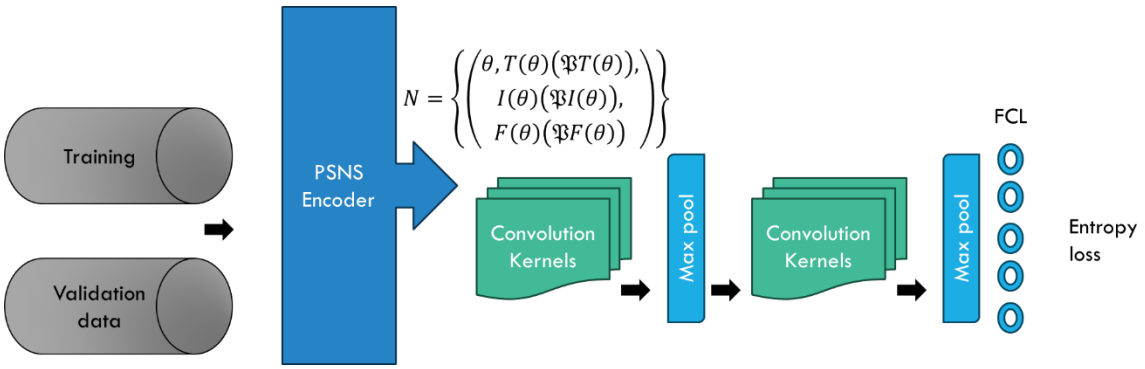

Figure 1. architecture of the proposed PSNS framework for art design evaluation

In step 1, we emphasize encoding art images into PSNS representation. Assume we have an art image $x \in X$ with RGB channels R, G, B, we compute the corresponding PSNS representation as follows:

We calculate the truth membership based on the average of values in RGB channels:

$$T_K(x) = \frac{R+G+B}{3} \tag{12}$$

We calculate indeterminacy membership based on the absolute difference between the green and blue channels.

$$I_K(x) = |G - B| \tag{13}$$

We calculate falsity membership complement of the truth membership

$$F_K(x) = 1 - T_K(x) \tag{14}$$

In step 2, we try to normalize values of T, I, F. But, we begin by adding ϵ as a small value (e.g., 10^{-6}) to avoid division by zero.

$$S = T_K(x) + I_K(x) + F_K(x) + \epsilon \tag{15}$$

Then, the normalized components are given as follows:

$$T'_K(x) = \frac{T_K(x)}{S},\ I'_K(x) = \frac{I_K(x)}{S},\ F'_K(x) = \frac{F_K(x)}{S} \tag{16}$$

Where the sum of new components T', I', F' is 1.

In step 3, we move foreward to calculate the probability of of truth, indeterminacy, and falsity using the softmax function:

$$P_{T_K}(x) = \frac{e^{T'_K(x)}}{e^{T'_K(x)} + e^{I'_K(x)} + e^{F'_K(x)}} \tag{17}$$

$$P_{I_K}(x) = \frac{e^{I'_K(x)}}{e^{T'_K(x)} + e^{I'_K(x)} + e^{F'_K(x)}} \tag{18}$$

$$P_{F_K}(x) = \frac{e^{F'_K(x)}}{e^{T'_K(x)} + e^{I'_K(x)} + e^{F'_K(x)}} \tag{19}$$

Again, this computed probabilities always sum to 1 :

$$P_{T_K}(x) + P_{I_K}(x) + P_{F_K}(x) = 1 \tag{20}$$

By the completion PSNS encoding that converts an image into a 6-channel tensor $(T', I', F', P_T, P_I, P_F)$, we we apply Convolutional Neural Network (CNN) to process these features, as shown in Figure 1.

Indeed, the convolution operation with input feature map X, kernel W, and bias b, is computed as follows:

$$Y = \text{ReLU}\left(\sum_{i=1}^{6} X_i * W_i + b\right) \tag{21}$$

where * symbolize convolution kernel. This is followed by max pooling layer to reduces dimensionality:

$$Y_{\text{pooled}}(i,j) = \max_{(m,n)\in \text{ window}} Y(i+m, j+n) \tag{22}$$

The about of the above layers are flattening, thend passed to linear layers:

$$\begin{aligned} Z_1 &= \text{ReLU}\left(W_1 Y + b_1\right) \\ Z_2 &= W_2 Z_1 + b_2 \end{aligned} \tag{23}$$

where Z_2 is the final logit vector for classification.

To convert the final logits into class probabilities:

$$P(y = c \mid x) = \frac{e^{Z_2^c}}{\sum_j e^{Z_2^j}} \tag{24}$$

where c represents class index. Using categorical cross-entropy loss:

$$\mathcal{L} = -\sum_{c=1}^{C} y_c \log P(y = c \mid x) \tag{25}$$

where C symolize number of classes. y_c symolize one-hot encoded ground-truth label. $P(y = c \mid x)$ symolize the predicted probability.

The gradient of the model is updated using the chain rule:

$$\frac{\partial \mathcal{L}}{\partial Z_2^c} = P(y = c \mid x) - y_c \tag{26}$$

where the gradient of linear layer is calculated as follows:

$$\frac{\partial \mathcal{L}}{\partial W_2} = \frac{\partial \mathcal{L}}{\partial Z_2} \cdot Z_1^T, \tag{27}$$

and the gradient of convolutional layer is calculated as follows:

$$\frac{\partial \mathcal{L}}{\partial W_i} = X_i * \frac{\partial \mathcal{L}}{\partial Y} \tag{28}$$

Using Adam optimizer, the model parameters are updated as follows:

$$m_t = \beta_1 m_{t-1} + (1-\beta_1)\nabla\mathcal{L} \tag{29}$$

$$v_t = \beta_2 v_{t-1} + (1-\beta_2)(\nabla\mathcal{L})^2 \tag{30}$$

$$\hat{m}_t = \frac{m_t}{1-\beta_1^t},\ \hat{v}_t = \frac{v_t}{1-\beta_2^t} \tag{31}$$

$$W_t = W_{t-1} - \eta \frac{\hat{m}_t}{\sqrt{\hat{v}_t}+\epsilon} \tag{32}$$

Where β_1, β_2 symolize decay rates (default: 0.9,0.999). η symolize learning rate.

IV. Implementation Setup & Case study

The procedure of assessing the classification performance is directed founded on common set of assessment metrics drived from particular relation between True positive (TP), False Positive (FP), True Negative (TN), and False Negative (FN).

$$\boldsymbol{Accuracy} = \frac{\boldsymbol{TP}+\boldsymbol{TN}}{\boldsymbol{TP}+\boldsymbol{TN}+\mathrm{F}\boldsymbol{P}+\mathrm{F}\boldsymbol{N}} \tag{33}$$

$$\boldsymbol{Precision} = \frac{\boldsymbol{TP}}{\boldsymbol{TP}+\mathrm{F}\boldsymbol{P}} \tag{34}$$

$$\boldsymbol{Recall} = \frac{\boldsymbol{TP}}{\boldsymbol{TP}+\mathrm{F}\boldsymbol{N}} \tag{35}$$

$$\mathrm{F}_{\boldsymbol{1}} - \boldsymbol{score} = \boldsymbol{2}\frac{\boldsymbol{precision}\times\boldsymbol{recall}}{\boldsymbol{precision}+\boldsymbol{recall}} \tag{36}$$

Proof-of-concept investigations are steered on a Dell workstation functioned with Windows 10 64 bit. This device was armed by 64 G RAM, and Intel(R) Core(TM) i5-7200U CPU @ 2.50GHz 2.71 GHz. The model implementation is performed using torch library running in python 11 environment. The performance metrics are implemented using Scikil-learn library.

To demonstrate the effectiveness of our approach, a public case study was prudently selected, in which a dataset containing about 9000 images was collected. This case study is highly relevant as it emphasizes evaluation and classifying different styles of art. The data contain 5

main categories that have been taken including drawings and watercolours, works of painting, sculpture, graphic art, and iconography (old Russian art) [11]. The dataset was split into training and validation sets. The class distribution for both sets is displayed in Figure 2.

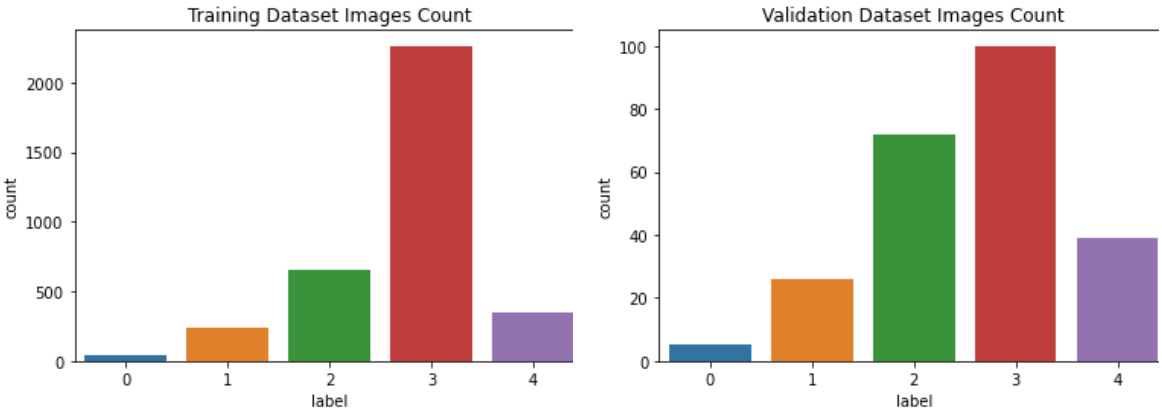

Figure 2. Class distribution for the case study.

Figure 3. Samples of artwork from the dataset case study.

IV. RESULTS AND ANALYSIS

The classification report, accessible in Table 1, provides a detailed performance analysis of our PSNS-based ML classifier across different evaluation metrics, including precision, recall, and F1-score for each class. These metrics help in assessing the model's ability to correctly classify instances while balancing false positives and false negatives. The macro-average and weighted-average scores offer a comprehensive view of the classifier's overall performance, ensuring robustness across both balanced and imbalanced datasets.

Table 1: Classification report showing performance metrics for each class

Class	Precision	Recall	F1-score
Drawings	**0.55**	**0.57**	**0.56**
Painting	**0.66**	**0.51**	**0.58**
Sculpture	**0.89**	**0.90**	**0.90**
Graphic Art	**0.90**	**0.85**	**0.88**
Iconography	**0.77**	**0.88**	**0.82**

The confusion matrix (CM), illustrated in Figure 4, offer a visual representation of the model's classification performance. This matrix help in understanding where the classifier performs well and where it struggles, providing insights into potential misclassifications. With the analysis the CM, we can identify patterns in errors and refine the model to enhance its predictive accuracy.

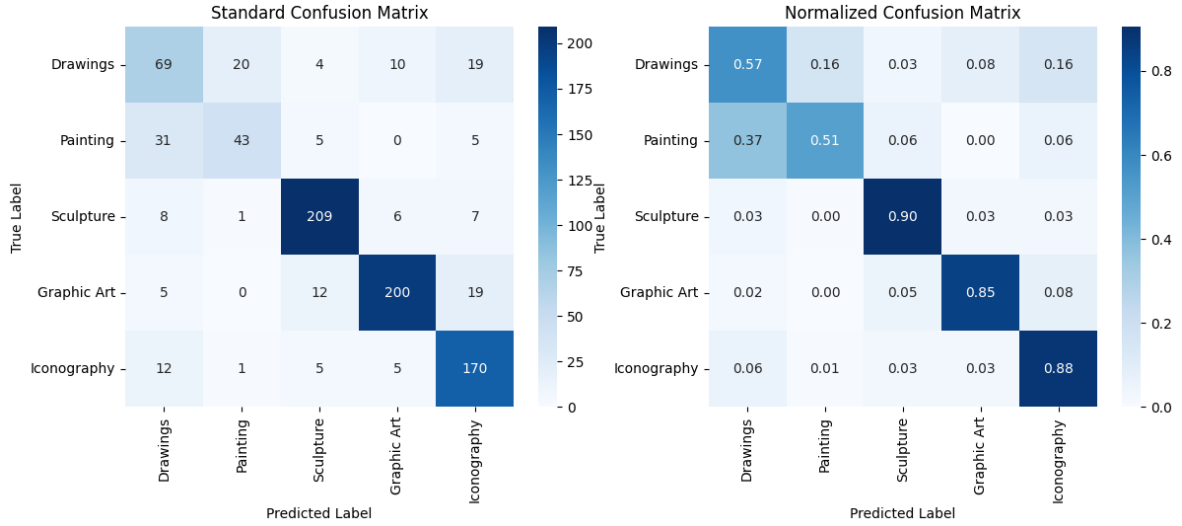

Figure 4: Confusion matrix for the PSNS-based classifier.

To further evaluates the ability of our model to distinguish between different classes, we present the Receiver Operating Characteristic (ROC) curves coupled with the Area Under the Curve (AUC) scores in Figure 5. The ROC curve plots the true positive rate (sensitivity) against the false positive rate, illustrating the trade-off between recall and specificity. The AUC score quantifies the overall model performance, where a higher AUC indicates better classification capability. This analysis is crucial for validating the reliability of our approach, especially in scenarios with varying class distributions.

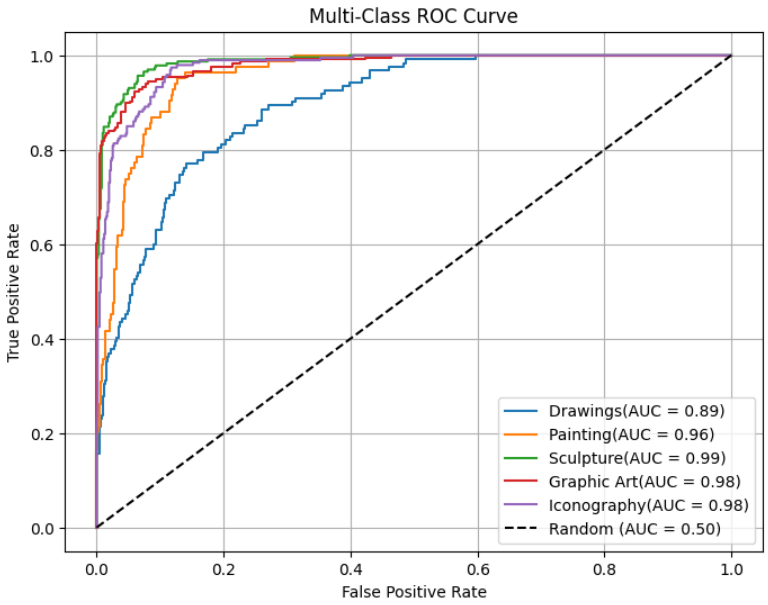

Figure 5: ROC curves and AUC scores to distinguish between different classes.

Furthermore, Precision-Recall (PR) curves and t-SNE (t-distributed Stochastic Neighbor Embedding) are displayed as powerful visualization to asses the performance of our approach with respect to arts data characteristics. As shown in Figure 6, PR curve explain the trade-off between precision and recall at various thresholds, showing that our model well distinguishes between differ classes of arts. On the other hand, Figure 7 show t-SNE for visualizing high-dimensional data in two dimensions by preserving the pairwise similarities between data points.

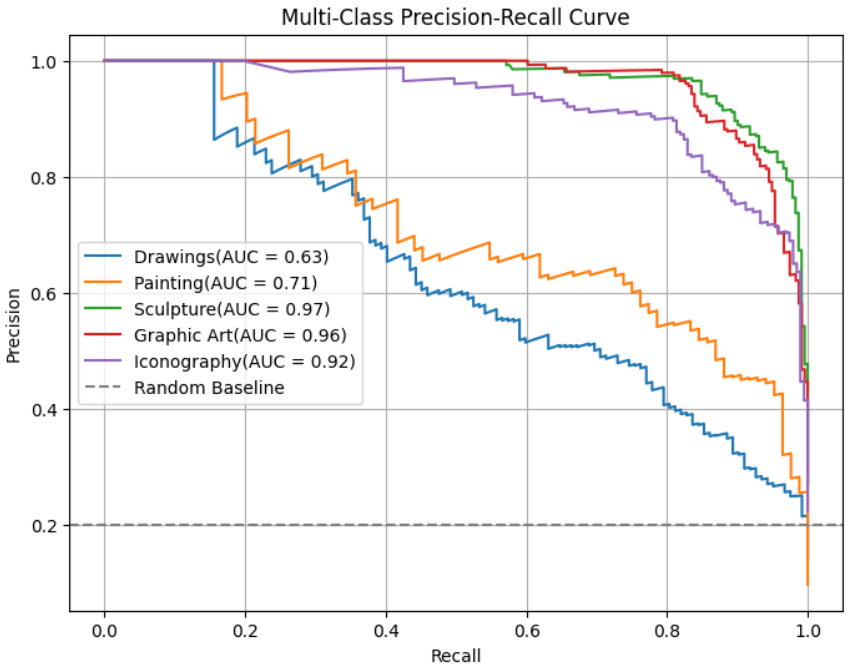

Figure 6: ROC curves and AUC scores to distinguish between different classes.

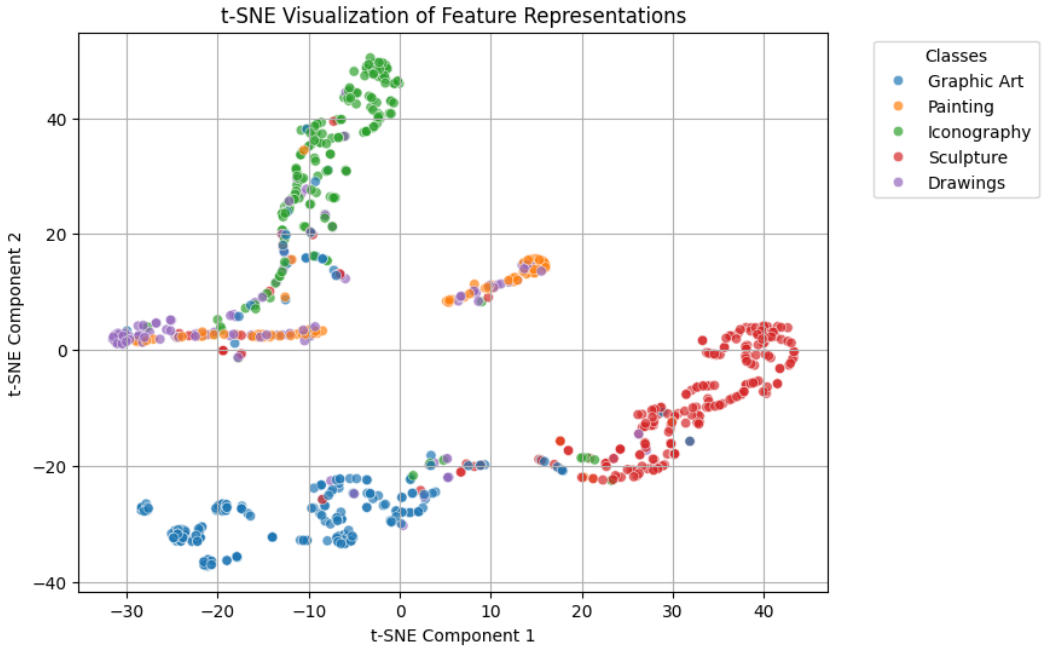

Figure 7: T-SNE plot predictions of different classes.

5. CONCLUSIONS

In this paper, a PSNS-based convolutional art classifier is introduced to take advantage of neutrosophic logic and deep learning to innovatively assess and categorize digital artworks. Our approach enables effective modeling of the uncertainty, indeterminacy, and imprecision intrinsic in artistic evaluation, which provides a vigorous and innovative framework to assess design styles. Through extensive experiments, we demonstrated the efficiency of our approach can classify digital media art with high accuracy and reliability. The quantitative results imply that our PSNS-based classifier can serve as a promising solution for art classification, and computerized design evaluations. Future work will emphasize enhancing the efficiency of the proposed integration by joining attention mechanisms, multimodal data fusion, and generative models to further refine and expand its capabilities.

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Fixed point results in complete neutrosophic fuzzy metric spaces for NF-L contractions

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Abstract. Fixed point theory occupies a significant position in both applied and pure mathematics due to its wide-ranging applications. The Banach contraction principle serves as a foundational theorem in metric spaces, which has been subject to numerous generalizations and applications. In this study, we explore the concept of neutrosophic fuzzy metric spaces and introduce the notion of NF-L contractions through a novel class of functions. We demonstrate that these contractions possess a unique fixed point. Initially, we present a crucial lemma that will be utilized in subsequent sections, followed by the proof of our primary result. Ultimately, we derive several fixed point results based on these types of contractions, which are derived from the main theorem.

Keywords: Fixed point; Neutrosophic set; Neutrosophic fuzzy metric space; NF-L contraction; Non linear contraction.

1. Introduction

The concept of Fuzzy Sets (FSs), initially proposed by Zadeh [1], has had a profound impact across various scientific disciplines since its inception. Although this framework holds significant relevance for practical applications, it has not always provided adequate solutions to numerous challenges over time. Consequently, there has been a resurgence of research aimed at addressing these issues. In this regard, Atanassov [2] introduced Intuitionistic Fuzzy Sets (IFSs) as a means to confront such challenges. In addition, Smarandache [3] introduced Neutrosophic Set (NS), represents a complex extension of conventional set theory.

Neutrosophic sets demonstrate a wide array of applications across various fields. For example, Barbosa and Smarandache [4] introduced the Neutrosophic One-Round Zero-Knowledge Proof protocol (N-1-R) ZKP, which expands upon the One-Round (1-R) ZKP framework by

incorporating Neutrosophic numbers. Additionally, the authors in [5] offer a detailed characterization of effective and optimally suitable solutions pertaining to scalar optimization challenges. They also delineate the Kuhn-Tucker conditions that are pertinent to both efficiency and proper efficiency. We cant make better decisions when temporal value is not considered, which leads to less accurate decision-making. This is made possible by including historical knowledge into the state representation. In the event that we wish to consider the opinions of many time periods, we should do so for both the current and prior eras (forecasting information). For instance, some researchers would want to think about using the influence of temporal variables in their calculations. The concept of time fuzzy soft sets and its use in design-making were presented by [6]. The effect of time on fuzzy soft expert sets was also discussed by [9]. [13] also introduced the concepts and applications of time-shadow soft sets, For a more in-depth investigation into the applications of neutrosophic sets and their extensive uses with time factor , it is advisable to refer to the literature cited in [12]. [14]Bipolar Complex Fuzzy Soft Sets and Their Application. We can consult the following for a more thorough examination of the applications of neutrosophic sets and their wide range of uses, we can refer to following, it is advisable to refer to the literature as [7] introduced Neutrosophic Logic to Navigate Uncertainty of Security Events in Mexico, [8] introduced Set and Machine Learning Approach for Breast Cancer Prediction, [10] introduced Multi Attribute Neutrosophic Optimization Technique for Optimal Crop Selection in Ariyalur District, [11] introduced Tele-Medical Realization via Integrating Vague T2NSs with OWCM-RAM Toward Intelligent Medical 4.0 Evaluator Framework.

[15] presented Utilizing a single-valued neutrosophic fuzzy soft expert setting, a strong framework for decision-making Applications of a novel extension of interval-valued Q-neutrosophic soft matrix Decision-making methods based on similarity metrics of possible neutrosophic soft expert sets were introduced by [17]. An approach for determining decisions based on similarity measures of possibility interval-valued neutrosophic soft setting settings was presented by [18]. The Banach fixed-point theorem [19], commonly known as the contraction mapping theorem, is a fundamental principle in the theory of metric spaces. It asserts that within any complete metric space, a contraction mapping—defined as a function that reduces the distance between points—will possess a unique fixed point. This theorem is significant because it provides a systematic approach to identifying fixed points, which are defined as points that remain invariant under a specific function. Moreover, it guarantees both the existence and uniqueness of such points under well-defined conditions.

The Banach fixed-point theorem is widely recognized for its significant applications in numerous fields, such as differential equations and numerical analysis. Its primary function lies

in demonstrating the existence and uniqueness of solutions to equations, as well as facilitating iterative methods for solution approximation. Initially proposed by Banach in 1922, this theorem has spurred a multitude of mathematicians to investigate various extensions and generalizations across diverse mathematical areas. For instance, the authors in [20, 21] have derived fixed-point results by introducing novel types of distance spaces associated with metric spaces. This line of inquiry was further advanced in [22], where the author established fixed-point results for contractions of Geraghty type. Additionally, Bataihah [23] contributed new fixed-point theorems through a distinct type of distance based on b-metric spaces.

Moreover, the theorem has inspired a wide array of generalizations and extensions. Aydi et al. [24] introduced triple coincidence point theorems for weak Φ-contractions within the framework of partially ordered metric spaces. The research referenced in [25, 26] presents various fixed-point results in b-metric spaces, while [27,28] discusses several fixed-point results in extended b-metric spaces. Furthermore, the authors in [29–31] explored fixed-point results in generalized metric spaces. For additional insights into fixed-point theory, we direct the reader to [32–34] and references therin.

Triangular norms (TN), introduced by Menger [35], generalize the triangle inequality using probability distributions instead of numerical distances. Their duals, triangular conorms (CN), are essential in fuzzy operations, particularly for intersections and unions.

In this document, we define the sets as follows: $\mathbb{R}^+ = [0, \infty)$ and $I_0 = [0, 1]$.

Definition 1.1. Let $\bullet : I_0 \times I_0 \to I_0$ be an operation. Then, $\bullet$ is said to be as continuous TN (CN) if it satisfies the following: for any $\varsigma, \varsigma', s, s' \in I_0$.

(1) $\varsigma \bullet 1 = \varsigma$,
(2) If $\varsigma \leq \varsigma'$ and $s \leq s'$, than $\varsigma \bullet s \leq \varsigma' \bullet s'$,
(3) $\bullet$ is continuous,
(4) $\bullet$ is associate and commutative.

Definition 1.2. Let $\diamond : I_0 \times I_0 \to I_0$ be an operation. Then, $\diamond$ is designated as continuous TN (TN) if it satisfies the following: for all $\varsigma, \varsigma', s, s' \in I_0$:

(1) $\varsigma \diamond 0 = \varsigma$,
(2) If $\varsigma \leq \varsigma'$ and $s \leq s'$, than $\varsigma \diamond s \leq \varsigma' \diamond s'$,
(3) $\diamond$ is continuous,
(4) $\diamond$ is associate and commutative.

The notion of neutrosophic metric space (NMS) was initially presented by Kirisci and Simsek. This setting has been utilized to investigate several fixed point theorems. The definition of neutrosophic metric spaces is outlined as follows.

Definition 1.3. [36] A 6-tuple $(\mathcal{W}, \Theta, \Phi, \Psi, \bullet, \diamond)$ is defined as a NMS when the set $\mathcal{W}$ constitutes a non-empty arbitrary collection. In this context, $\bullet$ represents a continuous t-norm, while $\diamond$ denotes a continuous t-conorm. Additionally, the elements Θ, Φ, and Ψ are three fuzzy sets defined on the Cartesian product $\mathcal{W}^2 \times (0, \infty)$. These components are required to fulfill specific conditions applicable to all elements $\zeta, \varrho, c \in \mathcal{W}$ and for all positive real numbers γ, ϖ.

(1) $0 \leq \Theta(\zeta, \varrho, \gamma) \leq 1$, $0 \leq \Phi(\zeta, \varrho, \gamma) \leq 1$, $0 \leq \Psi(\zeta, \varrho, \gamma) \leq 1$,
(2) $0 \leq \Theta(\zeta, \varrho, \gamma) + \Phi(\zeta, \varrho, \gamma) + \Psi(\zeta, \varrho, \gamma) \leq 3$,
(3) $\Theta(\zeta, \varrho, \gamma) = 1$, iff $\zeta = \varrho$
(4) $\Theta(\zeta, \varrho, \gamma) = H(\varrho, \zeta, \gamma)$, for $\gamma > 0$
(5) $\Theta(\zeta, \varrho, \gamma) \bullet \Theta(\varrho, c, \varpi) \leq \Theta(\zeta, c, \gamma + \varpi)$
(6) $\Theta(\zeta, \varrho, \cdot) : \mathbb{R}^+ \to I_0$ is continuous
(7) $\lim_{\gamma \to \infty} \Theta(\zeta, \varrho, \gamma) = 1$
(8) $\Phi(\zeta, \varrho, \gamma) = 0$ iff $\zeta = \varrho$
(9) $\Phi(\zeta, \varrho, \gamma) = \Phi(\varrho, \zeta, \gamma)$,
(10) $\Phi(\zeta, \varrho, \gamma) \diamond \Phi(\varrho, c, \varpi) \geq \Phi(\zeta, c, \gamma + \varpi)$,
(11) $\Phi(\zeta, \varrho, \cdot) : \mathbb{R}^+ \to I_0$ is continuous
(12) $\lim_{\gamma \to \infty} \Phi(\zeta, \varrho, \gamma) = 0$
(13) $\Psi(\zeta, \varrho, \gamma) = 0$, iff $\zeta = \varrho$
(14) $\Psi(\zeta, \varrho, \gamma) = \Psi(\varrho, \zeta, \gamma)$,
(15) $\Psi(\zeta, \varrho, \gamma) \diamond \Psi(\varrho, c, \varpi) \geq S(\zeta, c, \gamma + \varpi)$,
(16) $\Psi(\zeta, \varrho, \cdot) : \mathbb{R} \to I_0$ is continuous
(17) $\lim_{\gamma \to \infty} \Psi(\zeta, \varrho, \gamma) = 0$
(18) If $\gamma \leq 0$, then $\Theta(\zeta, \varrho, \gamma) = 0$, $\Phi(\zeta, \varrho, \gamma) = \Psi(\zeta, \varrho, \gamma) = 1$

The functions $\Theta(\zeta, \varrho, \gamma)$, $\Phi(\zeta, \varrho, \gamma)$, and $\Psi(\zeta, \varrho, \gamma)$ represent the degrees of nearness, neutralness, and non-nearness between the elements ζ and ϱ in relation to the parameter γ, respectively.

The convergence, Cauchy-ness, completeness in NMS are given as follows.

Definition 1.4. [36] Let (ζ_n) be a sequence in a NMS $(\mathcal{W}, \Theta, \Phi, \Psi, \bullet, \diamond)$. Then

(1) (ζ_n) converges to $\zeta \in \mathcal{W}$ if for any $\epsilon \in (0, 1)$, $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$

$$\Theta(\zeta_n, \zeta, \gamma) > 1 - \epsilon, \ \Phi(\zeta_n, \zeta, \gamma) < \epsilon, \ \Psi(\zeta_n, \zeta, \gamma) < \epsilon$$

i.e.,

$$\lim_{n \to \infty} \Theta(\zeta_n, \zeta, \gamma) = 1, \ \lim_{n \to \infty} \Phi(\zeta_n, \zeta, \gamma) = 0, \ \lim_{n \to \infty} \Psi(\zeta_n, \zeta, \gamma) = 0$$

(2) (ζ_n) is Cauchy if for any $\epsilon \in (0, 1)$, $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that for each $n, m \geq n_0$

$$\Theta(\zeta_n, \zeta_m, \gamma) > 1 - \epsilon, \ \Phi(\zeta_n, \zeta_m, \gamma) < \epsilon, \ \Psi(\zeta_n, \zeta_m, \gamma) < \epsilon$$

i.e.,

$$\lim_{n.m\to\infty}\Theta(\zeta_n,\zeta_m,\gamma)=1,\ \lim_{n,m\to\infty}\Phi(\zeta_n,\zeta_m,\gamma)=0,\ \lim_{n,m\to\infty}\Psi(\zeta_n,\zeta_m,\gamma)=0$$

(3) $(\mathcal{W},\Theta,\Phi,\Psi,\bullet,\diamond)$ is complete if every Cauchy sequence in $\mathcal{W}$ is convergent to an element in $\mathcal{W}$.

In the work of Simsek and Kirisci [37], NC-contractions were introduced within the context of neutrosophic metric spaces, demonstrating that each NC-contraction possesses a unique fixed point under specific conditions.

Definition 1.5. [37] Let $(\mathcal{W},\Theta,\Phi,\Psi,\bullet,\diamond)$ be a NMS. A mapping $f:\mathcal{W}\to\mathcal{W}$ is called neutrosophic contraction (NC) if there is $k\in(0,1)$ such that for each $\zeta,\varrho\in\mathcal{W}$ and $\gamma>0$, we have

$$\frac{1}{\Theta(f\zeta,f\varrho,\gamma)}-1\leq k\left(\frac{1}{\Theta(\zeta,\varrho,\gamma)}-1\right),$$
$$\frac{1}{\Phi(f\zeta,f\varrho,\gamma)}-1\geq k\left(\frac{1}{\Phi(\zeta,\varrho,\gamma)}-1\right),$$

and

$$\frac{1}{\Psi(f\zeta,f\varrho,\gamma)}-1\geq k\left(\frac{1}{\Psi(\zeta,\varrho,\gamma)}-1\right).$$

Recently, Ghosh et al. [38] introduced the concept of neutrosophic fuzzy metric spaces and presented various topological characteristics regarding to this concept.

Definition 1.6. [38] A 7-tuple $(\mathcal{W},\Theta,\Pi,\Phi,\Psi,\bullet,\diamond)$ is known as a Neutrophic Fuzzy Metric Space (NFMS) if $\mathcal{W}$ is an arbitary set, $\bullet$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and Θ,Π,Φ, and Ψ are fuzzy sets on $\mathcal{W}^2\times(0,\infty)$ satisfying the following conditions for all $\zeta,\varrho,c,\in\mathcal{W}$ and $\gamma,\varpi>0$.

(1) $0\leq\Theta(\zeta,\varrho,\gamma)\leq 1$, $0\leq\Pi(\zeta,\varrho,\gamma)\leq 1$, $0\leq\Phi(\zeta,\varrho,\gamma)\leq 1$, $0\leq\Psi(\zeta,\varrho,\gamma)\leq 1$,
(2) $0\leq\Theta(\zeta,\varrho,\gamma)+\Pi(\zeta,\varrho,\gamma)+\Phi(\zeta,\varrho,\gamma)+\Psi(\zeta,\varrho,\gamma)\leq 4$,
(3) $\Theta(\zeta,\varrho,\gamma)=1$, iff $\zeta=\varrho$
(4) $\Theta(\zeta,\varrho,\gamma)=H(\varrho,\zeta,\gamma)$,
(5) $\Theta(\zeta,\varrho,\gamma)\bullet\Theta(\varrho,c,\varpi)\leq\Theta(\zeta,c,\gamma+\varpi)$,
(6) $\Theta(\zeta,\varrho,\cdot):\mathbb{R}^+\to I_0$ is continuous
(7) $\lim_{\gamma\to\infty}\Theta(\zeta,\varrho,\gamma)=1$
(8) $\Pi(\zeta,\varrho,\gamma)=1$, iff $\zeta=\varrho$
(9) $\Pi(\zeta,\varrho,\gamma)=\Pi(\varrho,\zeta,\gamma)$, for $\gamma>0$
(10) $\Pi(\zeta,\varrho,\gamma)\bullet\Pi(\varrho,c,\varpi)\leq\Pi(\zeta,c,\gamma+\varpi)$,
(11) $\Pi(\zeta,\varrho,\cdot):\mathbb{R}^+\to I_0$ is continuous
(12) $\lim_{\gamma\to\infty}\Pi(\zeta,\varrho,\gamma)=1$
(13) $\Phi(\zeta,\varrho,\gamma)=0$, iff $\zeta=\varrho$

(14) $\Phi(\zeta, \varrho, \gamma) = \Phi(\varrho, \zeta, \gamma)$,

(15) $\Phi(\zeta, \varrho, \gamma) \diamond \Phi(\varrho, c, \varpi) \geq \Phi(\zeta, c, \gamma + \varpi)$,

(16) $\Phi(\zeta, \varrho, \cdot) : \mathbb{R}^+ \to I_0$ is continuous

(17) $\lim_{\gamma \to \infty} \Phi(\zeta, \varrho, \gamma) = 0$

(18) $\Psi(\zeta, \varrho, \gamma) = 0$, iff $\zeta = \varrho$

(19) $\Psi(\zeta, \varrho, \gamma) = \Psi(\varrho, \zeta, \gamma)$,

(20) $\Psi(\zeta, \varrho, \gamma) \diamond \Psi(\varrho, c, \varpi) \geq S(\zeta, c, \gamma + \varpi)$,

(21) $\Psi(\zeta, \varrho, \cdot) : \mathbb{R}^+ \to I_0$ is continuous

(22) $\lim_{\gamma \to \infty} \Psi(\zeta, \varrho, \gamma) = 0$

(23) If $\gamma \leq 0$, then $\Theta(\zeta, \varrho, \gamma) = \Pi(\zeta, \varrho, \gamma) = 0$, $\Phi(\zeta, \varrho, \gamma) = \Psi(\zeta, \varrho, \gamma) = 1$

In this context, $\Theta(\zeta, \varrho, \gamma)$ represents the certainity that distance between ζ and ϱ is less than γ, $\Pi(\zeta, \varrho, \gamma)$ represents the degree of nearness, $\Phi(\zeta, \varrho, \gamma)$ stadns for the degree of neutralness, and $\Psi(\zeta, \varrho, \gamma)$ denotes the degree of non nearness between ζ and ϱ with respect to γ, respectively.

The convergence, Cauchyness, completeness are given as follows.

Definition 1.7. [38] Let (ζ_n) be a sequence in a NFMS $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$. Then

(1) (ζ_n) converges to $\zeta \in \mathcal{W}$ if for any $\epsilon \in (0,1)$, $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$

$$\Theta(\zeta_n, \zeta, \gamma) > 1 - \epsilon,\ \Pi(\zeta_n, \zeta, \gamma) > 1 - \epsilon,\ \Phi(\zeta_n, \zeta, \gamma) < \epsilon,\ \Psi(\zeta_n, \zeta, \gamma) < \epsilon$$

i.e.,

$$\lim_{n\to\infty} \Theta(\zeta_n, \zeta, \gamma) = 1,\ \lim_{n\to\infty} \Pi(\zeta_n, \zeta, \gamma) = 1,\ \lim_{n\to\infty} \Phi(\zeta_n, \zeta, \gamma) = 0,\ \lim_{n\to\infty} \Psi(\zeta_n, \zeta, \gamma) = 0$$

(2) (ζ_n) is called Cauchy if any $\epsilon \in (0,1)$, $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that for each $n, m \geq n_0$

$$\Theta(\zeta_n, \zeta_m, \gamma) > 1 - \epsilon,\ \Pi(\zeta_n, \zeta_m, \gamma) > 1 - \epsilon,\ \Phi(\zeta_n, \zeta_m, \gamma) < \epsilon,\ \Psi(\zeta_n, \zeta_m, \gamma) < \epsilon$$

i.e.,

$$\lim_{n.m\to\infty} \Theta(\zeta_n, \zeta_m, \gamma) = 1,\ \lim_{n.m\to\infty} \Pi(\zeta_n, \zeta_m, \gamma) = 1,\ \lim_{n,m\to\infty} \Phi(\zeta_n, \zeta_m, \gamma) = 0,\ \lim_{n,m\to\infty} \Psi(\zeta_n, \zeta_m, \gamma) = 0$$

(3) $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$ is called complete if every Cauchy sequence in $\mathcal{W}$ is convergent to an element in $\mathcal{W}$.

In 2008, Popescu [39] introduced the innovative concept of P-contraction mappings in the realm of metric spaces. He demonstrated that any P-contraction mapping defined on complete metric spaces has a unique fixed point.

The definition of P-contractions is articulated as follows:

Definition 1.8. [39] Let $(\mathcal{W}, d)$ represent a metric space, and let f be a self-mapping on $\mathcal{W}$. The mapping f is classified as a P-contraction if

$$d(f\zeta, f\xi) \leq k\ \{d(\zeta, \xi) + |d(\zeta, f\zeta) - d(\xi, f\xi)|\ \},$$

holds for all $\zeta, \xi \in \mathcal{W}$.

It is noteworthy that when $k = 1$, the mapping f is referred to as p-contractive, as indicated in [40].

The theorem established by Popescu is articulated as follows.

Theorem 1.9. *[39] If $(\mathcal{W}, d)$ is a complete metric space and $f : \mathcal{W} \to \mathcal{W}$ is a P-contraction, then f possesses a unique fixed point.*

This study investigates fixed point theory within the context of neutrosophic metric spaces, drawing inspiration from previous works in the field, such as those by [41–43]. This paper draws inspiration from the P-contractions concept introduced by Popescu to establish a novel category of contractions, referred to as N-$P\mathcal{L}$-contractions. This new classification employs an auxiliary function within the neutrosophic metric space framework. We prove that these contractions possess a unique fixed point under specific conditions. The main findings of this study enable the formulation of various fixed point theorems.

2. Main Result

We commence with the subsequent essential lemmas that are required to derive our principal result.

Lemma 2.1. *Let $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$ be a NFMS. Then*

(1) $\Theta(\zeta, \varrho, \cdot) : \mathbb{R} \to \mathbb{R}$ *is non-decreasing*
(2) $\Pi(\zeta, \varrho, \cdot) : \mathbb{R} \to \mathbb{R}$ *is non-decreasing*
(3) $\Phi(\zeta, \varrho, \cdot) : \mathbb{R} \to \mathbb{R}$ *is non-increasing*
(4) $\Psi(\zeta, \varrho, \cdot) : \mathbb{R} \to \mathbb{R}$ *is non-increasing*

Proof. (1) Let $\gamma_1, \gamma_2 > 0$, with $\gamma_1 > \gamma_2$. Then, there is $\delta > 0$ such that $\gamma_1 = \gamma_2 + \delta$.
From (5), we get

$$\begin{aligned}\Theta(\zeta, \varrho, \gamma_1) &= \Theta(\zeta, \varrho, \gamma_2 + \delta)\\ &\geq \Theta(\zeta, \varrho, \gamma_2) \bullet \Theta(\varrho, \varrho, \delta)\\ &= \Theta(\zeta, \varrho, \gamma_2).\end{aligned}$$

The demonstrations for the remaining cases are analogous to that of (1). □

Lemma 2.2. *Let $\mathcal{H} : \mathcal{D} \to \mathcal{B}$ be a bounded real valued function, where $\mathcal{D}$, $\mathcal{B}$ are subsets of $\mathbb{R}$. Then*

(1) *If $\mathcal{H}$ is non increasing, then there is $c > 0$ such that for all $\lambda \in \mathcal{D}$*

$$\mathcal{H}\left(\frac{\lambda}{2}\right) \leq c\ \mathcal{H}(\lambda)$$

(2) *If $\mathcal{H}$ is non decreasing, then there is $\tau > 0$ such that for all $\lambda \in \mathcal{D}$*

$$\mathcal{H}\left(\frac{\lambda}{2}\right) \geq \tau\ \mathcal{H}(\lambda)$$

Proof. (1)Let $\lambda \in \mathcal{D}$ be arbitrary. Since $\frac{\lambda}{2} < \lambda$ and $\mathcal{H}$ is non increasing, then

$$\mathcal{H}\left(\frac{\lambda}{2}\right) \geq \mathcal{H}(\lambda).$$

So, there is $\xi_\lambda \geq 0$ such that

$$\mathcal{H}\left(\frac{\lambda}{2}\right) = \mathcal{H}(\lambda) + \xi_\lambda.$$

Thus,

$$\mathcal{H}\left(\frac{\lambda}{2}\right) \leq 2\max\{\mathcal{H}(\lambda), \xi_\lambda\}.$$

Now,

$$\mathcal{H}(\lambda).\xi_\lambda = c_\lambda \Longrightarrow c_\lambda.\mathcal{H}(\lambda) = \xi_\lambda.$$

So,

$$\mathcal{H}\left(\frac{\lambda}{2}\right) \leq 2\max\{\mathcal{H}(\lambda), c_\lambda.\mathcal{H}(\lambda)\}$$

$$\mathcal{H}\left(\frac{\lambda}{2}\right) \leq \max\{2, 2c_\lambda\}\mathcal{H}(\lambda).$$

Let $C = \max\{2, \max_{\lambda\in\mathcal{D}}\{2c_\lambda\}\}$. Then we get the result.
The proof of (2) is identical to that of (1).□

Remark 2.3. According to Lemma 2.1, and Lemma 2.2, there are $C_\Theta, C_\Pi, C_\Phi, C_\Psi$ such that

$$\Theta(\zeta, \varrho, \frac{\lambda}{2}) \geq C_\Theta\ \Theta(\zeta, \varrho, \lambda),$$

$$\Pi(\zeta, \varrho, \frac{\lambda}{2}) \geq C_\Pi\ \Pi(\zeta, \varrho, \lambda),$$

$$\Phi(\zeta, \varrho, \frac{\lambda}{2}) \leq C_\Phi\ \Phi(\zeta, \varrho, \lambda),$$

$$\Psi(\zeta, \varrho, \frac{\lambda}{2}) \leq C_\Psi\ \Psi(\zeta, \varrho, \lambda),$$

Definition 2.4. In this context, we define a real-valued function of three variables that operates over the domain $\mathcal{W}^2 \times (0, \infty)$, where $\mathcal{W}$ is any non-empty set. We refer to this function as $\mathcal{H}$ and claim that it possesses the property (UC) if, for any sequences (ζ_n) and (ϱ_n) contained within $\mathcal{W}$, the following equality holds:

$$\lim_{\lambda \to \lambda_0} \lim_{n \to \infty} \mathcal{H}(\zeta_n, \varrho_n, \lambda) = \lim_{n \to \infty} \lim_{\lambda \to \lambda_0} \mathcal{H}(\zeta_n, \varrho_n, \lambda).$$

Whenever both limits exist.

In the following sections of this study, we will proceed with the premise that each of the fuzzy sets $\Theta,\ \Phi,\ \Psi$ exhibits the UC property.

Through this context we need the following class of functions:

Definition 2.5. Let $\mathscr{L}$ denote the collection of all functions $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the condition

$$\mathcal{L}(a, b) \leq a + b.$$

We will now present the definition of NF-L contractions.

Definition 2.6. Let $(\mathcal{W}, \Theta, \Phi, \Psi, \bullet, \diamond)$ be an NFMS. A mapping $f : \mathcal{W} \to \mathcal{W}$ is called neutrosophic fuzzy p-contraction (shortly NF-L contraction) if there is $0 < k < \frac{1}{3}$ such that for each $\zeta, \varrho \in \mathcal{W}$ and each $\gamma > 0$, we have

$$\frac{1}{\Theta(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Theta(\zeta, \varrho, \gamma)} - 1 + \mathcal{L}\left(\frac{1}{\Theta(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\Theta(\varrho, f\varrho, \gamma)} - 1 \right) \right],$$

$$\frac{1}{\Pi(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Pi(\zeta, \varrho, \gamma)} - 1 + \mathcal{L}\left(\frac{1}{\Pi(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\Pi(\varrho, f\varrho, \gamma)} - 1 \right) \right],$$

$$\Phi(f\zeta, f\varrho, \gamma) \leq k \left[\Phi(\zeta, \varrho, \gamma) + \mathcal{L}\left(\Phi(\zeta, f\zeta, \gamma), \Phi(\varrho, f\varrho, \gamma) \right) \right],$$

and

$$\Psi(f\zeta, f\varrho, \gamma) \leq k \left[\Psi(\zeta, \varrho, \gamma) + \mathcal{L}\left(\Psi(\zeta, f\zeta, \gamma), \Psi(\varrho, f\varrho, \gamma) \right) \right].$$

Lemma 2.7. *Let $(\mathcal{W}, \Theta, \Phi, \Psi, \bullet, \diamond)$ be a complete NFMS, Suppose that $f : \mathcal{W} \to \mathcal{W}$ is NF-L contraction. Consequently, if f has a fixed point then it is unique.*

Proof. Assume that $\mu, \nu \in \mathcal{W}$ such that $f\mu = \mu$ and $f\nu = \nu$. Then, Definition 2.6 implies

$$\frac{1}{\Theta(\mu, \nu, \gamma)} - 1 = \frac{1}{\Theta(f\mu, f\nu, \gamma)} - 1 \leq k \left[\frac{1}{\Theta(\mu, \nu, \gamma)} - 1 + \mathcal{L}\left(\frac{1}{\Theta(\mu, \mu, \gamma)} - 1, \frac{1}{\Theta(\nu, \nu, \gamma)} - 1 \right) \right],$$

$$\frac{1}{\Pi(\mu, \nu, \gamma)} - 1 = \frac{1}{\Pi(f\mu, f\nu, \gamma)} - 1 \leq k \left[\frac{1}{\Pi(\mu, \nu, \gamma)} - 1 + \mathcal{L}\left(\frac{1}{\Pi(\mu, \mu, \gamma)} - 1, \frac{1}{\Pi(\nu, \nu, \gamma)} - 1 \right) \right],$$

$$\Phi(\mu,\nu,\gamma)=\Phi(f\mu,f\nu,\gamma)\leq k\left[\Phi(\mu,\nu,\gamma)+\mathcal{L}\left(\Phi(\mu,\mu,\gamma),\Phi(\nu,\nu,\gamma)\right)\right],$$

and

$$\Psi(\mu,\nu,\gamma)=\Psi(f\mu,f\nu,\gamma)\leq k\left[\Psi(\mu,\nu,\gamma)+\mathcal{L}\left(\Psi(\mu,\mu,\gamma),\Psi(\nu,\nu,\gamma)\right)\right].$$

So, $\Theta(\mu,\nu,\gamma)=1$, $\Phi(\mu,\nu,\gamma)=0$, and $\Psi(\mu,\nu,\gamma)=0$, and hence $\mu=\nu$. □

Lemma 2.8. *Let $(\mathcal{W},\Theta,\mathcal{B},\Phi,\Psi,\diamond,\bullet)$ be a NFMS, and let (ζ_n) be a sequence such that for $\gamma>0$*

$$\begin{aligned}
\Theta(\zeta_p,\zeta_q,\gamma)&\geq\Theta(\zeta_{p-1},\zeta_{q-1},\gamma)\\
\Pi(\zeta_p,\zeta_q,\gamma)&\geq\Pi(\zeta_{p-1},\zeta_{q-1},\gamma)\\
\Phi(\zeta_p,\zeta_q,\gamma)&\leq\Phi(\zeta_{p-1},\zeta_{q-1},\gamma)\\
\Psi(\zeta_p,\zeta_q,\gamma)&\leq\Psi(\zeta_{p-1},\zeta_{q-1},\gamma)
\end{aligned}\tag{1}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\Theta(\zeta_n,\zeta_{n+1},\gamma)&=1,\\
\lim_{n\to\infty}\Pi(\zeta_n,\zeta_{n+1},\gamma)&=1,\\
\lim_{n\to\infty}\Phi(\zeta_n,\zeta_{n+1},\gamma)&=0,\\
\lim_{n\to\infty}\Psi(\zeta_n,\zeta_{n+1},\gamma)&=0.
\end{aligned}\tag{2}$$

If (ζ_n) is not Cauchy, then there exist an $1>\epsilon>0$ and $\gamma>0$ along with two subsequences (ζ_{n_k}) and (ζ_{m_k}) derived from (ζ_n), where (m_k) such that one at least of the following holds.

$$\begin{aligned}
\lim_{k\to\infty}\Theta(\zeta_{n_k},\zeta_{m_k},\gamma)&=1-\epsilon,\\
\lim_{k\to\infty}\Pi(\zeta_{n_k},\zeta_{m_k},\gamma)&=1-\epsilon,\\
\lim_{k\to\infty}\Phi(\zeta_{n_k},\zeta_{m_k},\gamma)&=\epsilon,\\
\lim_{k\to\infty}\Psi(\zeta_{n_k},\zeta_{m_k},\gamma)&=\epsilon.
\end{aligned}$$

Proof. If (ζ_n) is not Cauchy, then for each $\gamma>0$

$$\begin{aligned}
\lim_{n,m\to\infty}\Theta(\zeta_n,\zeta_m,\gamma)&\neq 1,\\
\lim_{n,m\to\infty}\Pi(\zeta_n,\zeta_m,\gamma)&\neq 1,\\
\lim_{n,m\to\infty}\Phi(\zeta_n,\zeta_m,\gamma)&\neq 0,
\end{aligned}$$

or

$$\lim_{n,m\to\infty}\Psi(\zeta_n,\zeta_m,\gamma)\neq 0.$$

Case 1: If $\lim_{n,m\to\infty} \Theta(\zeta_n, \zeta_m, \gamma) \neq 1$, then there are $\gamma > 0$, and $\epsilon > 0$ along with two subsequences (ζ_{n_k}) and (ζ_{m_k}) derived from (ζ_n), where (m_k) is chosen as the smallest index which satisfying the followiing condition.

$$\Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma) \leq 1 - \epsilon, \quad m_k > n_k > k. \tag{3}$$

This implies that

$$\Theta(\zeta_{n_k}, \zeta_{m_k-1}, \gamma) > 1 - \epsilon. \tag{4}$$

chose $\delta > 0$. Then

$$\begin{aligned} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma + \delta) &\geq \Theta(\zeta_{n_k}, \zeta_{m_k-1}, \gamma) \diamond \Theta(\zeta_{m_k-1}, \zeta_{m_k}, \delta) \\ &> (1 - \epsilon) \diamond \Theta(\zeta_{m_k-1}, \zeta_{m_k}, \delta). \end{aligned}$$

Using Equation 2, we get

$$\liminf_{k\to\infty} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma + \delta) \geq (1 - \epsilon).$$

Also,

$$\begin{aligned} (1 - \epsilon) &\leq \lim_{\delta\to 0^+} \liminf_{k\to\infty} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma + \delta) \\ &= \liminf_{k\to\infty} \lim_{\delta\to 0^+} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma + \delta) \\ &= \liminf_{k\to\infty} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma). \end{aligned}$$

Also, from 3, it follows

$$\limsup_{k\to\infty} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma) \leq (1 - \epsilon).$$

So, we get

$$\lim_{k\to\infty} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma) = (1 - \epsilon).$$

Again, we have

$$\begin{aligned} \Theta(\zeta_{n_k-1}, \zeta_{m_k-1}, \gamma + \delta) &\geq \Theta(\zeta_{n_k-1}, \zeta_{n_k}, \delta) \diamond \Theta(\zeta_{n_k}, \zeta_{m_k-1}, \gamma) \\ &> \Theta(\zeta_{n_k-1}, \zeta_{n_k}, \delta) \diamond (1 - \epsilon). \end{aligned}$$

Using Equation 2, we get $\liminf_{k\to\infty} \Theta(\zeta_{n_k-1}, \zeta_{m_k-1}, \gamma + \delta) \geq (1 - \epsilon)$.
Also,

$$\begin{aligned}(1-\epsilon) &\leq \lim_{\delta\to 0^+}\liminf_{k\to\infty}\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma+\delta)\\ &=\liminf_{k\to\infty}\lim_{\delta\to 0^+}\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma+\delta)\\ &=\liminf_{k\to\infty}\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma).\end{aligned}$$

From Eq 3, we get

$$\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)\leq\Theta(\zeta_{n_k},\zeta_{m_k},\gamma)\leq(1-\epsilon).$$

So,

$$\limsup_{k\to\infty}\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)\leq(1-\epsilon).$$

Hence,

$$\lim_{k\to\infty}\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)=(1-\epsilon).$$

The demonstration for the remaining cases is Similar to that of Case (1). □

Theorem 2.9. *Let $(\mathcal{W},\Theta,\Pi,\Phi,\Psi,\bullet,\diamond)$ be a complete NFMS, Suppose that $f:\mathcal{W}\to\mathcal{W}$ is NF-L contraction where $k.C_\Psi<1$. Therefor, the function f has a unique fixed point.*

Proof. Let $\zeta_0\in\mathcal{W}$ denote an arbitrary point. We consider the Picard sequence (ζ_n) defined by the relation $\zeta_{n+1}=f^n(\zeta_0)$ for all $n\geq 0$. According to Definition 2.6, this holds for each $n\in\mathbb{N}$.

$$\frac{1}{\Theta(\zeta_n,\zeta_{n+1},\gamma)}-1\leq k\left[\frac{1}{\Theta(\zeta_{n-1},\zeta_n,\gamma)}-1+\mathcal{L}\left(\frac{1}{\Theta(\zeta_{n-1},\zeta_n,\gamma)}-1,\frac{1}{\Theta(\zeta_n,\zeta_{n+1},\gamma)}-1\right)\right],$$

$$\frac{1}{\Pi(\zeta_n,\zeta_{n+1},\gamma)}-1\leq k\left[\frac{1}{\Pi(\zeta_{n-1},\zeta_n,\gamma)}-1+\mathcal{L}\left(\frac{1}{\Pi(\zeta_{n-1},\zeta_n,\gamma)}-1,\frac{1}{\Pi(\zeta_n,\zeta_{n+1},\gamma)}-1\right)\right],$$

$$\Phi(\zeta_n,\zeta_{n+1},\gamma)\leq k\left[\Phi(\zeta_{n-1},\zeta_n,\gamma)+\mathcal{L}\left(\Phi(\zeta_{n-1},\zeta_n,\gamma),\Phi(\zeta_n,\zeta_{n+1},\gamma)\right)\right],$$

and

$$\Psi(\zeta_n,\zeta_{n+1},\gamma)\leq k\left[\Psi(\zeta_{n-1},\zeta_n,\gamma)+\mathcal{L}\left(\Psi(\zeta_{n-1},\zeta_n,\gamma),\Psi(\zeta_n,\zeta_{n+1},\gamma)\right)\right].$$

From Definition 2.5, we get

$$\frac{1}{\Theta(\zeta_n,\zeta_{n+1},\gamma)}-1\leq\frac{2k}{1-k}\left(\frac{1}{\Theta(\zeta_{n-1},\zeta_n,\gamma)}-1\right),$$

$$\frac{1}{\Pi(\zeta_n,\zeta_{n+1},\gamma)}-1\leq\frac{2k}{1-k}\left(\frac{1}{\Pi(\zeta_{n-1},\zeta_n,\gamma)}-1\right),$$

$$\Phi(\zeta_n,\zeta_{n+1},\gamma)\leq\frac{2k}{1-k}\left(\Phi(\zeta_{n-1},\zeta_n,\gamma)\right),$$

and

$$\Psi(\zeta_n,\zeta_{n+1},\gamma)\leq\frac{2k}{1-k}\left(\Psi(\zeta_{n-1},\zeta_n,\gamma)\right).$$

By proceeding this process, we get

$$\frac{1}{\Theta(\zeta_n,\zeta_{n+1},\gamma)}-1\leq\left(\frac{2k}{1-k}\right)^n\left(\frac{1}{\Theta(\zeta_0,\zeta_1,\gamma)}-1\right),$$

$$\frac{1}{\Pi(\zeta_n,\zeta_{n+1},\gamma)}-1\leq\left(\frac{2k}{1-k}\right)^n\left(\frac{1}{\Pi(\zeta_0,\zeta_1,\gamma)}-1\right),$$

$$\Phi(\zeta_n,\zeta_{n+1},\gamma)\leq\left(\frac{2k}{1-k}\right)^n\left(\Phi(\zeta_0,\zeta_1,\gamma)\right),$$

and

$$\Psi(\zeta_n,\zeta_{n+1},\gamma)\leq\left(\frac{2k}{1-k}\right)^n\left(\Psi(\zeta_0,\zeta_1,\gamma)\right).$$

Given that $k<\frac{1}{3}$, we can determine the limit as follows.

$$\begin{aligned}
&\lim_{n\to\infty}\Theta(\zeta_n,\zeta_{n+1},\gamma)=1,\\
&\lim_{n\to\infty}\Pi(\zeta_n,\zeta_{n+1},\gamma)=1,\\
&\lim_{n\to\infty}\Phi(\zeta_n,\zeta_{n+1},\gamma)=0,\\
&\lim_{n\to\infty}\Psi(\zeta_n,\zeta_{n+1},\gamma)=0.
\end{aligned}\tag{5}$$

We assert that the sequence (ζ_n) is a Cauchy sequence. Suppose the opposite is true. By Lemma 2.8, there exist $1>\epsilon>0$ and $\gamma>0$, along with two subsequences (ζ_{n_k}) and (ζ_{m_k}) extracted from (ζ_n), where (m_k) is such that at least one of the following conditions is satisfied.

$$\begin{aligned}
&\lim_{k\to\infty}\Theta(\zeta_{n_k},\zeta_{m_k},\gamma)=1-\epsilon,\\
&\lim_{k\to\infty}\Pi(\zeta_{n_k},\zeta_{m_k},\gamma)=1-\epsilon,\\
&\lim_{k\to\infty}\Phi(\zeta_{n_k},\zeta_{m_k},\gamma)=\epsilon,\\
&\lim_{k\to\infty}\Psi(\zeta_{n_k},\zeta_{m_k},\gamma)=\epsilon.
\end{aligned}$$

Definition 2.6 implies that

$$\frac{1}{\Theta(\zeta_{n_k},\zeta_{m_k},\gamma)}-1\leq k\left[\frac{1}{\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)}-1+\mathcal{L}\left(\frac{1}{\Theta(\zeta_{n_k-1},\zeta_{n_k},\gamma)}-1,\frac{1}{\Theta(\zeta_{m_k-1},\zeta_{m_k},\gamma)}-1\right)\right],$$

$$\frac{1}{\Pi(\zeta_{n_k},\zeta_{m_k},\gamma)}-1\leq k\left[\frac{1}{\Pi(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)}-1+\mathcal{L}\left(\frac{1}{\Pi(\zeta_{n_k-1},\zeta_{n_k},\gamma)}-1,\frac{1}{\Pi(\zeta_{m_k-1},\zeta_{m_k},\gamma)}-1\right)\right],$$

$$\Phi(\zeta_{n_k},\zeta_{m_k},\gamma)\leq k\left[\Phi(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)+\mathcal{L}\left(\Phi(\zeta_{n_k-1},\zeta_{n_k},\gamma),\Phi(\zeta_{m_k-1},\zeta_{m_k},\gamma)\right)\right],$$

and

$$\Psi(\zeta_{n_k},\zeta_{m_k},\gamma)\leq k\left[\Psi(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)+\mathcal{L}\left(\Psi(\zeta_{n_k-1},\zeta_{n_k},\gamma),\Psi(\zeta_{m_k-1},\zeta_{m_k},\gamma)\right)\right].$$

Hence,

$$\frac{1}{\Theta(\zeta_{n_k},\zeta_{m_k},\gamma)}-1\leq k\left[\frac{1}{\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)}-1+\frac{1}{\Theta(\zeta_{n_k-1},\zeta_{n_k},\gamma)}-1+\frac{1}{\Theta(\zeta_{m_k-1},\zeta_{m_k},\gamma)}-1\right],$$

$$\frac{1}{\Pi(\zeta_{n_k},\zeta_{m_k},\gamma)}-1\leq k\left[\frac{1}{\Pi(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)}-1+\frac{1}{\Pi(\zeta_{n_k-1},\zeta_{n_k},\gamma)}-1+\frac{1}{\Pi(\zeta_{m_k-1},\zeta_{m_k},\gamma)}-1\right],$$

$$\Phi(\zeta_{n_k},\zeta_{m_k},\gamma)\leq k\left[\Phi(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)+\Phi(\zeta_{n_k-1},\zeta_{n_k},\gamma)+\Phi(\zeta_{m_k-1},\zeta_{m_k},\gamma)\right],$$

and

$$\Psi(\zeta_{n_k},\zeta_{m_k},\gamma)\leq k\left[\Psi(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)+\Psi(\zeta_{n_k-1},\zeta_{n_k},\gamma)+\Psi(\zeta_{m_k-1},\zeta_{m_k},\gamma)\right].$$

By taking the limit when $k\to\infty$, we get

$$\frac{\epsilon}{1-\epsilon}\leq\frac{k\ \epsilon}{1-\epsilon},$$

$$\frac{\epsilon}{1-\epsilon}\leq\frac{k\ \epsilon}{1-\epsilon},$$

$$\epsilon\leq k\ \epsilon,$$

and

$$\epsilon\leq k\ \epsilon.$$

Which is a contradiction because $k<\frac{1}{3}$. Hence (ζ_n) is a Cauchy sequence, thus, there is $\mu\in\mathcal{W}$ such that $\zeta_n\to\mu$.

From Definition 2.6, we have for each $n\in\mathbb{N}$

$$\begin{aligned}\Psi(f\mu,\zeta_{n+1},\gamma)&\leq k\left[\Psi(\mu,\zeta_n,\gamma)+\mathcal{L}\left(\Psi(\mu,f\mu,\gamma),\Psi(\zeta_n,\zeta_{n+1},\gamma)\right)\right]\\&\leq k\left[\Psi(\mu,\zeta_n,\gamma)+\Psi(\mu,f\mu,\gamma)+\Psi(\zeta_n,\zeta_{n+1},\gamma)\right]\end{aligned}$$

By taking the limit, we obtain

$$\lim_{n\to\infty} \Psi(f\mu, \zeta_{n+1}, \gamma) \leq k\ \Psi(\mu, f\mu, \gamma). \tag{6}$$

On the other hand, using Remark 2.3, we get

$$\begin{aligned}\Psi(\mu, f\mu, \gamma) &\leq \Psi(\mu, \zeta_n, \tfrac{\gamma}{2}) \bullet \Psi(\zeta_n, f\mu, \tfrac{\gamma}{2}) \\ &\leq \Psi(\mu, \zeta_n, \tfrac{\gamma}{2}) \bullet C_\Psi\ \Psi(\zeta_n, f\mu, \gamma).\end{aligned}$$

So,

$$\Psi(\mu, f\mu, \gamma) \leq C_\Psi \lim_{n\to\infty} \Psi(\zeta_n, f\mu, \gamma). \tag{7}$$

From Eq 6, Eq 7, we get

$$\lim_{n\to\infty} \Psi(\zeta_n, f\mu, \gamma) \leq k\ C_\Psi \lim_{n\to\infty} \Psi(\zeta_n, f\mu, \gamma).$$

So, $\lim_{n\to\infty} \Psi(\zeta_n, f\mu, \gamma) = 0$, and hence (ζ_n) converges to $f\mu$. Hence $\mu = f\mu$. The uniqueness follows from Lemma 2.7 □

By establishing the function $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the definition $\mathcal{L}(a, b) = 0$, we derive the subsequent result.

Corollary 2.10. *Let $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$ be a complete NFMS and $f : \mathcal{W} \to \mathcal{W}$ be a self map. Assume that there is $0 < k < \min\{\frac{1}{3}, \frac{1}{C_\Psi}\}$ such that for each $\zeta, \varrho \in \mathcal{W}$ and each $\gamma > 0$, f satisfies the following:*

$$\frac{1}{\Theta(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Theta(\zeta, \varrho, \gamma)} - 1\right],$$

$$\frac{1}{\Pi(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Pi(\zeta, \varrho, \gamma)} - 1\right],$$

$$\Phi(f\zeta, f\varrho, \gamma) \leq k\,[\Phi(\zeta, \varrho, \gamma)],$$

and

$$\Psi(f\zeta, f\varrho, \gamma) \leq k\,[\Psi(\zeta, \varrho, \gamma)].$$

Therefor, the function f has a unique fixed point.

By establishing the function $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the definition $\mathcal{L}(a, b) = |a - b|$, we derive the subsequent result.

Corollary 2.11. *Let $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$ be a complete NFMS and $f : \mathcal{W} \to \mathcal{W}$ be a self map. Assume that there is $0 < k < \min\{\frac{1}{3}, \frac{1}{C_\Psi}\}$ such that for each $\zeta, \varrho \in \mathcal{W}$ and each $\gamma > 0$, f satisfies the following:*

$$\frac{1}{\Theta(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Theta(\zeta, \varrho, \gamma)} - 1 + \left| \frac{1}{\Theta(\zeta, f\zeta, \gamma)} - \frac{1}{\Theta(\varrho, f\varrho, \gamma)} \right| \right],$$

$$\frac{1}{\Pi(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Pi(\zeta, \varrho, \gamma)} - 1 + \left| \frac{1}{\Pi(\zeta, f\zeta, \gamma)} - \frac{1}{\Pi(\varrho, f\varrho, \gamma)} \right| \right],$$

$$\Phi(f\zeta, f\varrho, \gamma) \leq k \left[\Phi(\zeta, \varrho, \gamma) + |\Phi(\zeta, f\zeta, \gamma) - \Phi(\varrho, f\varrho, \gamma)| \right],$$

and

$$\Psi(f\zeta, f\varrho, \gamma) \leq k \left[\Psi(\zeta, \varrho, \gamma) + |\Psi(\zeta, f\zeta, \gamma) - \Psi(\varrho, f\varrho, \gamma)| \right].$$

Therefor, the function f has a unique fixed point.

By establishing the function $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the definition $\mathcal{L}(a, b) = a + b$, we derive the subsequent result.

Corollary 2.12. *Let $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$ be a complete NFMS and $f : \mathcal{W} \to \mathcal{W}$ be a self map. Assume that there is $0 < k < \min\{\frac{1}{3}, \frac{1}{C_\Psi}\}$ such that for each $\zeta, \varrho \in \mathcal{W}$ and each $\gamma > 0$, f satisfies the following:*

$$\frac{1}{\Theta(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Theta(\zeta, \varrho, \gamma)} - 1 + \frac{1}{\Theta(\zeta, f\zeta, \gamma)} - 1 + \frac{1}{\Theta(\varrho, f\varrho, \gamma)} - 1 \right],$$

$$\frac{1}{\Pi(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Pi(\zeta, \varrho, \gamma)} - 1 + \frac{1}{\Pi(\zeta, f\zeta, \gamma)} - 1 + \frac{1}{\Pi(\varrho, f\varrho, \gamma)} - 1 \right],$$

$$\Phi(f\zeta, f\varrho, \gamma) \leq k \left[\Phi(\zeta, \varrho, \gamma) + \Phi(\zeta, f\zeta, \gamma) + \Phi(\varrho, f\varrho, \gamma) \right],$$

and

$$\Psi(f\zeta, f\varrho, \gamma) \leq k \left[\Psi(\zeta, \varrho, \gamma) + \Psi(\zeta, f\zeta, \gamma) + \Psi(\varrho, f\varrho, \gamma) \right].$$

Therefor, the function f has a unique fixed point.

By establishing the function $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the definition $\mathcal{L}(a, b) = \max\{a, b\}$, we derive the subsequent result.

Corollary 2.13. *Let $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$ be a complete NFMS and $f : \mathcal{W} \to \mathcal{W}$ be a self map. Assume that there is $0 < k < \min\{\frac{1}{3}, \frac{1}{C_\Psi}\}$ such that for each $\zeta, \varrho \in \mathcal{W}$ and each $\gamma > 0$, f satisfies the following:*

$$\frac{1}{\Theta(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Theta(\zeta, \varrho, \gamma)} - 1 + \max \left\{ \frac{1}{\Theta(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\Theta(\varrho, f\varrho, \gamma)} - 1 \right\} \right],$$

$$\frac{1}{\Pi(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Pi(\zeta, \varrho, \gamma)} - 1 + \max \left\{ \frac{1}{\Pi(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\Pi(\varrho, f\varrho, \gamma)} - 1 \right\} \right],$$

$$\Phi(f\zeta, f\varrho, \gamma) \leq k\left[\Phi(\zeta, \varrho, \gamma) + \max\left\{\Phi(\zeta, f\zeta, \gamma), \Phi(\varrho, f\varrho, \gamma)\right\}\right],$$

and

$$\Psi(f\zeta, f\varrho, \gamma) \leq k\left[\Psi(\zeta, \varrho, \gamma) + \max\left\{\Psi(\zeta, f\zeta, \gamma), \Psi(\varrho, f\varrho, \gamma)\right\}\right].$$

Therefor, the function f has a unique fixed point.

By establishing the function $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the definition $\mathcal{L}(a, b) = 2\min\{a, b\}$, we derive the subsequent result.

Corollary 2.14. *Let $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$ be a complete NFMS and $f : \mathcal{W} \to \mathcal{W}$ be a self map. Assume that there is $0 < k < \min\{\frac{1}{3}, \frac{1}{C_\Psi}\}$ such that for each $\zeta, \varrho \in \mathcal{W}$ and each $\gamma > 0$, f satisfies the following:*

$$\frac{1}{\Theta(f\zeta, f\varrho, \gamma)} - 1 \leq k\left[\frac{1}{\Theta(\zeta, \varrho, \gamma)} - 1 + 2\min\left\{\frac{1}{\Theta(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\Theta(\varrho, f\varrho, \gamma)} - 1\right\}\right],$$

$$\frac{1}{\Pi(f\zeta, f\varrho, \gamma)} - 1 \leq k\left[\frac{1}{\Pi(\zeta, \varrho, \gamma)} - 1 + 2\min\left\{\frac{1}{\Pi(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\Pi(\varrho, f\varrho, \gamma)} - 1\right\}\right],$$

$$\Phi(f\zeta, f\varrho, \gamma) \leq k\left[\Phi(\zeta, \varrho, \gamma) + 2\min\left\{\Phi(\zeta, f\zeta, \gamma), \Phi(\varrho, f\varrho, \gamma)\right\}\right],$$

and

$$\Psi(f\zeta, f\varrho, \gamma) \leq k\left[\Psi(\zeta, \varrho, \gamma) + 2\min\left\{\Psi(\zeta, f\zeta, \gamma), \Psi(\varrho, f\varrho, \gamma)\right\}\right].$$

Therefor, the function f has a unique fixed point.

Conclusion

Fixed-point theory broadly encompasses a range of theorems that address the behavior of transformations applied to points within a given set, guaranteeing the existence of at least one invariant point. These theorems play a pivotal role in establishing the existence of solutions to various equations and systems across multiple branches of mathematics. A notable example is Banach's Fixed Point Theorem, a cornerstone in the field of analysis, which asserts that any contraction mapping from a complete metric space to itself possesses a unique fixed point. Such theorems are essential in diverse areas, including differential equations, economics, and computer science, as they facilitate the identification of equilibria and solutions. In summary, fixed-point theorems serve as vital instruments in both theoretical and applied mathematics, offering foundational knowledge and effective strategies for addressing intricate problems by confirming the existence and, in some cases, the uniqueness of solutions.

In this study, we introduced neutrosophic P$\mathcal{L}$-contractions within the framework of neutrosophic fuzzy metric spaces, demonstrating that these contractions yield a unique fixed point.

This primary finding paves the way for numerous subsequent results. Future research may aim to extend our findings to other types of distance spaces or enhance them by incorporating additional appropriate conditions.

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Comprehensive Evaluation of Safety Emergency Plans for Petrochemical Enterprises under Interval Valued Trapezoidal Neutrosophic Set to Enhance Preparedness and Risk Mitigation

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Abstract: Safety production incidents have had a significant impact on people's livelihoods and property as the petrochemical industry has grown. To prevent and respond to various types of mishaps, numerous nations have created emergency response systems. This study proposes a decision-making approach for evaluation of safety emergency plans for petrochemical enterprises under the uncertainty framework. We use the neutrosophic sets to deal with the uncertainty and vague information. This study uses the interval valued trapezoidal neutrosophic set (IVTNS). We define a set of definition related to IVTNS and their operations. Two decision making methods are used in this study such as LBWA to compute the criteria weights and CoCoSo method to rank the alternatives. This study uses nine criteria and eleven alternatives to be evaluated in the decision-making problem. The sensitivity analysis shows the ranks of the alternatives are stable under different cases.

Keywords: Interval Valued Trapezoidal Neutrosophic Set (IVTNS); Petrochemical Enterprises; Emergency; Safety.

1. Introduction and Related Work

To address the imprecision, incompleteness, and ambiguity in the information, Zadeh created the fuzzy set theory. In the event where the grade of membership is ambiguous and cannot be represented by a precise value, Zadeh [1] later introduced the interval valued fuzzy sets (IVFS). Based on fuzzy set theory, Atanassov [2] created an intuitionistic fuzzy set (IFS) and an interval-valued IFS. Numerous scholars have investigated the application of IFSs in MCDM scenarios[3].

Liu and Yuan [4] created triangular intuitionistic fuzzy sets (TIFS) by fusing the ideas of IFS and triangular fuzzy numbers (TFN). Additionally, the interval valued intuitionistic fuzzy set (IVIFS)

was introduced by Atanassov and Gargov, [5] who integrated the IFS and IVFS. Additionally, MADM and multi attribute group decision making (MAGDM) scenarios were used to illustrate the application of IVIFS. Wang used triangular intuitionistic fuzzy sets to propose the weighted geometric and hybrid geometric operators. Additionally, he used both operators to solve MAGDM issues.

Based on fuzzy number intuitionistic fuzzy numbers, Wei et al. [6] developed an induced ordered weighted geometric operator and presented a method for resolving group decision-making issues using the suggested operator. Ye [7] introduced the trapezoidal intuitionistic fuzzy set (TrIFS) as an extension of the TIFS for encoding the membership and non-membership values as a trapezoid. Even though many academics have proposed different forms of fuzzy sets, the impact of this ignorance should also be considered when making a judgment if an expert is not quite certain about their choice.

Experts may occasionally be unsure about the outcome but nonetheless reach a decision that is either positive (acceptable) or negative (rejection). Together with membership and non-membership functions, the unknowingness/indeterminacy component must be considered in this situation. To deal with imprecise, incomplete, and uncertain information, Smarandache introduced the neutrosophic set (NS) and expanded the concepts of classic, fuzzy, and IFS. Later, a single-valued neutrosophic set (SVNS), a version of an NS, is presented that has practical applications. Ye presented the TrNS as an expansion of the SVNS and trapezoidal fuzzy numbers (TrFN). Additionally, he presented the geometric averaging operator and weighted arithmetic based on the trapezoidal neutrosophic number.

Furthermore, he presented a technique to deal with MADM difficulties utilizing these operators. As previously mentioned, researchers have proposed a number of methods based on IVIFS, TrIFS, and TrNS sets to deal with inconsistency, impreciseness, uncertainty, incompleteness, and indeterminacy in information that is either (1) intuitionistic fuzzy and in the unit interval of real numbers and can be represented as a triangle/trapezoid, or (2) neutrosophic and can be represented as a trapezoid. However, the suggested approach deals with data that is neutrosophic in nature, falls inside the real number unit interval [8], and can be shown as a triangle or a trapezoid.

Based on the combination of IVIFN and TrNS, Kiran Khatter [9] suggested an interval valued trapezoidal neutrosophic set (IVTNS). The operating laws for IVTNN are also introduced in the study. Additionally, to incorporate the neutrosophic trapezoidal information in the unit interval of real number is introduced. To demonstrate the applicability of the created method, a mathematical illustration is presented after a method is developed to manage the issues in the MADM system utilizing the IVTNN operator.

It has been shown that a single valued neutrosophic set may encompass the indeterminacy in real-world decision-making problems. It is an expansion of the neutrosophic set with interval values. Most real-world issues contain some degree of uncertainty, and determining the

network's shortest path is one of the most well-known examples. Broumi et al. [10] proposed interval valued neutrosophic numbers to solve shortest path and a new score function for these numbers is proposed. By considering interval-valued neutrosophic numbers, trapezoidal, and triangular interval-valued neutrosophic numbers for the path length in a network with an illustrative example, new algorithms are also presented to determine the neutrosophic shortest path.

2. Interval valued trapezoidal neutrosophic set (IVTNS)

The interval valued trapezoidal neutrosophic number (IVTNN) can be defined as[9]:

$$N^L = \left\{\left(x, T_N^L(x), I_N^L(x), F_N^L(x)\right); x \in X\right\} \tag{1}$$

$$T_N^L(x) \subset [0,1] \tag{2}$$

$$I_N^L(x) \subset [0,1] \tag{3}$$

$$F_N^L(x) \subset [0,1] \tag{4}$$

These numbers refer to the trapezoidal neutrosophic fuzzy numbers

$$T_N^L(x) = \left(t^{L1}{}_N(x), t^{L2}{}_N(x), t^{L3}{}_N(x), t^{L4}{}_N(x)\right) x \to [0,1] \tag{5}$$

$$F_N^L(x) = \left(f^{L1}{}_N(x), f^{L2}{}_N(x), f^{L3}{}_N(x), f^{L4}{}_N(x)\right) x \to [0,1] \tag{6}$$

$$T_N^L(x) = \left(i^{L1}{}_N(x), i^{L2}{}_N(x), i^{L3}{}_N(x), i^{L4}{}_N(x)\right) x \to [0,1] \tag{7}$$

$$0 \le t^{L4}{}_N(x) + i^{L4}{}_N(x) + f^{L4}{}_N(x) \le 3 \tag{8}$$

$$N^L = \left\{\left((a,b,c,d),(,e,f,g,h),(l,l,n,p)\right) > X \to [0,1]\right\} \tag{9}$$

We can define the $T_N^L(x), I_N^L(x), F_N^L(x)$

$$T_N^L(x) = \begin{cases} \frac{x-a}{b-a} T_N^L & a \le x \le b \\ T_N^L & b \le x \le c \\ \frac{d-x}{d-c} T_N^L & c \le x \le d \\ 0 & otherwise \end{cases} \tag{10}$$

$$I_N^L(x) = \begin{cases} \frac{f-x+I_N^L(x-e)}{f-e} & e \le x \le f \\ I_N^L & f \le x \le g \\ \frac{x-g+I_N^L(h-x)}{h-g} & g \le x \le h \\ 1 & otherwise \end{cases} \tag{11}$$

$$F_N^L(x) = \begin{cases} \frac{m-x+F_N^L(x-l)}{m-l} & l \le x \le m \\ F_N^L & m \le x \le n \\ \frac{x-n+F_N^L(p-x)}{p-n} & n \le x \le p \\ 1 & otherwise \end{cases} \tag{12}$$

Definition

Let N^U be upper trapezoidal neutrosophic fuzzy number then $T^U{}_N(x), I^U{}_N(x)\ and\ F^U{}_N(x)$

$$T^U{}_N(x) = \begin{cases} \frac{x-a}{b-a} & a \le x < b \\ T^U{}_N & b \le x < c \\ \frac{d-x}{d-c} & c \le x < d \\ 0 & otherwise \end{cases} \tag{13}$$

$$I^U{}_N(x) = \begin{cases} \frac{f-x+I^U{}_N(x-e)}{f-e} & e \le x < f \\ I^U{}_N & f \le x < g \\ \frac{x-g+I^U{}_N(h-x)}{h-g} & g \le x < h \\ 1 & otherwise \end{cases} \tag{14}$$

$$F^U{}_N(x) = \begin{cases} \frac{m-x+F^U{}_N(x-l)}{m-l} & l \le x < m \\ F^U{}_N & m \le x < n \\ \frac{x-n+F^U{}_N(p-x)}{p-n} & n \le x < p \\ 1 & otherwise \end{cases} \tag{15}$$

Definition

Let two IVTNNs can be defined such as:

$$n_1 = \begin{Bmatrix} \left(\left[(a_1, b_1, c_1, d_1), \left(\overline{a_1}, \overline{b_1}, \overline{c_1}, \overline{d_1}\right)\right]; T_{n_1}\right), \\ \left(\left[(e_1, f_1, g_1, h_1), \left(\overline{e_1}, \overline{f_1}, \overline{g_1}, \overline{h_1}\right)\right]; I_{n_1}\right), \\ \left(\left[(l_1, m_1, n_1, p_1), \left(\overline{l_1}, \overline{m_1}, \overline{n_1}, \overline{p_1}\right)\right]; F_{n_1}\right) \end{Bmatrix} \tag{16}$$

$$n_2 = \begin{Bmatrix} \left(\left[(a_2, b_2, c_2, d_2), \left(\overline{a_2}, \overline{b_2}, \overline{c_2}, \overline{d_2}\right)\right]; T_{n_2}\right), \\ \left(\left[(e_2, f_2, g_2, h_2), \left(\overline{e_2}, \overline{f_2}, \overline{g_2}, \overline{h_2}\right)\right]; I_{n_2}\right), \\ \left(\left[(l_2, m_2, n_2, p_2), \left(\overline{l_2}, \overline{m_2}, \overline{n_2}, \overline{p_2}\right)\right]; F_{n_2}\right) \end{Bmatrix} \tag{17}$$

$$n_1 \oplus n_2 = \left(\begin{array}{c} \left[\begin{pmatrix} a_1 + a_2 - a_1 a_2, b_1 + b_2 - b_1 b_2 \\ c_1 + c_2 - c_1 c_2 + d_1 + d_2 - d_1 d_2 \end{pmatrix}, \right] \\ \left[\begin{pmatrix} \overline{a_1} + \overline{a_2} - \overline{a_1 a_2}, \overline{b_1} + \overline{b_2}, \overline{b_1 b_2}, \\ \overline{c_1} + \overline{c_2} - \overline{c_1 c_2}, \overline{d_1} + \overline{d_2} - \overline{d_1 d_2} \end{pmatrix} \right], \\ \left[(e_1 e_2, f_1 f_2, g_1 g_2, h_1 h_2), \right. \\ \left. \left(\overline{e_1 e_2}, \overline{f_1 f_2}, \overline{g_1 g_2}, \overline{h_1 h_2} \right) \right], \\ \left[(l_1 l_2, m_1 m_2, n_1 n_2, p_1 p_2), \right. \\ \left. \left(\overline{l_1 l_2}, \overline{m_1 m_2}, \overline{n_1 n_2}, \overline{p_1 p_2} \right) \right] \end{array} \right) \tag{18}$$

$$n_1 \otimes n_2 = \left\{ \begin{array}{c} \left[(a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2), \right. \\ \left. \left(\overline{a_1 a_2}, \overline{b_1 b_2}, \overline{c_1 c_2}, \overline{d_1 d_2} \right) \right], \\ \left[\begin{pmatrix} e_1 + e_2 - e_1 e_2, f_1 + f_1 - f_1 f_2, \\ g_1 + g_2 - g_1 g_2, h_1 + h_2 - h_1 h_2 \end{pmatrix}, \right] \\ \left[\begin{pmatrix} \overline{e_1} + \overline{e_1} - \overline{e_1 e_2}, \overline{f_1} + \overline{f_2} - \overline{f_1 f_2} \\ , \overline{g_1} + \overline{g_2} - \overline{g_1 g_2}, \overline{h_1} + \overline{h_2} - \overline{h_1 h_2} \end{pmatrix} \right], \\ \left[\begin{pmatrix} l_1 + l_2 - l_1 l_2, m_1 + m_2 - m_1 m_2, \\ n_1 + n_2 - n_1 n_2, p_1 + p_2 - p_1 p_2 \end{pmatrix}, \right] \\ \left[\begin{pmatrix} \overline{l_1} + \overline{l_2} - \overline{l_1 l_2}, \overline{m_1} + \overline{m_2} - \overline{m_1 m_2}, \\ \overline{n_1} + \overline{n_2} - \overline{n_1 n_2}, \overline{p_1} + \overline{p_2} - \overline{p_1 p_2} \end{pmatrix} \right] \end{array} \right\} \tag{19}$$

$$\varphi n_1 = \left(\begin{array}{c} \left[\begin{pmatrix} 1 - (1 - a_1)^{\varphi}, 1 - (1 - b_1)^{\varphi}, \\ 1 - (1 - c_1)^{\varphi}, 1 - (1 - d_1)^{\varphi} \end{pmatrix}, \right] \\ \left[\begin{pmatrix} 1 - (1 - \overline{a_1})^{\varphi}, 1 - \left(1 - \overline{b_1}\right)^{\varphi}, \\ 1 - (1 - \overline{c_1})^{\varphi}, 1 - \left(1 - \overline{d_1}\right)^{\varphi} \end{pmatrix} \right], \\ \left[\left(e_1^{\varphi}, f_1^{\varphi}, g_1^{\varphi}, h_1^{\varphi}\right), \left(\overline{e_1}^{\varphi}, \overline{f_1}^{\varphi}, \overline{g_1}^{\varphi}, \overline{h_1}^{\varphi}\right) \right], \\ \left[\left(l_1^{\varphi}, m_1^{\varphi}, n_1^{\varphi}, p_1^{\varphi}\right), \left(\overline{l_1}^{\varphi}, \overline{m_1}^{\varphi}, \overline{n_1}^{\varphi}, \overline{p_1}^{\varphi}\right) \right] \end{array} \right) \tag{20}$$

$$n_1^{\varphi} = \left(\begin{array}{c} \left[\left(a_1^{\varphi}, b_1^{\varphi}, c_1^{\varphi}, d_1^{\varphi}\right), \left(\overline{a_1}^{\varphi}, \overline{b_1}^{\varphi}, \overline{c_1}^{\varphi}, \overline{d_1}^{\varphi}\right) \right], \\ \left[\begin{pmatrix} 1 - (1 - e_1)^{\varphi}, 1 - (1 - f_1)^{\varphi}, \\ 1 - (1 - g_1)^{\varphi}, 1 - (1 - h_1)^{\varphi} \end{pmatrix}, \right] \\ \left[\begin{pmatrix} 1 - (1 - \overline{e_1})^{\varphi}, 1 - \left(1 - \overline{f_1}\right)^{\varphi}, \\ 1 - (1 - \overline{g_1})^{\varphi}, 1 - \left(1 - \overline{h_1}\right)^{\varphi} \end{pmatrix} \right], \\ \left[\begin{pmatrix} 1 - (1 - l_1)^{\varphi}, 1 - (1 - m_1)^{\varphi}, \\ 1 - (1 - n_1)^{\varphi}, 1 - (1 - p_1)^{\varphi} \end{pmatrix}, \right] \\ \left[\begin{pmatrix} 1 - \left(1 - \overline{l_1}\right)^{\varphi}, 1 - (1 - \overline{m_1})^{\varphi}, \\ 1 - (1 - \overline{n_1})^{\varphi}, 1 - (1 - \overline{p_1})^{\varphi} \end{pmatrix} \right] \end{array} \right) \tag{21}$$

3. IVTN-LBWA-CoCoSo

Another new subjective weighting technique is the LBWA method. Its foundation is the pairwise evaluation of criteria at criteria significance levels using non-decreasing strings. The approach does not require redefining the ordinal scale for pairwise criteria comparison because it employs

a novel algorithm for grouping criteria according to their significance levels. Following level-by-level categorization, criteria's relevance is determined in relation to DM preferences. LBWA has recently been used to address decision-making problems[11], [12].

The steps of this methodology are organized as follows:

1. Experts define the most important criterion

These criteria can be defined such as: $\{C_1, \dots, C_n\}$

2. Rest the criteria are classified into different levels

Level 1: Criteria in this level are either up to twice as important as the most important criterion or equally so.

Level 2: Criteria with a significance level that is precisely twice or even three times lower than the most crucial criterion are positioned in this level.

Level 3: Criteria that are precisely k times less significant than the most important criterion or up to k + 1 times less significant than the most essential criterion are placed in this level. By assigning the criteria to varying degrees of significance, DMs utilize this grouping process to create a rough classification of the criteria.

$$S = S_1 \cup S_2 \dots \cap S_k \tag{22}$$

3. Each criterion assigned into integral value

$$r = \max\{|S_1|, |S_2|, \dots, |S_k|\} \tag{23}$$

4. The value of r should meet $r_0 > r \; where \; r_0$ is defined as the elasticity coefficient.

5. The influence function is computed such as:

$$f(C_{ip}) = \frac{r_0}{i(r_0)+I_{ip}} \tag{24}$$

6. Compute the criteria weights such as:

$$w_j = \frac{1}{1+f(C_j)+\cdots+f(C_n)} \tag{25}$$

Then we applied the steps of the CoCoSo method to rank the alternatives

1. Build the decision matrix

The decision matrix is built between the criteria and alternatives by a set of experts

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix} \tag{26}$$

2. Normalize the decision matrix based on the positive and negative criteria such as:

$$n_{ij} = \frac{x_{ij} - \min_i x_{ij}}{\max_i x_{ij} - \min_i x_{ij}} \quad (27)$$

$$n_{ij} = \frac{\min_i x_{ij} - x_{ij}}{\max_i x_{ij} - \min_i x_{ij}} \quad (28)$$

3. Compute the sum of weighted comparability and power weighted

$$a_i = \sum_{j=1}^{n} (w_j n_{ij}) \quad (29)$$

$$b_i = \sum_{j=1}^{n} (w_j)^{n_{ij}} \quad (30)$$

4. Compute appraisal scores such as:

$$U_{ia} = \frac{a_i + b_i}{\sum_{i=1}^{m} (a_i + b_i)} \quad (31)$$

$$U_{ib} = \frac{a_i}{\min a_i} + \frac{b_i}{\min b_i} \quad (32)$$

$$U_{ic} = \frac{\pi(a_i) + (1-\pi)(b_i)}{\pi \max a_i + (1-\pi) \max b_i} \quad (33)$$

Where $0 \leq \pi \leq 1$

5. Compute the combined score

$$U_i = (U_{ia} U_{ib} U_{ic})^{(1/3)} + \frac{1}{3}(U_{ia} + U_{ib} + U_{ic}) \quad (34)$$

6. Rank the alternatives

4. Results

Petrochemical companies have grown to represent a significant sector of the national economy, and their growth has increased the risk of accidents. According to statistics, 10% of incidents in the petrochemical sector from 1985 to 2002 caused environmental damage, while 20% of accidents resulted in human fatalities, according to Nivolianitou et al[13]. Regretfully, many accidents and fatalities still occur that are not reported. There were 5207 hazardous chemical leakage incidents in China between 2009 and 2018, with varying degrees of financial losses, fatalities, and environmental harm[14]. Environmentally speaking, unexpected air pollution incidents can release a lot of toxic and dangerous pollutants quickly, resulting in significant harm to the environment and fatalities. They are also challenging to control. The petrochemical sector is prone to accidents, which have a more significant effect on the environment, property, and worker safety. Numerous studies show that it is challenging to combine the cause of an accident and the post-event emergency procedures[15], [16]. This section shows the results of evaluation of safety emergency plans for petrochemical enterprises with the neutrosophic sets. Three experts have evaluated a set of criteria and alternatives. This study uses nine criteria and ten alternatives to evaluate this problem as shown in Fig 1.

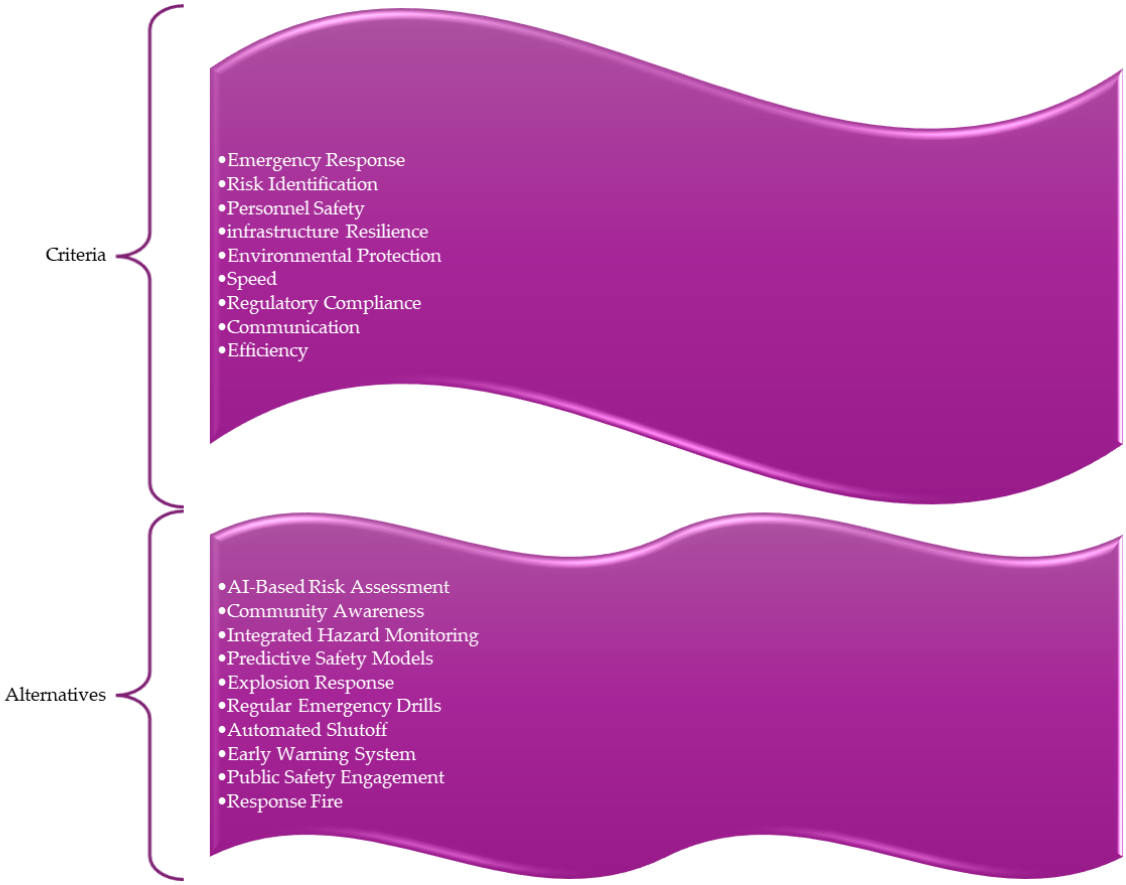

Fig 1. The safety emergency plans for petrochemical enterprises criteria.

1. Three experts are evaluated the criteria.

2. We put other criteria with set of levels. In this study we put two levels.

3. Eq. (3) is used to each criterion assigned into integral value

4. We defined r_0 is defined as the elasticity coefficient.

5. Eq. (24) is used to compute the influence function

6. Eq. (25) is used to compute the criteria weights as shown in Fig 2.

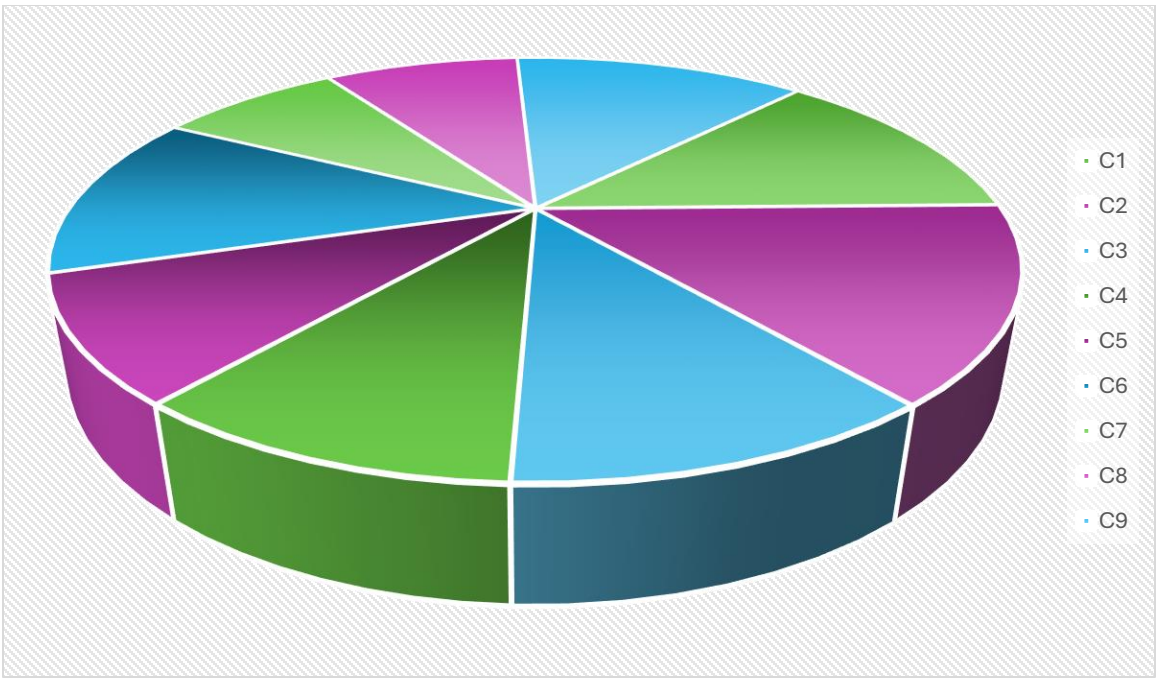

Fig 2. safety emergency plans for petrochemical enterprises criteria weights.

Then we applied the steps of the CoCoSo method to rank the alternatives

1. We used the IVTNNs to evaluate the criteria and alternatives as shown in Tables 1-3.

2. Eq. (27) is used to normalize the decision matrix as shown in Table 4. All criteria are beneficial criteria.

3. Eqs. (29 and 30) are used to compute the sum of weighted comparability and power weighted.

4. Eqs. (31, 32, and 33) are used to compute appraisal scores. We put π value with 0.5.

5. Eq. (34) is used to compute the combined score as shown in Fig 3.

6. Then we ranked the alternatives as shown in Fig 4.

Table 1. The first IVTNNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A_2	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1	[((0,0.1,0.1,0.2),(0.1,0.1,0.1	[((0.4,0.5,0.6,0.7),(0,0.1,0.2	[((0.2,0.3,0.4,0.5),(0,0.1,0.2	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1	[((0,0.1,0.1,0.2),(0.1,0.1,0.1	[((0.4,0.5,0.6,0.7),(0,0.1,0.2

	,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]
A3	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A4	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A5	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A6	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A7	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A8	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A9	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A10	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0

	.1,0.1,0.1,0.1))]	(0.1,0.1,0.1,0. 1))]	.1,0.1,0.1,0.1))]	(0.1,0.1,0.1,0. 1))]	(0.1,0.1,0.1,0. 1))]	.1,0.1,0.1,0.1))]	(0.1,0.1,0.1,0. 1))]	.1,0.1,0.1,0.1))]	.1,0.1,0.1,0.1))]

Table 2. The second IVTNNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.7,0 .7,0.7,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0.0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.4 ,0.5,0.6,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]
A_2	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.7,0 .7,0.7,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.4 ,0.5,0.6,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.4 ,0.5,0.6,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]
A_3	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.7,0 .7,0.7,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.4 ,0.5,0.6,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.4 ,0.5,0.6,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.4 ,0.5,0.6,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.7,0 .7,0.7,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.7,0 .7,0.7,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]
A_4	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.7,0 .7,0.7,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]
A_5	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.4 ,0.5,0.6,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.4 ,0.5,0.6,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]
A_6	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.7,0 .7,0.7,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]
A_7	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7 ,0.7,0.7,0.7),(0,0.1,0.2,0.3), (0.1,0.1,0.1,0. 1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0 .5,0.6,0.7),(0, 0.1,0.2,0.3),(0 .1,0.1,0.1,0.1))]
A_8	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.4,0	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.7,0	[((0.4,0.5,0.6, 0.7),(0,0.1,0.2 ,0.3),(0.1,0.1, 0.1,0.1)),((0.7	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.7,0	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.4	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.7,0	[((0.2,0.3,0.4, 0.5),(0,0.1,0.2 ,0.3),(0,0.1,0. 2,0.2)),((0.7,0	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7	[((0,0.1,0.1,0. 2),(0.1,0.1,0.1 ,0.1),(0.6,0.7, 0.8,0.9)),((0.7

	.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]	,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)]
A_9	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A_{10}	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]

Table 3. The third IVTNNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A_2	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A_3	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A_4	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0,0.1,0.2,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A_5	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A_6	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),	[((0.0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),

	(0.1,0.1,0.1,0.1))]	(0.1,0.1,0.1,0.1))]	.1,0.1,0.1,0.1))]	(0.1,0.1,0.1,0.1))]	(0.1,0.1,0.1,0.1))]	(0.1,0.1,0.1,0.1))]	.1,0.1,0.1,0.1))]	(0.1,0.1,0.1,0.1))]	(0.1,0.1,0.1,0.1))]
A_7	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A_8	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A_9	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,1),(0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]
A_{10}	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.9)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.4,0.5,0.6,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]	[((0.2,0.3,0.4,0.5),(0,0.1,0.2,0.3),(0,0.1,0.2,0.2)),((0.7,0.7,0.7,0.7),(0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1))]

Table 4. The normalized IVTNNs.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.311594	1	0.840909	0.130435	0.69697	0	0.934783	1	0
A_2	0	0.651515	0	1	0.840909	0.139535	0.130435	0	1
A_3	0.956522	0.886364	0.840909	0.130435	1	0.953488	0.57971	0.569767	0.804348
A_4	0.355072	0.30303	0.742424	0	0.651515	0.356589	1	0.069767	0.130435
A_5	0.57971	0.606061	0.371212	0.666667	0.045455	0.689922	0	0.895349	0.934783
A_6	1	0.045455	0.954545	0.268116	0.628788	0.333333	0.934783	0.569767	0
A_7	0	0.863636	0.045455	0.978261	0.977273	0.689922	0.333333	0	0.934783
A_8	0.934783	0.931818	1	0.311594	0.325758	0.930233	0.934783	0.5	0
A_9	0	0	0.280303	0.043478	0.136364	0.093023	0.57971	0	1
A_{10}	0.934783	0.090909	0.606061	1	0	1	0.398551	0.569767	0.804348

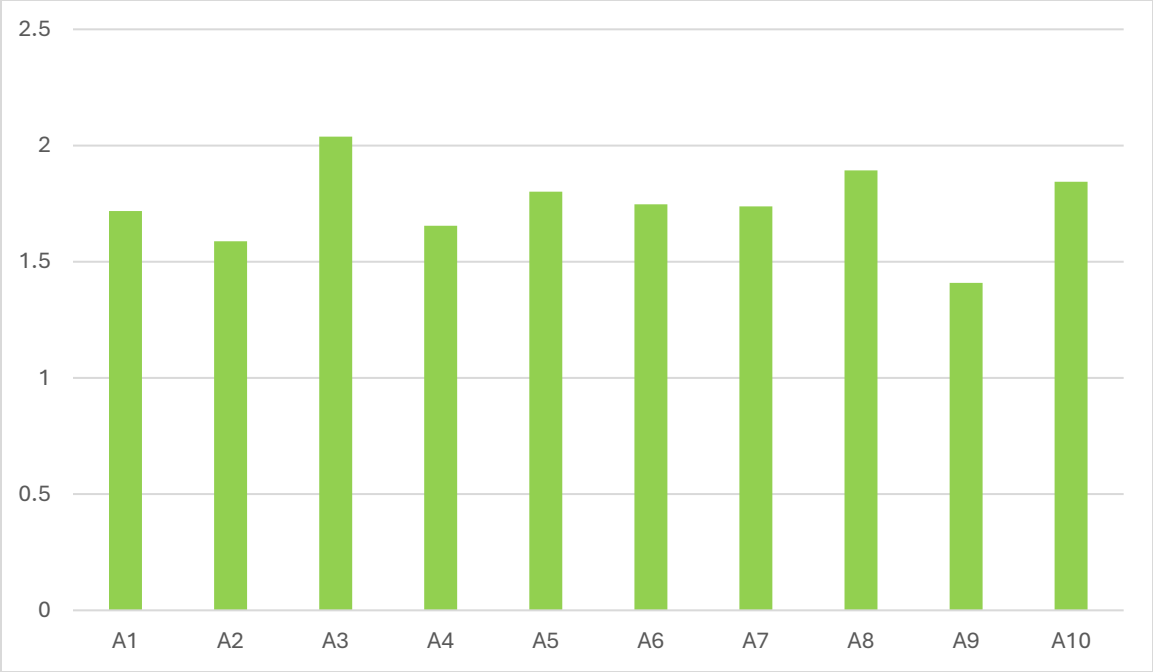

Fig 3. The combined score of each alternative.

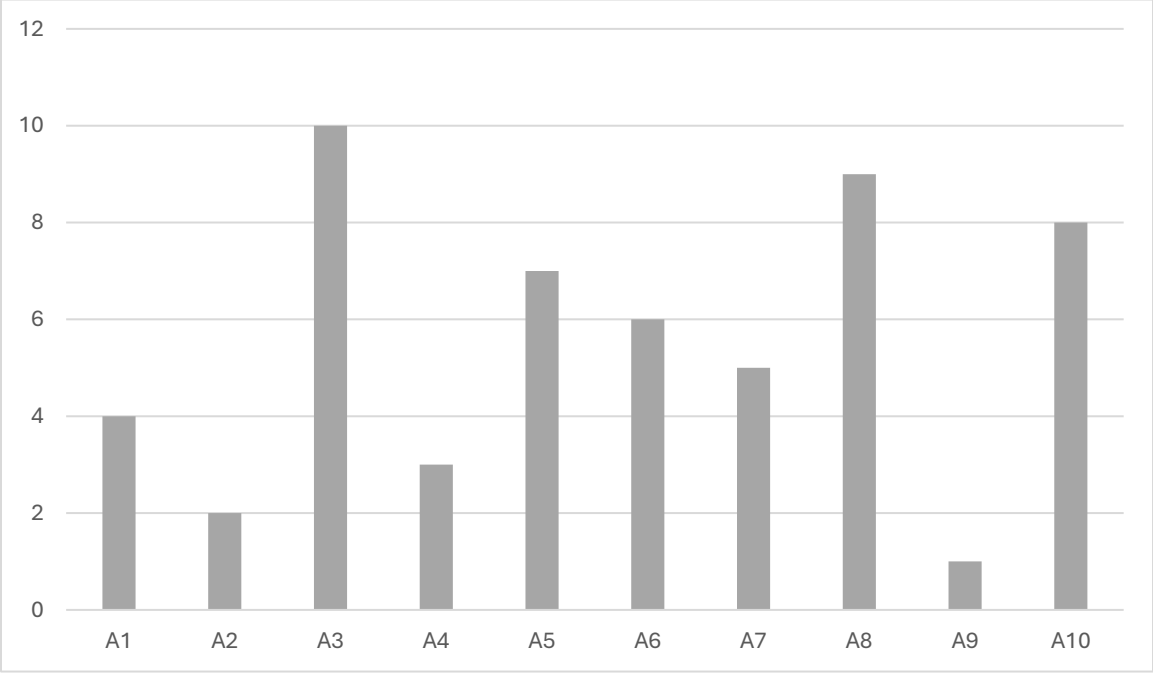

Fig 4. The rank of alternatives.

Sensitivity analysis

This section shows the ranks of alternatives under different cases. The aim of this sensitivity analysis is to show the stability of the rank under different cases. We change value of π value between 0 and 1. Then we rank the alternatives under different cases. Figs 5 and 6 show the combined scores and ranks of alternatives.

In case 1, we show alternative 3 is the best, followed by alternative 8, alternative 10, and alternative 5. We show the alternative 9 is the worst. In case 2, we show alternative 3 is the best, followed by alternative 8, alternative 10, and alternative 5. We show the alternative 9 is the worst. In case 3, we show alternative 3 is the best, followed by alternative 8, alternative 10, and alternative 5. We show the alternative 9 is the worst. In case 4, we show alternative 3 is the best, followed by alternative 8, alternative 10, and alternative 5. We show the alternative 9 is the worst.

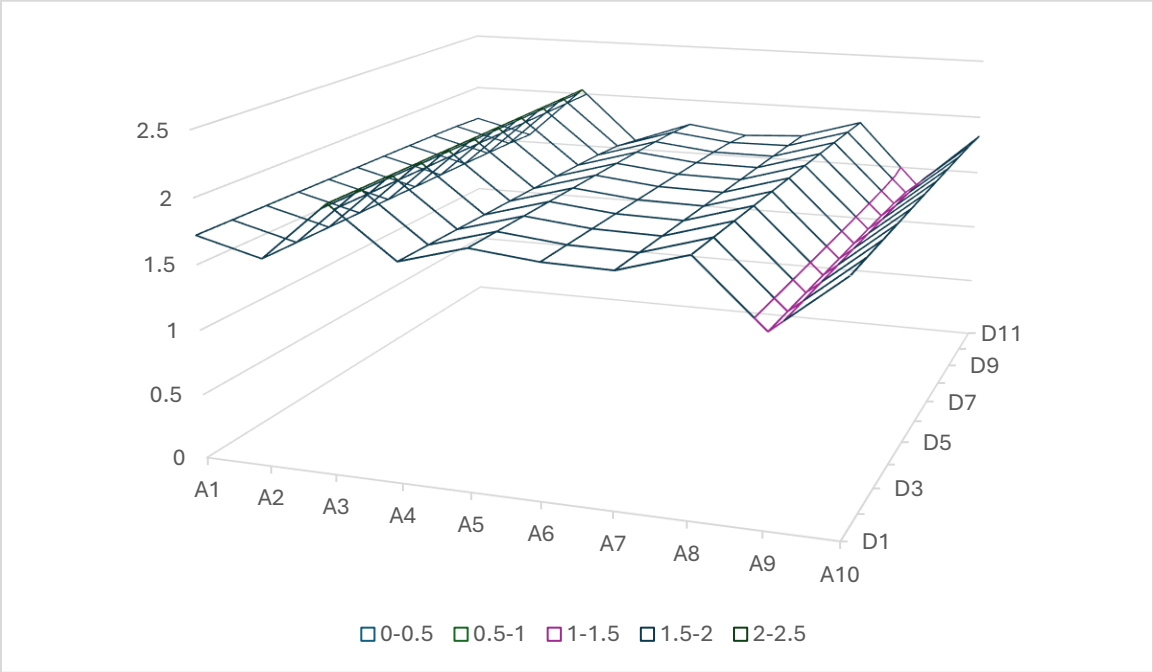

Fig 5. The different combined scores

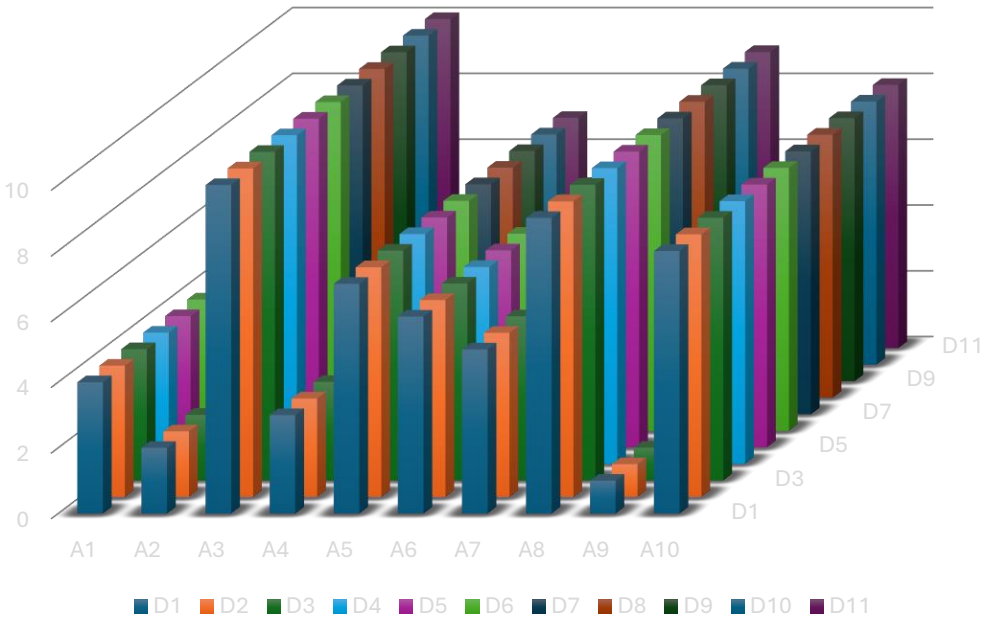

Fig 6. The different ranks.

5. Conclusions

This paper suggested the framework for evaluation of safety emergency plans for petrochemical enterprises. We used two decision making approaches in this study. We used LBWA to compute the criteria weights. Then we used the CoCoSo method to rank the alternatives. Two methods are used under the interval valued trapezoidal neutrosophic sets to dela with uncertainty and vague information. The criteria weights are computed such as Risk Identification which identify as the highest weights and the Regulatory Compliance criterion is defined as the lowest weights. The results show the alternative 3 is the best and alternative 9 is the worst. The sensitivity analysis was conducted between the ranks of the alternative. We proposed eleven cases, then we ranked the proposed approach into eleven cases. The results show the rank of alternatives is stable in different cases.

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Uncertainty-Based Measurement of Job Satisfaction Evaluation of University Teachers: An Improved Bipolar Neutrosophic Multi-Criteria Approach for Enhanced Decision-Making

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Abstract

Assessment of Job satisfaction in higher education persists as a complicated process involving many uncertain and regularly incompatible aspects. Old-style methods for quantifying job satisfaction are no longer able to account for growing uncertainty, paradoxes, and bipolarity in human opinions, a mix of satisfaction and dissatisfaction feedback. To fill this gap, we propose on bipolar neutrosophic decision-making (BNDM) approach to quantify job satisfaction within the vague, and bipolar nature of academic life. Our approach proposes a weighted bipolar neutrosophic cross-entropy technique to quantify optimization-based weighting of criteria. Then, we drive a weighted decision matrix and propose a bi-polar edition of evaluation based on distance from the average solution method to rank alternatives in a logical and appropriate way. Besides, we introduce a case study on job satisfaction among university teachers in the Middle East based on a questionnaire survey conducted across different universities. Proof-of-concept results are derived from the application of our approach to collected data to provide an in-depth analysis of aspects that might affect job satisfaction under uncertain and contradictory perceptions.

Keywords: Neutrosophic Logic, Bi-polar Neutrosophic Sets (BNS), Job Satisfaction, Higher Education, Uncertainty Quantification, Multi-Criteria Decision Making (MCDM)

1. Introduction

Job satisfaction is a critical factor in higher education institutions, influencing faculty performance, institutional reputation, and overall workplace harmony [1]. Numerous factors—including workload, institutional policies, leadership styles, and career growth—shape job satisfaction, often leading to conflicting emotions and perceptions among educators. In real-world scenarios, job satisfaction is rarely binary; it exists on a spectrum between satisfaction and

dissatisfaction [2]. An academic professional may feel satisfied with career growth opportunities but dissatisfied with administrative support at the same time. Such bipolar tendencies demand a more flexible and uncertainty-resilient evaluation framework [3]. Traditional models for measuring job satisfaction rely on deterministic and probabilistic approaches, which fail to account for the inherent uncertainty, vagueness, and contradictions in human perception [4].

To handle the ambiguity and contradiction in decision-making, this study adopts Neutrosophic Logic (NL)—a mathematical framework introduced by Florentin Smarandache to extend fuzzy and intuitionistic fuzzy logic [5], [6]. Unlike classical logic, which operates on absolute truth (0 or 1), and fuzzy logic, which represents truth as a degree between 0 and 1, NL introduces three independent degrees including Truth (T), Falsity (F), and Indeterminacy (I) [7]. With the integration of $t, f, and\ I$ in an independent and dynamic manner, NL can provide a more comprehensive and exact approach to modelling human acuity and decision-making hard ambiguity. This makes it particularly useful for evaluating job satisfaction, where individuals may simultaneously experience positive and negative emotions, alongside uncertainty. The NL theory to deliberate the evolution of dissimilar types sets namely single-valued neutrosophic sets (SVNS) [8], interval-valued neutrosophic sets (IVNS) [9], triangular neutrosophic sets (TNS) [10], bi-polar neutrosophic sets (BNSs) [11], and others.

In job satisfaction assessment, existing methods such as Likert scales, fuzzy logic, and traditional multi-criteria decision-making (MCDM) techniques struggle with the dual nature of satisfaction and dissatisfaction [12]. These methods typically assume a unipolar structure, treating satisfaction as a linear scale from negative to positive, without accounting for simultaneous conflicting perceptions.

To address this limitation, we propose an Improved Bi-Polar Neutrosophic Decision-Making (BNCDM) for uncertainty-based job satisfaction assessment in higher education. The key contributions of this study are as follows. First, we propose to integrate BNS principles to handle imprecise, contradictory, and uncertain feedback in faculty surveys, aiming to enhance the reliability of job satisfaction assessment. Second, as an improvement to [13], we proposed and Bi-Polar Neutrosophic weighted-entropy based Inter-criteria Correlation method to appropriately decide the criterion weights while considering bipolar uncertainty in BNCDM. Third, we introduce an extended edition of the Evaluation Based on the Distance from Average Solution (EDAS) method that integrates the power of bipolar neutrosophic theory to obtain better decisions about the ranking of alternatives in uncertain environments. Finally, we present a case study for an academic job satisfaction-based survey from five universities to investigate real-world factors to provide empirical validation of the proposed approach. Figure 1 displays the outline of the paper.

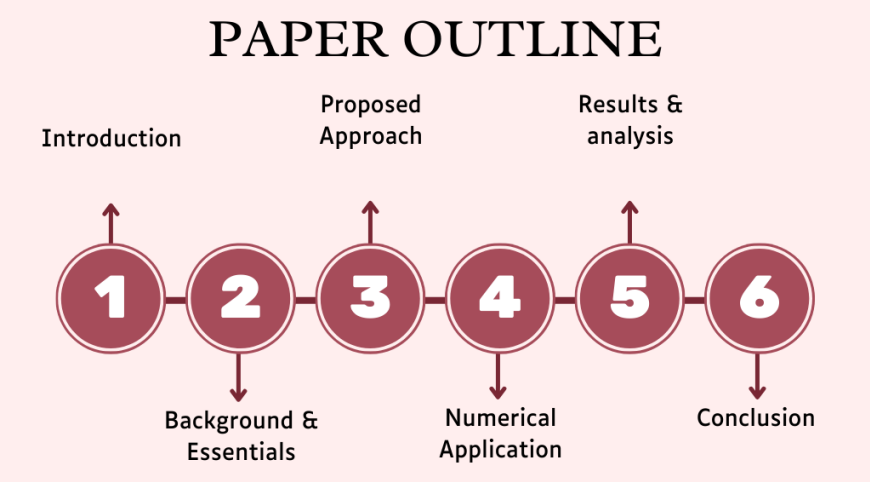

Figure 1. The visual organization of our study

2. Background & Essentials

Definition 1: A Neutrosophic Set (NS) is a mathematical framework extending the classical, fuzzy, and intuitionistic fuzzy sets with the inclusion of three independent membership functions:

$$A = \{\langle \mathfrak{y}, (\mathfrak{t}_A(\mathfrak{y}), \mathfrak{I}_A(\mathfrak{y}), \mathfrak{f}_A(\mathfrak{y}))\rangle : \mathfrak{y} \in \mathfrak{Y}\}. \quad (1)$$

These functions are defined as follows:

$$\mathfrak{t}_A : \mathfrak{Y} \to]^-0, 1^+[, \mathfrak{I}_A : \mathfrak{Y} \to]^-0, 1^+[, \mathfrak{f}_A : \mathfrak{Y} \to]^-0, 1^+[\quad \text{such that} \quad 0^- \leq \mathfrak{t}_A(\mathfrak{y}) + \mathfrak{I}_A(\mathfrak{y}) + \mathfrak{f}_A(\mathfrak{y}) \leq 3^+ \quad (2)$$

Definition 2: A Single-Valued NSs (SVNSs) is a special case of the NS that constrains the truth, indeterminacy, and falsity membership values to the unit interval [0,1].

$$\text{SVNS} = \{\langle \mathfrak{y}, (\mathfrak{t}_A(\mathfrak{y}), \mathfrak{I}_A(\mathfrak{y}), \mathfrak{f}_A(\mathfrak{y}))\rangle : \mathfrak{y} \in \mathfrak{Y}\}, \quad (3)$$

where

$$\mathfrak{t}_A : \mathfrak{Y} \to [0,1], \mathfrak{I}_A : \mathfrak{Y} \to [0,1], \mathfrak{f}_A : \mathfrak{y} \to [0,1] \; such\; that \; 0 \leq \mathfrak{t}_A(\mathfrak{y}) + \mathfrak{I}_A(\mathfrak{y}) + \mathfrak{f}_A(\mathfrak{y}) \leq 3 \quad (4)$$

Definition 3: A BNS is an extension of traditional NS by introducing both positive and negative membership functions disjointedly.

$$A = \{\langle \mathfrak{y}, \mathfrak{t}^+(\mathfrak{y}), \mathfrak{I}^+(\mathfrak{y}), \mathfrak{f}^+(\mathfrak{y}), \mathfrak{t}^-(\mathfrak{y}), \mathfrak{I}^-(\mathfrak{y}), \mathfrak{f}^-(\mathfrak{y})\rangle : \mathfrak{y} \in \mathfrak{Y}\}. \quad (5)$$

where

$$\mathfrak{t}^+, \mathfrak{I}^+, \mathfrak{f}^+ : \mathfrak{Y} \to [0,1], \mathfrak{t}^-, \mathfrak{I}^-, \mathfrak{f}^- : X \to [-1,0]. \quad (6)$$

Definition 4: Given two BNSs $A_1 = \langle \mathfrak{y}, \mathfrak{t}_1^+(\mathfrak{y}), \mathfrak{I}_1^+(\mathfrak{y}), \mathfrak{f}_1^+(\mathfrak{y}), \mathfrak{t}_1^-(\mathfrak{y}), \mathfrak{I}_1^-(\mathfrak{y}), \mathfrak{f}_1^-(\mathfrak{y})\rangle$, and $A_2 = \langle \mathfrak{y}, \mathfrak{t}_2^+(\mathfrak{y}), \mathfrak{I}_2^+(\mathfrak{y}), \mathfrak{f}_2^+(\mathfrak{y}), \mathfrak{t}_2^-(\mathfrak{y}), \mathfrak{I}_2^-(\mathfrak{y}), \mathfrak{f}_2^-(\mathfrak{y})\rangle$ in the discourse $\mathfrak{Y}$. Then the following operations are defined as follows:

A. Equality

$$A_1 = A_2 \; iff \\ \mathfrak{t}_1^+(\mathfrak{y}) = \mathfrak{t}_2^+(\mathfrak{y}), \mathfrak{I}_1^+(\mathfrak{y}) = \mathfrak{I}_2^+(\mathfrak{y}), \mathfrak{f}_1^+(\mathfrak{y}) = \mathfrak{f}_2^+(\mathfrak{y}), \mathfrak{t}_1^-(\mathfrak{y}) = \mathfrak{t}_2^-(\mathfrak{y}), \mathfrak{I}_1^-(\mathfrak{y}) = \mathfrak{I}_2^-(\mathfrak{y}), \mathfrak{f}_1^-(\mathfrak{y}) = \mathfrak{f}_2^-(\mathfrak{y}). \tag{7}$$

B. Union

$$A_1 \cup A_2 = \left\{ \left\langle \begin{matrix} \mathfrak{y}, max\left(\mathfrak{t}_1^+(\mathfrak{y}), \mathfrak{t}_2^+(\mathfrak{y})\right), \dfrac{\mathfrak{I}_1^+(\mathfrak{y}) + \mathfrak{I}_2^+(\mathfrak{y})}{2}, min\left(\mathfrak{f}_1^+(\mathfrak{y}), \mathfrak{f}_2^+(\mathfrak{y})\right), \\ min\left(\mathfrak{t}_1^-(\mathfrak{y}), \mathfrak{t}_2^-(\mathfrak{y})\right), \dfrac{\mathfrak{I}_1^-(\mathfrak{y}) + \mathfrak{I}_2^-(\mathfrak{y})}{2}, max\left(\mathfrak{f}_1^-(\mathfrak{y}), \mathfrak{f}_2^-(\mathfrak{y})\right) \end{matrix} \right\rangle \right\} \tag{8}$$

C. Intersection

$$A_1 \cap A_2 = \left\{ \left\langle \begin{matrix} \mathfrak{y}, min\left(\mathfrak{t}_1^+(\mathfrak{y}), \mathfrak{t}_2^+(\mathfrak{y})\right), \dfrac{\mathfrak{I}_1^+(\mathfrak{y}) + \mathfrak{I}_2^+(\mathfrak{y})}{2}, max\left(\mathfrak{f}_1^+(\mathfrak{y}), \mathfrak{f}_2^+(\mathfrak{y})\right), \\ max(\mathfrak{t}_1^-(\mathfrak{y}), \mathfrak{t}_2^-(\mathfrak{y})), \dfrac{\mathfrak{I}_1^-(\mathfrak{y}) + \mathfrak{I}_2^-(\mathfrak{y})}{2}, min(\mathfrak{f}_1^-(\mathfrak{y}), \mathfrak{f}_2^-(\mathfrak{y})) \end{matrix} \right\rangle \right\} \tag{9}$$

D. Subset

$$A_1 \subseteq A_2 \; iff \\ \mathfrak{t}_1^+(\mathfrak{y}) \le \mathfrak{t}_2^+(\mathfrak{y}), \mathfrak{I}_1^+(\mathfrak{y}) \le \mathfrak{I}_2^+(\mathfrak{y}), \mathfrak{f}_1^+(\mathfrak{y}) \ge \mathfrak{f}_2^+(\mathfrak{y}) \\ \&\& \\ \mathfrak{t}_1^-(\mathfrak{y}) \ge \mathfrak{t}_2^-(\mathfrak{y}), \mathfrak{I}_1^-(\mathfrak{y}) \ge \mathfrak{I}_2^-(\mathfrak{y}), \mathfrak{f}_1^-(\mathfrak{y}) \le \mathfrak{f}_2^-(\mathfrak{y}). \tag{10}$$

E. Complement

$$A_1^c = \{\langle \mathfrak{y}, 1 - \mathfrak{t}_1^+(\mathfrak{y}), 1 - \mathfrak{I}_1^+(\mathfrak{y}), 1 - \mathfrak{f}_1^+(\mathfrak{y}), 1 - \mathfrak{t}_1^-(\mathfrak{y}), 1 - \mathfrak{I}_1^-(\mathfrak{y}), 1 - \mathfrak{f}_1^-(\mathfrak{y}) \rangle\} \tag{11}$$

Definition 5: Given two BNSs $A_1 = \langle \mathfrak{y}, \mathfrak{t}_1^+(\mathfrak{y}), \mathfrak{I}_1^+(\mathfrak{y}), \mathfrak{f}_1^+(\mathfrak{y}), \mathfrak{t}_1^-(\mathfrak{y}), \mathfrak{I}_1^-(\mathfrak{y}), \mathfrak{f}_1^-(\mathfrak{y}) \rangle$, and $A_2 = \langle \mathfrak{y}, \mathfrak{t}_2^+(\mathfrak{y}), \mathfrak{I}_2^+(\mathfrak{y}), \mathfrak{f}_2^+(\mathfrak{y}), \mathfrak{t}_2^-(\mathfrak{y}), \mathfrak{I}_2^-(\mathfrak{y}), \mathfrak{f}_2^-(\mathfrak{y}) \rangle$ in discourse $\mathfrak{Y}$, then the following operators of BNSs are depicted below:

$$\lambda \mathrm{A}_1 = \langle 1 - (1 - \mathfrak{t}_1^+)^\lambda, (\mathfrak{I}_1^+)^\lambda, (\mathfrak{f}_1^+)^\lambda, -(-\mathfrak{t}_1^-)^\lambda, -(-\mathfrak{I}_1^-)^\lambda, -\left(1 - \left(1 - (\mathfrak{f}_1^-)\right)\right)^\lambda \rangle \tag{12}$$

$$\mathrm{A}_1^\lambda = \langle (\mathfrak{t}_1^+)^\lambda, 1 - (1 - \mathfrak{I}_1^+)^\lambda, 1 - (1 - \mathfrak{f}_1^+)^\lambda, -(1 - (1 - (-\mathfrak{t}_1^-)))^\lambda, -(-\mathfrak{I}_1^-)^\lambda, -(-\mathfrak{f}_1^-)^\lambda \rangle \tag{13}$$

$$\mathrm{A}_1 + \mathrm{A}_2 = \langle \mathfrak{t}_1^+ + \mathfrak{t}_2^+ - \mathfrak{t}_1^+\mathfrak{t}_2^+, \mathfrak{I}_1^+\mathfrak{I}_2^+, \mathfrak{f}_1^+\mathfrak{f}_2^+, -\mathfrak{t}_1^-\mathfrak{t}_2^-, -(-\mathfrak{I}_1^- - \mathfrak{I}_2^- - \mathfrak{I}_1^-\mathfrak{I}_2^-), -(-\mathfrak{f}_1^- - \mathfrak{f}_2^- - \mathfrak{f}_1^-\mathfrak{f}_2^-) \rangle \tag{14}$$

$$\mathrm{A}_1\mathrm{A}_2 = \langle \mathfrak{t}_1^+\mathfrak{t}_2^+, \mathfrak{I}_1^+ + \mathfrak{I}_2^+ - \mathfrak{I}_1^+\mathfrak{I}_2^+, \mathfrak{f}_1^+ + \mathfrak{f}_2^+ - \mathfrak{f}_1^+\mathfrak{f}_2^+, -(-\mathfrak{t}_1^- - \mathfrak{t}_2^- - \mathfrak{t}_1^-\mathfrak{t}_2^-), -\mathfrak{I}_1^-\mathfrak{I}_2^-, -\mathfrak{f}_1^-\mathfrak{f}_2^- \rangle \tag{15}$$

Definition 6: Given a BNS $A_1 = \langle \mathfrak{y}, \mathfrak{t}_1^+(\mathfrak{y}), \mathfrak{I}_1^+(\mathfrak{y}), \mathfrak{f}_1^+(\mathfrak{y}), \mathfrak{t}_1^-(\mathfrak{y}), \mathfrak{I}_1^-(\mathfrak{y}), \mathfrak{f}_1^-(\mathfrak{y}) \rangle$, then, the score function $s(\tilde{b1})$, accuracy function $a(\tilde{b1})$ and certainty function $c(\tilde{b1})$ are defined as follows:

$$s(\tilde{b}_1) = \frac{(\mathfrak{t}_1^+ + 1 - \mathfrak{I}_1^+ + 1 - \mathfrak{f}_1^+ + 1 + \mathfrak{t}_1^- - \mathfrak{I}_1^- - \mathfrak{f}_1^-)}{6} \tag{16}$$

$$a(\tilde{b}_1) = \mathfrak{t}_1^+ - \mathfrak{f}_1^+ + \mathfrak{t}_1^- - \mathfrak{I}_1^- \tag{17}$$

$$c(\tilde{b}_1) = \mathfrak{t}_1^+ - \mathfrak{f}_1^+ \tag{18}$$

Definition 7: Given a set of BNSs $\widetilde{\mathcal{B}} = \{B_1, B_2, B_3, \dots, B_n\}$, and the corresponding weights $W = \{w_1, w_2, \dots, w_n\}$ such that $0 \le w_j \le 1 \&\& \sum_{k=1}^{n} w_k = 1$, then, the weighted average of this set is formulated as follows:

$$\begin{aligned} \mathrm{WA}_{\mathcal{W}}^{\mathrm{BNS}}(\widetilde{\mathcal{B}}) &= \sum_{k=1}^{n} w_k t_k \\ &= \left\langle \begin{array}{c} 1 - \prod_{k=1}^{n} \left(1 - \mathfrak{t}_{\theta_k}^+\right)^{w_k}, \prod_{k=1}^{n} \mathfrak{I}_{\theta_k}^{+w_k}, \prod_{k=1}^{n} \mathfrak{f}_{\theta_k}^{+w_k}, \\ -\prod_{k=1}^{n} \left(-\mathfrak{t}_{\theta_k}^-\right)^{w_k}, \\ -\left(1 - \prod_{k=1}^{n} \left(1 - \left(-\mathfrak{I}_{\theta_{t_k}^-}\right)\right)^{w_k}\right), -\left(1 - \prod_{k=1}^{n} \left(1 - \left(-\mathfrak{f}_{\theta_k}^-\right)\right)^{w_k}\right) \end{array} \right\rangle. \end{aligned} \tag{19}$$

Definition 8: Given a set of BNSs $\widetilde{\mathcal{B}} = \{B_1, B_2, B_3, \dots, B_n\}$, and the corresponding weights $W = \{w_1, w_2, \dots, w_n\}$ such that $0 \le w_j \le 1 \&\& \sum_{k=1}^{n} w_k = 1$, then, the weighted geometric operator of this set is formulated as follows:

$$\begin{aligned} \mathrm{WG}_{\mathcal{W}}^{\mathrm{BNS}}(\widetilde{\mathcal{B}}) &= \prod_{k=1}^{n} \theta_k^{w_k} \\ &= \left\langle \begin{array}{c} \prod_{k=1}^{n} \mathfrak{t}_{\theta_k}^{+w_k}, 1 - \prod_{k=1}^{n} \left(1 - \mathfrak{I}_{\theta_k}^+\right)^{w_k}, 1 - \prod_{k=1}^{n} \left(1 - \mathfrak{f}_{\theta_k}^+\right)^{w_k}, \\ -\left(1 - \prod_{k=1}^{n} \left(1 - \left(-\mathfrak{t}_{\theta_k}^-\right)\right)^{w_k}\right), -\prod_{k=1}^{n} \left(-\mathfrak{I}_{\theta_k}^-\right)^{w_k}, -\prod_{k=1}^{n} \left(-\mathfrak{f}_{\theta_k}^-\right)^{w_k} \end{array} \right\rangle \end{aligned} \tag{20}$$

Definition 9: Given two BNSs $A_1 = \langle \mathfrak{y}, \mathfrak{t}_1^+(\mathfrak{y}), \mathfrak{I}_1^+(\mathfrak{y}), \mathfrak{f}_1^+(\mathfrak{y}), \mathfrak{t}_1^-(\mathfrak{y}), \mathfrak{I}_1^-(\mathfrak{y}), \mathfrak{f}_1^-(\mathfrak{y}) \rangle$, and $A_2 = \langle \mathfrak{y}, \mathfrak{t}_2^+(\mathfrak{y}), \mathfrak{I}_2^+(\mathfrak{y}), \mathfrak{f}_2^+(\mathfrak{y}), \mathfrak{t}_2^-(\mathfrak{y}), \mathfrak{I}_2^-(\mathfrak{y}), \mathfrak{f}_2^-(\mathfrak{y}) \rangle$, the dice similarity [14] between these two sets can be computed as follows:

$$Dice(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{2\left(\mathfrak{t}_A(\mathfrak{y}_i)\mathfrak{t}_B(\mathfrak{y}_i) + \mathfrak{I}_A(\mathfrak{y}_i)\mathfrak{I}_B(\mathfrak{y}_i) + \mathfrak{f}_A(\mathfrak{y}_i)\mathfrak{f}_B(\mathfrak{y}_i)\right)}{\left[\begin{matrix}\left((\mathfrak{t}_A)^2(\mathfrak{y}_i) + (\mathfrak{I}_A)^2(\mathfrak{y}_i) + (\mathfrak{f}_A)^2(\mathfrak{y}_i)\right) + \\ \left((\mathfrak{t}_B)^2(\mathfrak{y}_i) + (\mathfrak{I}_B)^2(\mathfrak{y}_i) + (\mathfrak{f}_B)^2(\mathfrak{y}_i)\right)\end{matrix}\right]}\right) \tag{21}$$

where

$$Dice(A_1, A_2) = Dice(A_1, A_2)\,;$$

$$Dice(A_1, A_2) = 1 \text{ for } A_1 = A_2$$

$$\mathfrak{t}_{A_1}(\mathfrak{y}_i) = \mathfrak{t}_{A_2}(\mathfrak{y}_i), \mathfrak{I}_{A_1}(\mathfrak{y}_i) = \mathfrak{I}_{A_2}(\mathfrak{y}_i), \mathfrak{f}_{A_1}(\mathfrak{y}_i) = \mathfrak{f}_{A_2}(\mathfrak{y}_i)(i = 1,2 \dots, n)\ \forall\ \mathfrak{y}_i(i = 1,2, \dots, n) \in \mathfrak{Y}.$$

Definition 10: Given two BNSs $\mathrm{A} = \langle \mathfrak{y}, \mathfrak{t}_1^+(\mathfrak{y}), \mathfrak{I}_1^+(\mathfrak{y}), \mathfrak{f}_1^+(\mathfrak{y}), \mathfrak{t}_1^-(\mathfrak{y}), \mathfrak{I}_1^-(\mathfrak{y}), \mathfrak{f}_1^-(\mathfrak{y}) \rangle$, and $B = \langle \mathfrak{y}, \mathfrak{t}_2^+(\mathfrak{y}), \mathfrak{I}_2^+(\mathfrak{y}), \mathfrak{f}_2^+(\mathfrak{y}), \mathfrak{t}_2^-(\mathfrak{y}), \mathfrak{I}_2^-(\mathfrak{y}), \mathfrak{f}_2^-(\mathfrak{y}) \rangle$, the Hybrid vector similarity (HVS) measures [15] between these two sets can be computed as follows:

$$\begin{aligned} &HVS(A,B) \\ &= \lambda\left(\frac{2\left(\mathfrak{t}_A(\mathfrak{y}_i)\mathfrak{t}_B(\mathfrak{y}_i) + \mathfrak{I}_A(\mathfrak{y}_i)\mathfrak{I}_B(\mathfrak{y}_i) + \mathfrak{f}_A(\mathfrak{y}_i)\mathfrak{f}_B(\mathfrak{y}_i)\right)}{[((\mathfrak{t}_A)^2(\mathfrak{y}_i) + (\mathfrak{I}_A)^2(\mathfrak{y}_i) + (\mathfrak{f}_A)^2(\mathfrak{y}_i)) + ((\mathfrak{t}_B)^2(\mathfrak{y}_i) + (\mathfrak{I}_B)^2(\mathfrak{y}_i) + (\mathfrak{f}_B)^2(\mathfrak{y}_i))]}\right) \\ &+ (1-\lambda)\left(\frac{\mathfrak{t}_A(\mathfrak{y}_i)\mathfrak{t}_B(\mathfrak{y}_i) + \mathfrak{I}_A(\mathfrak{y}_i)\mathfrak{I}_B(\mathfrak{y}_i) + \mathfrak{f}_A(\mathfrak{y}_i)\mathfrak{f}_B(\mathfrak{y}_i)}{\left[\sqrt{((\mathfrak{t}_A)^2(\mathfrak{y}_i) + (\mathfrak{I}_A)^2(\mathfrak{y}_i) + (\mathfrak{f}_A)^2(\mathfrak{y}_i))}.\sqrt{((\mathfrak{t}_B)^2(\mathfrak{y}_i) + (\mathfrak{I}_B)^2(\mathfrak{y}_i) + (\mathfrak{f}_B)^2(\mathfrak{y}_i))}\right]}\right) \end{aligned} \tag{22}$$

where

$$0 \le HVS(A,B) \le 1;$$

$$HVS(A,B) = HVS(B,A);$$

$$HVS(A,B) = 1\ for\ B = A\ \mathfrak{y}_i = y_i(i = 1,2 \dots, n)\forall \mathfrak{y}_i \in \mathfrak{Y}.$$

Definition 12: Given two BNSs $A_1 = \langle \mathfrak{y}, \mathfrak{t}_1^+(\mathfrak{y}), \mathfrak{I}_1^+(\mathfrak{y}), \mathfrak{f}_1^+(\mathfrak{y}), \mathfrak{t}_1^-(\mathfrak{y}), \mathfrak{I}_1^-(\mathfrak{y}), \mathfrak{f}_1^-(\mathfrak{y}) \rangle$, and $A_2 = \langle \mathfrak{y}, \mathfrak{t}_2^+(\mathfrak{y}), \mathfrak{I}_2^+(\mathfrak{y}), \mathfrak{f}_2^+(\mathfrak{y}), \mathfrak{t}_2^-(\mathfrak{y}), \mathfrak{I}_2^-(\mathfrak{y}), \mathfrak{f}_2^-(\mathfrak{y}) \rangle$, the weighted dice similarity [14] between these two sets can be computed as follows:

$$Dice_w(A,B) = \frac{1}{n}\sum_{i=1}^{n} w_i \times \left(\frac{2\left(\mathfrak{t}_A(\mathfrak{y}_i)\mathfrak{t}_B(\mathfrak{y}_i) + \mathfrak{I}_A(\mathfrak{y}_i)\mathfrak{I}_B(\mathfrak{y}_i) + \mathfrak{f}_A(\mathfrak{y}_i)\mathfrak{f}_B(\mathfrak{y}_i)\right)}{\left[\begin{matrix}\left((\mathfrak{t}_A)^2(\mathfrak{y}_i) + (\mathfrak{I}_A)^2(\mathfrak{y}_i) + (\mathfrak{f}_A)^2(\mathfrak{y}_i)\right) + \\ \left((\mathfrak{t}_B)^2(\mathfrak{y}_i) + (\mathfrak{I}_B)^2(\mathfrak{y}_i) + (\mathfrak{f}_B)^2(\mathfrak{y}_i)\right)\end{matrix}\right]}\right) \tag{23}$$

Definition 13: Given two BNSs $\mathrm{A} = \langle \mathfrak{y}, \mathfrak{t}_1^+(\mathfrak{y}), \mathfrak{I}_1^+(\mathfrak{y}), \mathfrak{f}_1^+(\mathfrak{y}), \mathfrak{t}_1^-(\mathfrak{y}), \mathfrak{I}_1^-(\mathfrak{y}), \mathfrak{f}_1^-(\mathfrak{y}) \rangle$, and $B = \langle \mathfrak{y}, \mathfrak{t}_2^+(\mathfrak{y}), \mathfrak{I}_2^+(\mathfrak{y}), \mathfrak{f}_2^+(\mathfrak{y}), \mathfrak{t}_2^-(\mathfrak{y}), \mathfrak{I}_2^-(\mathfrak{y}), \mathfrak{f}_2^-(\mathfrak{y}) \rangle$, the weighted Hybrid vector similarity (HVS_w) measures [15] between these two sets can be computed as follows:

$$HVS_w(A,B) = \lambda \sum_{i=1}^{n} w_i \times \left(\frac{2\left(\mathfrak{t}_A(\mathfrak{y}_i)\mathfrak{t}_B(\mathfrak{y}_i) + \mathfrak{I}_A(\mathfrak{y}_i)\mathfrak{I}_B(\mathfrak{y}_i) + \mathfrak{f}_A(\mathfrak{y}_i)\mathfrak{f}_B(\mathfrak{y}_i)\right)}{\left[\begin{array}{c}\left((\mathfrak{t}_A)^2(\mathfrak{y}_i) + (\mathfrak{I}_A)^2(\mathfrak{y}_i) + (\mathfrak{f}_A)^2(\mathfrak{y}_i)\right) + \\ \left((\mathfrak{t}_B)^2(\mathfrak{y}_i) + (\mathfrak{I}_B)^2(\mathfrak{y}_i) + (\mathfrak{f}_B)^2(\mathfrak{y}_i)\right)\end{array}\right]} \right) + (1 - \lambda) \sum_{i=1}^{n} w_i \times \left(\frac{\mathfrak{t}_A(\mathfrak{y}_i)\mathfrak{t}_B(\mathfrak{y}_i) + \mathfrak{I}_A(\mathfrak{y}_i)\mathfrak{I}_B(\mathfrak{y}_i) + \mathfrak{f}_A(\mathfrak{y}_i)\mathfrak{f}_B(\mathfrak{y}_i)}{\left[\begin{array}{c}\sqrt{\left((\mathfrak{t}_A)^2(\mathfrak{y}_i) + (\mathfrak{I}_A)^2(\mathfrak{y}_i) + (\mathfrak{f}_A)^2(\mathfrak{y}_i)\right)} \\ \times \\ \sqrt{\left((\mathfrak{t}_B)^2(\mathfrak{y}_i) + (\mathfrak{I}_B)^2(\mathfrak{y}_i) + (\mathfrak{f}_B)^2(\mathfrak{y}_i)\right)}\end{array}\right]} \right) \tag{24}$$

3. Proposed Approach

In Step 1, we propose to normalize each BNS component for each pair of alternatives and criteria, in the decision matrix.

$$N_{i,j} = \left\langle t_{i,j}^{+}, l_{i,j}^{+}, f_{i,j}^{+}, t_{i,j}^{-}, l_{i,j}^{-}, f_{i,j}^{-} \right\rangle,$$

such that

$$t_{i,j}^{+} = \frac{t_{i,j}^{+}}{\sum_{i=1}^{m} t_{i,j}^{+}},\ l_{i,j}^{+} = \frac{l_{i,j}^{+}}{\sum_{i=1}^{m} l_{i,j}^{+}},\ f_{i,j}^{+} = \frac{f_{i,j}^{+}}{\sum_{i=1}^{m} f_{i,j}^{+}}$$
$$t_{i,j}^{-} = -\left|\frac{t_{i,j}^{-}}{\sum_{i=1}^{m} t_{i,j}^{-}}\right|,\ l_{i,j}^{-} = -\left|\frac{l_{i,j}^{-}}{\sum_{i=1}^{m} l_{i,j}^{-}}\right|,\ f_{i,j}^{-} = -\left|\frac{f_{i,j}^{-}}{\sum_{i=1}^{m} f_{i,j}^{-}}\right| \tag{25}$$

This operation helps ensure that all BNS components lie within the same scale, which makes them suitable for further analysis.

In step 2, for each criterion c_j, we propose the weighted BNS Cross-Entropy instead of standard entropy:

$$E_j = \left\langle E_j^{t^+}, E_j^{l^+}, E_j^{f^+}, E_j^{t^-}, E_j^{l^-}, E_j^{f^-} \right\rangle,$$

such that

$$E_j^k = -\frac{1}{\ln(m)} \sum_{i=1}^{m} ww_{i,j}^k \cdot X_{i,j}^k \cdot \ln\left(X_{i,j}^k\right), \tag{26}$$

with $k \in \{t^+, l^+, f^+, t^-, l^-, f^-\}$ represents each BNN component. $X_{i,j}^k$ is the normalized value for alternative y_i under criterion c_j. $W_{i,j}^k$ represents the adaptive weight, defined as:

$$ww_{i,j}^k = \left(\frac{1}{2} + \frac{\left|t_{i,j}^+ - t_{i,j}^-\right| + \left|f_{i,j}^+ - f_{i,j}^-\right|}{2}\right) \tag{27}$$

By applying this weight, we can amplify the entropy donations from exceedingly contradictory positive-negative interdependency (e.g., job satisfaction vs. dissatisfaction), while enhancing sensitivity for extreme paradoxes (such as high truth but high falsity).

In step 3, we compute objective weights determining the degree of reputation for each BNS component for each pair of alternatives and criteria, in the decision matrix.

$$d_j = \left\langle d_j^{t^+}, d_j^{l^+}, d_j^{f^+}, d_j^{t^-}, d_j^{l^-}, d_j^{f^-} \right\rangle,$$

such that:

$$\begin{aligned} d_j^{t^+} &= 1 - E_j^{t^+}, \quad d_j^{l^+} = 1 - E_j^{l^+}, \quad d_j^{f^+} = 1 - E_j^{f^+} \\ d_j^{t^-} &= 1 - E_j^{t^-}, \quad d_j^{l^-} = 1 - E_j^{l^-}, \quad d_j^{f^-} = 1 - E_j^{f^-}, \end{aligned} \tag{28}$$

By summing all components, we can reach the following:

$$w_j = \frac{\sum_k d_j^k}{\sum_{j=1}^n \sum_k d_j^k}, \tag{29}$$

where k represents BNS components.

In step 4, we propose to perform adjustment for BNS of cost criteria by transforming its constituting values using:

$$A'_{i,j} = \left\langle \frac{\min\left(t_{i,j}^+\right)}{t_{i,j}^+}, \frac{\min\left(l_{i,j}^+\right)}{l_{i,j}^+}, \frac{\min\left(f_{i,j}^+\right)}{f_{i,j}^+}, \frac{\max\left(t_{i,j}^-\right)}{t_{i,j}^-}, \frac{\max\left(l_{i,j}^-\right)}{l_{i,j}^-}, \frac{\max\left(f_{i,j}^-\right)}{f_{i,j}^-} \right\rangle \tag{30}$$

In step 5, we re-normalize the adjusted dataset by computing the normalized values:

$$X_{i,j} = \langle t'^{+}_{i,j}, l'^{+}_{i,j}, f'^{+}_{i,j}, t'^{-}_{i,j}, l'^{-}_{i,j}, f'^{-}_{i,j} \rangle$$

where

$$t'^{+}_{i,j} = \frac{t'^{+}_{i,j}}{\sum_{i=1}^{m} t'^{+}_{i,j}}, \text{ (similarly for } l'^{+}, f'^{+}, t'^{-}, l'^{-}, f'^{-}) \tag{31}$$

In step 6, we pairwise comparison matrix for each BNS component *k* in the decision matrix.

$$A_{i,j} = \left\langle \begin{matrix} \max(X_{i,,,t^{+}}), \max(X_{i,,,l^{+}}), \max(X_{i,,,f^{+}}), \\ \max(X_{i,,,t^{-}}), \max(X_{i,,,l^{-}}), \max(X_{i,,,f^{-}}) \end{matrix} \right\rangle \tag{32}$$

where diagonal elements are set to the maximum values in each column.

In step 7, we perform optimization of the weight distribution using a penalty matrix defined as follows:

$$P_{i,j} = \left\langle P^{t^{+}}_{i,j}, P^{l^{+}}_{i,j}, P^{f^{+}}_{i,j}, P^{t^{-}}_{i,j}, P^{l^{-}}_{i,j}, P^{f^{-}}_{i,j} \right\rangle$$

where

$$P^{k}_{i,j} = \frac{-A^{k}_{i,j} + \max(A^{k}_{:,j})}{\max(A^{k}_{:,j})} \tag{33}$$

In step 9, the weighted penalty function is to be minimized:

$$f(\mathrm{w}) = \sum_{k} \sum_{i=1}^{n} \sum_{j=1}^{n} P^{k}_{i,j} w_j,$$

such that (34)

$$\min_{\mathrm{w}} f(\mathrm{w}), \text{ subject to } \sum_{j=1}^{n} w_j = 1,\ w_j > 0$$

Then, the final weights of each criterion are derived as:

$$w^{*}_{j} = \frac{w_j}{\sum_{j=1}^{n} w_j}, \tag{35}$$

In step 10, using the calculated weights, we could calculate a weighted decision matrix as follows:

$$A_{ij}^{w_j} = \left\{ t_{ij}^{w_j+}, l_{ij}^{w_j+}, f_{ij}^{w_j+}, t_{ij}^{w_j-}, l_{ij}^{w_j-}, f_{ij}^{w_j-} \right\} = \begin{Bmatrix} 1-\left(1-t_{ij}^{+}\right)^{w_j}, \left(l_{ij}^{+}\right)^{w_j}, \left(f_{ij}^{+}\right)^{w_j}, \\ -\left(-t_{ij}^{-}\right)^{w_j}, -\left(-l_{ij}^{-}\right)^{w_j}, -\left(\left(1-\left(1-f_{ij}^{-}\right)\right)^{w_j}\right) \end{Bmatrix} \quad (36)$$

In step 11, we compute the average BNS across all alternatives with respect to each criterion c_j:

$$AV_j = \left\langle AV_j^{t^+}, AV_j^{l^+}, AV_j^{f^+}, AV_j^{t^-}, AV_j^{l^-}, AV_j^{f^-} \right\rangle$$

where:

$$AV_j^{t^+} = \frac{1}{m}\sum_{i=1}^{m} t_{ij}^{(w_j)+},\ AV_j^{l^+} = \frac{1}{m}\sum_{i=1}^{m} l_{ij}^{(w_j)+},\ AV_j^{f^+} = \frac{1}{m}\sum_{i=1}^{m} f_{ij}^{(w_j)+}$$
$$AV_j^{t^-} = \frac{1}{m}\sum_{i=1}^{m} t_{ij}^{(w_j)-},\ AV_j^{l^-} = \frac{1}{m}\sum_{i=1}^{m} l_{ij}^{(w_j)-},\ AV_j^{f^-} = \frac{1}{m}\sum_{i=1}^{m} f_{ij}^{(w_j)-} \quad (37)$$

The computed average solution acts as a reference point for evaluating the degree to which alternative deviates [16].

In step 12, we calculate the Positive Distance from Average (PDA) for each alternative y_i and criterion c_j:

$$PDA_{ij} = \left\langle PDA_{ij}^{t^+}, PDA_{ij}^{l^+}, PDA_{ij}^{f^+}, PDA_{ij}^{t^-}, PDA_{ij}^{l^-}, PDA_{ij}^{f^-} \right\rangle$$

where:

$$PDA_{ij}^{t^+} = max\left(0, t_{ij}^{(w_j)+} - AV_j^{t^+}\right),\ PDA_{ij}^{l^+} = max\left(0, AV_j^{l^+} - l_{ij}^{(w_j)+}\right),\ PDA_{ij}^{f^+} = max\left(0, AV_j^{f^+} - f_{ij}^{(w_j)+}\right) \quad (38)$$

$$PDA_{ij}^{t^-} = min\left(0, t_{ij}^{(w_j)-} - AV_j^{t^-}\right),\ PDA_{ij}^{l^-} = min\left(0, AV_j^{l^-} - l_{ij}^{(w_j)^-}\right),\ PDA_{ij}^{f^-} = min\left(0, AV_j^{f^-} - f_{ij}^{(w_j)^-}\right)$$

In step 13, we calculate the Negative Distance from Average (NDA) for each alternative y_i and criterion c_j:

$$NDA_{ij} = \left\langle NDA_{ij}^{t^+}, NDA_{ij}^{l^+}, NDA_{ij}^{f^+}, NDA_{ij}^{t^-}, NDA_{ij}^{l^-}, NDA_{ij}^{f^-} \right\rangle$$

where:

$$\begin{aligned} NDA_{ij}^{t^+} &= max\left(0, AV_j^{t^+} - t_{ij}^{(w_j)+}\right), NDA_{ij}^{l^+} \\ &= max\left(0, l_{ij}^{(w_j)+} - AV_j^{l^+}\right), NDA_{ij}^{f^+} \\ &= max\left(0, f_{ij}^{(w_j)+} - AV_j^{f^+}\right) \end{aligned} \tag{39}$$

$$\begin{aligned} NDA_{ij}^{t^-} &= min\left(0, AV_j^{t^-} - t_{ij}^{(w_j)-}\right), NDA_{ij}^{l^-} \\ &= min\left(0, l_{ij}^{(w_j)-} - AV_j^{l^-}\right), NDA_{ij}^{f^-} \\ &= min\left(0, f_{ij}^{(w_j)-} - AV_j^{f^-}\right) \end{aligned}$$

In step 13, we calculate the aggregated PDA and NDA for each alternative

$$\begin{aligned} SPDA_i &= \sum_{j=1}^{n} w_j \cdot s\left(PDA_{ij}^{t^+}, PDA_{ij}^{l^+}, PDA_{ij}^{f^+}, PDA_{ij}^{t^-}, PDA_{ij}^{l^-}, PDA_{ij}^{f^-}\right) \\ SNDA_i &= \sum_{j=1}^{n} w_j \cdot s\left(PDA_{ij}^{t^+}, PDA_{ij}^{l^+}, PDA_{ij}^{f^+}, PDA_{ij}^{t^-}, PDA_{ij}^{l^-}, PDA_{ij}^{f^-}\right) \end{aligned} \tag{40}$$

Finally, we compute the final ranking score of each alternative:

$$S_i = \frac{SPDA_i}{max(SPDA)} - \frac{SNDA_i}{max(SNDA)} \tag{41}$$

Accordingly, we find out that higher values of S_i indicates better alternative ↓.

4. Numerical Application

This research utilizes structured questionnaires to measure job satisfaction from different perspectives including but not limited to work environment, salary, career growth, academic autonomy, and institutional policies. The survey captures both qualitative and quantitative answers to amount ambiguity in faculty members' satisfaction degrees. The dataset includes answers from university faculty members within various age groups, genders, academic positions, and years of experience. Table 1 show tabular distribution of participants based on key demographic variables.

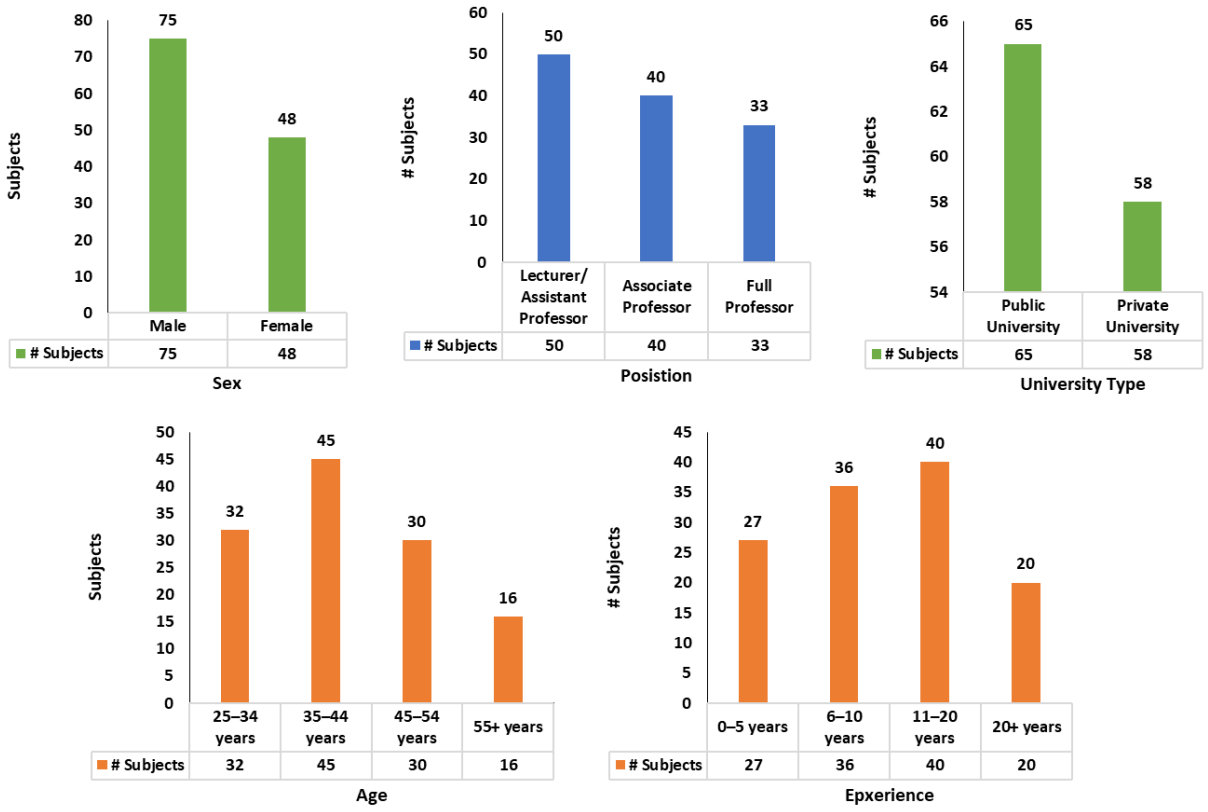

Figure 2: demographic distribution of respondents

The data collection in our case study includes broad range of criteria. Some needs to be maximized to indicate better satisfaction, and other that should be minimized to achieve better job satisfaction. These criteria include the following criteria Salary & Compensation, Career Growth Opportunities, Work Environment, Research Support [17], Teaching Autonomy, Work-Life Balance, Job Insecurity, and Bureaucratic Constraints. As part of our case study, we selected five distinctive universities from different regions in the Middle East to analyze job satisfaction. Each university vary from another one in different dimensions, thereby serving as an alternative for teachers in our MCDM along with the abovementioned criteria. We anonymously refer to these alternatives using their general university type Public Research University (U1), Private University (U2), Public Teaching University (U4), International Private Institution (U5), Newly Established University (U3). Table 1 show the summarize the aggregated BNS decision matrix from different experts.

Table 1. Aggregated BNS decision matrix for Job Satisfaction Assessment

	U1	U2	U3	U4	U5

C 1	(0.887, 0.063, 0.014, -0.097, -0.09, -0.915)	(0.614, 0.221, 0.244, -0.326, -0.275, -0.593)	(0.834, 0.088, 0.081, -0.208, -0.231, -0.843)	(0.949, 0.193, 0.06, -0.053, -0.067, -0.929)	(0.127, 0.147, 0.876, -0.806, -0.173, -0.179)
C 2	(0.632, 0.326, 0.318, -0.241, -0.3, -0.665)	(0.894, 0.196, 0.021, -0.06, -0.088, -0.979)	(0.444, 0.257, 0.541, -0.54, -0.333, -0.292)	(0.733, 0.279, 0.184, -0.126, -0.21, -0.806)	(0.171, 0.105, 0.9, -0.829, -0.162, -0.076)
C 3	(0.173, 0.169, 0.938, -0.821, -0.173, -0.168)	(0.361, 0.348, 0.53, -0.467, -0.241, -0.362)	(0.912, 0.115, 0.092, -0.16, -0.133, -0.93)	(0.241, 0.158, 0.808, -0.837, -0.272, -0.16)	(0.941, 0.06, 0.131, -0.105, -0.09, -0.889)
C 4	(1.019, 0.04, 0.131, -0.191, -0.17, -0.909)	(0.806, 0.262, 0.113, -0.17, -0.112, -0.937)	(0.584, 0.395, 0.243, -0.266, -0.376, -0.582)	(0.386, 0.308, 0.578, -0.501, -0.274, -0.386)	(0.068, 0.159, 0.928, -0.828, -0.152, -0.219)
C 5	(0.975, 0.168, 0.111, -0.148, -0.153, -0.854)	(0.907, 0.188, 0.137, -0.173, -0.23, -0.928)	(0.526, 0.348, 0.276, -0.263, -0.401, -0.467)	(0.388, 0.312, 0.595, -0.496, -0.298, -0.42)	(0.177, 0.233, 0.961, -0.863, -0.081, -0.155)
C 6	(0.17, 0.203, 0.755, -0.841, -0.205, -0.11)	(0.693, 0.202, 0.17, -0.225, -0.292, -0.776)	(0.366, 0.317, 0.471, -0.558, -0.266, -0.399)	(0.15, 0.065, 0.881, -0.765, -0.213, -0.116)	(0.738, 0.28, 0.174, -0.186, -0.153, -0.835)
C 7	(0.888, 0.113, 0.101, -0.049, -0.169, -0.988)	(0.892, 0.23, 0.224, -0.18, -0.169, -0.823)	(0.11, 0.207, 0.783, -0.854, -0.11, -0.129)	(0.375, 0.292, 0.577, -0.576, -0.281, -0.304)	(0.205, 0.252, 0.863, -0.931, -0.124, -0.161)
C 8	(0.935, 0.089, 0.032, -0.098, -0.068, -0.892)	(0.915, 0.079, 0.153, -0.178, -0.134, -0.802)	(0.543, 0.387, 0.257, -0.279, -0.388, -0.503)	(0.293, 0.37, 0.518, -0.479, -0.2, -0.357)	(0.198, 0.124, 0.755, -0.936, -0.157, -0.079)

5. Results & analysis

This section presents the numerical and analytical findings of our approach for evaluating job satisfaction among university faculty members across different set of universities, bearing in mind different criteria.

First, we drive a normalized decision matrix in Table 2, where the scaled BNSs is presented for each alternative across different criteria.

Table 2: Normalized BNS Decision Matrix for Job Satisfaction Assessment

	U1	**U2**	**U3**	**U4**	**U5**
C 1	(0.26, 0.161, 0.17, -0.149, -0.173, -0.265)	(0.18, 0.218, 0.132, -0.164, -0.217, -0.171)	(0.245, 0.285, 0.152, -0.105, -0.183, -0.244)	(0.278, 0.191, 0.073, -0.177, -0.29, -0.269)	(0.037, 0.145, 0.473, -0.405, -0.137, -0.052)

C 2	(0.206, 0.26, 0.162, -0.117, -0.207, -0.2)	(0.291, 0.156, 0.011, -0.154, -0.323, -0.295)	(0.144, 0.205, 0.275, -0.264, -0.221, -0.088)	(0.238, 0.223, 0.094, -0.061, -0.142, -0.243)	(0.121, 0.156, 0.458, -0.404, -0.107, -0.174)
C 3	(0.066, 0.168, 0.347, -0.34, -0.18, -0.067)	(0.137, 0.346, 0.197, -0.193, -0.251, -0.144)	(0.347, 0.114, 0.108, -0.068, -0.139, -0.37)	(0.092, 0.157, 0.299, -0.347, -0.283, -0.066)	(0.358, 0.215, 0.048, -0.052, -0.147, -0.353)
C 4	(0.319, 0.16, 0.066, -0.097, -0.158, -0.3)	(0.269, 0.196, 0.057, -0.089, -0.103, -0.309)	(0.195, 0.295, 0.122, -0.136, -0.346, -0.192)	(0.129, 0.23, 0.29, -0.255, -0.252, -0.127)	(0.089, 0.119, 0.466, -0.422, -0.14, -0.072)
C 5	(0.328, 0.135, 0.053, -0.076, -0.112, -0.302)	(0.305, 0.151, 0.066, -0.089, -0.17, -0.328)	(0.177, 0.279, 0.133, -0.135, -0.294, -0.165)	(0.131, 0.25, 0.286, -0.255, -0.218, -0.149)	(0.06, 0.187, 0.462, -0.444, -0.206, -0.055)
C 6	(0.081, 0.16, 0.307, -0.327, -0.182, -0.091)	(0.327, 0.159, 0.073, -0.087, -0.259, -0.332)	(0.173, 0.249, 0.191, -0.217, -0.236, -0.171)	(0.072, 0.208, 0.358, -0.297, -0.189, -0.05)	(0.348, 0.224, 0.071, -0.072, -0.136, -0.357)
C 7	(0.078, 0.348, 0.534, -0.298, -0.185, -0.054)	(0.077, 0.17, 0.241, -0.406, -0.185, -0.062)	(0.324, 0.19, 0.069, -0.087, -0.265, -0.397)	(0.184, 0.135, 0.093, -0.129, -0.111, -0.168)	(0.337, 0.156, 0.062, -0.08, -0.253, -0.318)
C 8	(0.086, 0.126, 0.181, -0.166, -0.072, -0.104)	(0.087, 0.223, 0.392, -0.379, -0.331, -0.116)	(0.148, 0.126, 0.233, -0.242, -0.114, -0.185)	(0.274, 0.131, 0.116, -0.141, -0.201, -0.261)	(0.405, 0.394, 0.079, -0.072, -0.282, -0.334)

Then, we drive pairwise comparison matrix, as depicted in Table 3, which explain the relative performance of each alternative in the normalized matrix.

Table 3: Pairwise Comparison Matrix for Job Satisfaction Assessment

	C 1	**C 2**	**C 3**	**C 4**
C 1	(0.278, 0.285, 0.473, -0.105, -0.137, -0.052)	(0.238, 0.205, 0.458, -0.264, -0.107, -0.174)	(0.092, 0.114, 0.048, -0.068, -0.147, -0.353)	(0.129, 0.295, 0.466, -0.136, -0.14, -0.072)
C 2	(0.18, 0.161, 0.473, -0.177, -0.137, -0.244)	(0.291, 0.26, 0.458, -0.061, -0.107, -0.088)	(0.137, 0.168, 0.048, -0.347, -0.147, -0.37)	(0.269, 0.16, 0.466, -0.255, -0.14, -0.192)
C 3	(0.037, 0.218, 0.17, -0.405, -0.183, -0.269)	(0.121, 0.156, 0.162, -0.404, -0.221, -0.243)	(0.358, 0.346, 0.347, -0.052, -0.139, -0.066)	(0.089, 0.196, 0.066, -0.422, -0.346, -0.127)
C 4	(0.26, 0.285, 0.473, -0.164, -0.217, -0.052)	(0.206, 0.205, 0.458, -0.154, -0.323, -0.174)	(0.066, 0.114, 0.048, -0.193, -0.251, -0.353)	(0.319, 0.295, 0.466, -0.089, -0.103, -0.072)
C 5	(0.26, 0.285, 0.473, -0.149, -0.173, -0.052)	(0.206, 0.205, 0.458, -0.117, -0.207, -0.174)	(0.066, 0.114, 0.048, -0.34, -0.18, -0.353)	(0.319, 0.295, 0.466, -0.097, -0.158, -0.072)
C 6	(0.037, 0.285, 0.073, -0.405, -0.137, -0.269)	(0.121, 0.205, 0.094, -0.404, -0.107, -0.243)	(0.358, 0.114, 0.299, -0.052, -0.147, -0.066)	(0.089, 0.295, 0.29, -0.422, -0.14, -0.127)
C 7	(0.037, 0.161, 0.17, -0.405, -0.29, -0.265)	(0.121, 0.26, 0.162, -0.404, -0.142, -0.2)	(0.358, 0.168, 0.347, -0.052, -0.283, -0.067)	(0.089, 0.16, 0.066, -0.422, -0.252, -0.3)
C 8	(0.037, 0.145, 0.132, -0.405, -0.173, -0.265)	(0.121, 0.156, 0.011, -0.404, -0.207, -0.2)	(0.358, 0.215, 0.197, -0.052, -0.18, -0.067)	(0.089, 0.119, 0.057, -0.422, -0.158, -0.3)

	C 5	**C 6**	**C 7**	**C 8**

C 1	(0.131, 0.279, 0.462, -0.135, -0.206, -0.055)	(0.072, 0.249, 0.071, -0.217, -0.136, -0.357)	(0.184, 0.19, 0.062, -0.087, -0.253, -0.318)	(0.274, 0.126, 0.079, -0.242, -0.282, -0.334)
C 2	(0.305, 0.135, 0.462, -0.255, -0.206, -0.165)	(0.327, 0.16, 0.071, -0.297, -0.136, -0.171)	(0.077, 0.348, 0.062, -0.129, -0.253, -0.397)	(0.087, 0.126, 0.079, -0.141, -0.282, -0.185)
C 3	(0.06, 0.151, 0.053, -0.444, -0.294, -0.149)	(0.348, 0.159, 0.307, -0.072, -0.236, -0.05)	(0.337, 0.17, 0.534, -0.08, -0.265, -0.168)	(0.405, 0.223, 0.181, -0.072, -0.114, -0.261)
C 4	(0.328, 0.279, 0.462, -0.089, -0.17, -0.055)	(0.081, 0.249, 0.071, -0.087, -0.259, -0.357)	(0.078, 0.19, 0.062, -0.406, -0.185, -0.318)	(0.086, 0.126, 0.079, -0.379, -0.331, -0.334)
C 5	(0.328, 0.279, 0.462, -0.076, -0.112, -0.055)	(0.081, 0.249, 0.071, -0.327, -0.182, -0.357)	(0.078, 0.19, 0.062, -0.298, -0.185, -0.318)	(0.086, 0.126, 0.079, -0.166, -0.072, -0.334)
C 6	(0.06, 0.279, 0.286, -0.444, -0.206, -0.149)	(0.348, 0.249, 0.358, -0.072, -0.136, -0.05)	(0.337, 0.19, 0.093, -0.08, -0.253, -0.168)	(0.405, 0.126, 0.116, -0.072, -0.282, -0.261)
C 7	(0.06, 0.135, 0.053, -0.444, -0.218, -0.302)	(0.348, 0.16, 0.307, -0.072, -0.189, -0.091)	(0.337, 0.348, 0.534, -0.08, -0.111, -0.054)	(0.405, 0.126, 0.181, -0.072, -0.201, -0.104)
C 8	(0.06, 0.187, 0.066, -0.444, -0.112, -0.302)	(0.348, 0.224, 0.073, -0.072, -0.182, -0.091)	(0.337, 0.156, 0.241, -0.08, -0.185, -0.054)	(0.405, 0.394, 0.392, -0.072, -0.072, -0.104)

Following, we drive the penalty matrix in Table 4, in which we quantify the deviation of each alternative from the most favorable performance across all criteria, highlighting the degree of underperformance against to the best-achieving university.

Table 4: Penalty Matrix for Job Satisfaction Assessment

	C 1	**C 2**	**C 3**	**C 4**
C 1	(0.0, 0.0, 0.0, 0.0, 0.0, 0.0)	(0.182, 0.212, 0.0, -3.328, 0.0, -0.977)	(0.743, 0.671, 0.862, -0.308, -0.058, -4.348)	(0.596, 0.0, 0.0, -0.528, -0.359, 0.0)
C 2	(0.353, 0.435, 0.0, -0.686, 0.0, -3.692)	(0.0, 0.0, 0.0, 0.0, 0.0, 0.0)	(0.617, 0.514, 0.862, -5.673, -0.058, -4.606)	(0.157, 0.458, 0.0, -1.865, -0.359, -1.667)
C 3	(0.867, 0.235, 0.641, -2.857, -0.336, -4.173)	(0.584, 0.4, 0.646, -5.623, -1.065, -1.761)	(0.0, 0.0, 0.0, 0.0, 0.0, 0.0)	(0.721, 0.336, 0.858, -3.742, -2.359, -0.764)
C 4	(0.065, 0.0, 0.0, -0.562, -0.584, 0.0)	(0.292, 0.212, 0.0, -1.525, -2.019, -0.977)	(0.816, 0.671, 0.862, -2.712, -0.806, -4.348)	(0.0, 0.0, 0.0, 0.0, 0.0, 0.0)
C 5	(0.065, 0.0, 0.0, -0.419, -0.263, 0.0)	(0.292, 0.212, 0.0, -0.918, -0.935, -0.977)	(0.816, 0.671, 0.862, -5.538, -0.295, -4.348)	(0.0, 0.0, 0.0, -0.09, -0.534, 0.0)
C 6	(0.867, 0.0, 0.846, -2.857, 0.0, -4.173)	(0.584, 0.212, 0.795, -5.623, 0.0, -1.761)	(0.0, 0.671, 0.138, 0.0, -0.058, 0.0)	(0.721, 0.0, 0.378, -3.742, -0.359, -0.764)
C 7	(0.867, 0.435, 0.641, -2.857, -1.117, -4.096)	(0.584, 0.0, 0.646, -5.623, -0.327, -1.273)	(0.0, 0.514, 0.0, 0.0, -1.036, -0.015)	(0.721, 0.458, 0.858, -3.742, -1.447, -3.167)
C 8	(0.867, 0.491, 0.721, -2.857, -0.263, -4.096)	(0.584, 0.4, 0.976, -5.623, -0.935, -1.273)	(0.0, 0.379, 0.432, 0.0, -0.295, -0.015)	(0.721, 0.597, 0.878, -3.742, -0.534, -3.167)

	C 5	**C 6**	**C 7**	**C 8**
C 1	(0.601, 0.0, 0.0, -0.776, -0.839, 0.0)	(0.793, 0.0, 0.802, -2.014, 0.0, -6.14)	(0.454, 0.454, 0.884, -0.087, -1.279, -4.889)	(0.323, 0.68, 0.798, -2.361, -2.917, -2.212)

C 2	(0.07, 0.516, 0.0, -2.355, -0.839, -2.0)	(0.06, 0.357, 0.802, -3.125, 0.0, -2.42)	(0.772, 0.0, 0.884, -0.612, -1.279, -6.352)	(0.785, 0.68, 0.798, -0.958, -2.917, -0.779)
C 3	(0.817, 0.459, 0.885, -4.842, -1.625, -1.709)	(0.0, 0.361, 0.142, 0.0, -0.735, 0.0)	(0.0, 0.511, 0.0, 0.0, -1.387, -2.111)	(0.0, 0.434, 0.538, 0.0, -0.583, -1.51)
C 4	(0.0, 0.0, 0.0, -0.171, -0.518, 0.0)	(0.767, 0.0, 0.802, -0.208, -0.904, -6.14)	(0.769, 0.454, 0.884, -4.075, -0.667, -4.889)	(0.788, 0.68, 0.798, -4.264, -3.597, -2.212)
C 5	(0.0, 0.0, 0.0, 0.0, 0.0, 0.0)	(0.767, 0.0, 0.802, -3.542, -0.338, -6.14)	(0.769, 0.454, 0.884, -2.725, -0.667, -4.889)	(0.788, 0.68, 0.798, -1.306, 0.0, -2.212)
C 6	(0.817, 0.0, 0.381, -4.842, -0.839, -1.709)	(0.0, 0.0, 0.0, 0.0, 0.0, 0.0)	(0.0, 0.454, 0.826, 0.0, -1.279, -2.111)	(0.0, 0.68, 0.704, 0.0, -2.917, -1.51)
C 7	(0.817, 0.516, 0.885, -4.842, -0.946, -4.491)	(0.0, 0.357, 0.142, 0.0, -0.39, -0.82)	(0.0, 0.0, 0.0, 0.0, 0.0, 0.0)	(0.0, 0.68, 0.538, 0.0, -1.792, 0.0)
C 8	(0.817, 0.33, 0.857, -4.842, 0.0, -4.491)	(0.0, 0.1, 0.796, 0.0, -0.338, -0.82)	(0.0, 0.552, 0.549, 0.0, -0.667, 0.0)	(0.0, 0.0, 0.0, 0.0, 0.0, 0.0)

Based on the above sequence of calculations, we drive the weights for job satisfaction criteria as shown in Table 5.

Table 5: Weights of Criteria for Job Satisfaction Assessment

Criteria	**C 1**	**C 2**	**C 3**	**C 4**	**C 5**	**C 6**	**C 7**	**C 8**
Weights	0.11411	0.11812	0.13514	0.12713	0.13013	0.12212	0.13213	0.12112

Based on the above sequence of calculations, we drive PDA for job satisfaction as shown in Table 6.

Table 6: PDA for Job Satisfaction Assessment

	U1	**U2**	**U3**	**U4**	**U5**
C 1	(0.058, 0.0, 0.001, 0.0, 0.0, -0.045)	(0.0, 0.011, 0.0, 0.0, -0.011, 0.0)	(0.023, 0.037, 0.0, 0.0, 0.0, -0.036)	(0.126, 0.0, 0.0, 0.0, -0.04, -0.047)	(0.0, 0.0, 0.11, -0.086, 0.0, 0.0)
C 2	(0.0, 0.029, 0.024, 0.0, -0.009, -0.008)	(0.111, 0.0, 0.0, 0.0, -0.057, -0.052)	(0.0, 0.005, 0.081, -0.048, -0.016, 0.0)	(0.022, 0.013, 0.0, 0.0, 0.0, -0.03)	(0.0, 0.0, 0.139, -0.096, 0.0, 0.0)
C 3	(0.0, 0.0, 0.093, -0.096, 0.0, 0.0)	(0.0, 0.069, 0.02, -0.024, -0.029, 0.0)	(0.136, 0.0, 0.0, 0.0, 0.0, -0.107)	(0.0, 0.0, 0.074, -0.098, -0.043, 0.0)	(0.174, 0.015, 0.0, 0.0, 0.0, -0.101)
C 4	(0.185, 0.0, 0.0, 0.0, 0.0, -0.063)	(0.043, 0.002, 0.0, 0.0, 0.0, -0.067)	(0.0, 0.048, 0.0, 0.0, -0.068, -0.009)	(0.0, 0.02, 0.075, -0.046, -0.033, 0.0)	(0.0, 0.0, 0.133, -0.106, 0.0, 0.0)

C 5	(0.216, 0.0, 0.0, 0.0, 0.0, -0.069)	(0.101, 0.0, 0.0, 0.0, 0.0, -0.079)	(0.0, 0.041, 0.0, 0.0, -0.048, 0.0)	(0.0, 0.028, 0.075, -0.051, -0.014, 0.0)	(0.0, 0.0, 0.135, -0.119, -0.008, 0.0)
C 6	(0.0, 0.0, 0.07, -0.074, 0.0, 0.0)	(0.058, 0.0, 0.0, 0.0, -0.028, -0.082)	(0.0, 0.024, 0.016, -0.026, -0.019, -0.006)	(0.0, 0.005, 0.089, -0.063, 0.0, 0.0)	(0.075, 0.013, 0.0, 0.0, 0.0, -0.09)
C 7	(0.126, 0.0, 0.0, 0.0, -0.003, -0.119)	(0.13, 0.01, 0.0, 0.0, -0.003, -0.101)	(0.0, 0.0, 0.08, -0.073, 0.0, 0.0)	(0.0, 0.036, 0.042, -0.024, -0.058, 0.0)	(0.0, 0.02, 0.093, -0.085, 0.0, 0.0)
C 8	(0.143, 0.036, 0.0, -0.003, -0.093, -0.062)	(0.119, 0.0, 0.0, 0.0, 0.0, -0.05)	(0.0, 0.035, 0.0, 0.0, -0.042, 0.0)	(0.0, 0.031, 0.041, -0.021, 0.0, 0.0)	(0.0, 0.0, 0.085, -0.098, 0.0, 0.0)

Based on the above sequence of calculations, we drive NDA for job satisfaction as shown in Table 7.

Table 7: NDA for Job Satisfaction Assessment

	U1	**U2**	**U3**	**U4**	**U5**
C 1	(0.0, 0.018, 0.0, -0.019, -0.011, 0.0)	(0.059, 0.0, 0.024, -0.01, 0.0, -0.003)	(0.0, 0.0, 0.01, -0.054, -0.006, 0.0)	(0.0, 0.002, 0.079, -0.002, 0.0, 0.0)	(0.147, 0.028, 0.0, 0.0, -0.033, -0.123)
C 2	(0.011, 0.0, 0.0, -0.037, 0.0, 0.0)	(0.0, 0.022, 0.215, -0.009, 0.0, 0.0)	(0.055, 0.0, 0.0, 0.0, 0.0, -0.08)	(0.0, 0.0, 0.03, -0.099, -0.028, 0.0)	(0.069, 0.023, 0.0, 0.0, -0.055, -0.008)
C 3	(0.119, 0.012, 0.0, 0.0, -0.007, -0.097)	(0.085, 0.0, 0.0, 0.0, 0.0, -0.011)	(0.0, 0.051, 0.051, -0.095, -0.035, 0.0)	(0.107, 0.019, 0.0, 0.0, 0.0, -0.099)	(0.0, 0.0, 0.138, -0.123, -0.029, 0.0)
C 4	(0.0, 0.019, 0.086, -0.06, -0.016, 0.0)	(0.0, 0.0, 0.1, -0.069, -0.058, 0.0)	(0.039, 0.0, 0.023, -0.025, 0.0, 0.0)	(0.085, 0.0, 0.0, 0.0, 0.0, -0.039)	(0.106, 0.049, 0.0, 0.0, -0.028, -0.101)
C 5	(0.0, 0.038, 0.109, -0.082, -0.057, 0.0)	(0.0, 0.026, 0.088, -0.066, -0.013, 0.0)	(0.072, 0.0, 0.014, -0.022, 0.0, -0.005)	(0.103, 0.0, 0.0, 0.0, 0.0, -0.017)	(0.14, 0.004, 0.0, 0.0, 0.0, -0.126)
C 6	(0.053, 0.022, 0.0, 0.0, -0.008, -0.061)	(0.0, 0.022, 0.085, -0.072, 0.0, 0.0)	(0.022, 0.0, 0.0, 0.0, 0.0, 0.0)	(0.056, 0.0, 0.0, 0.0, -0.004, -0.119)	(0.0, 0.0, 0.088, -0.091, -0.037, 0.0)
C 7	(0.0, 0.064, 0.149, -0.074, 0.0, 0.0)	(0.0, 0.0, 0.067, -0.107, 0.0, 0.0)	(0.094, 0.002, 0.0, 0.0, -0.034, -0.111)	(0.065, 0.0, 0.0, 0.0, 0.0, -0.02)	(0.095, 0.0, 0.0, 0.0, -0.029, -0.088)
C 8	(0.0, 0.0, 0.007, 0.0, 0.0, 0.0)	(0.0, 0.024, 0.085, -0.083, -0.066, 0.0)	(0.049, 0.0, 0.034, -0.037, 0.0, -0.004)	(0.098, 0.0, 0.0, 0.0, -0.017, -0.041)	(0.113, 0.079, 0.0, 0.0, -0.051, -0.067)

In Figure 3, we plot SPDA and SNDA scores for each university to offer a comparative evaluation of faculty job satisfaction across different institutions by explaining the level of deviation from the average satisfaction level, indicate fewer dissatisfaction factors.

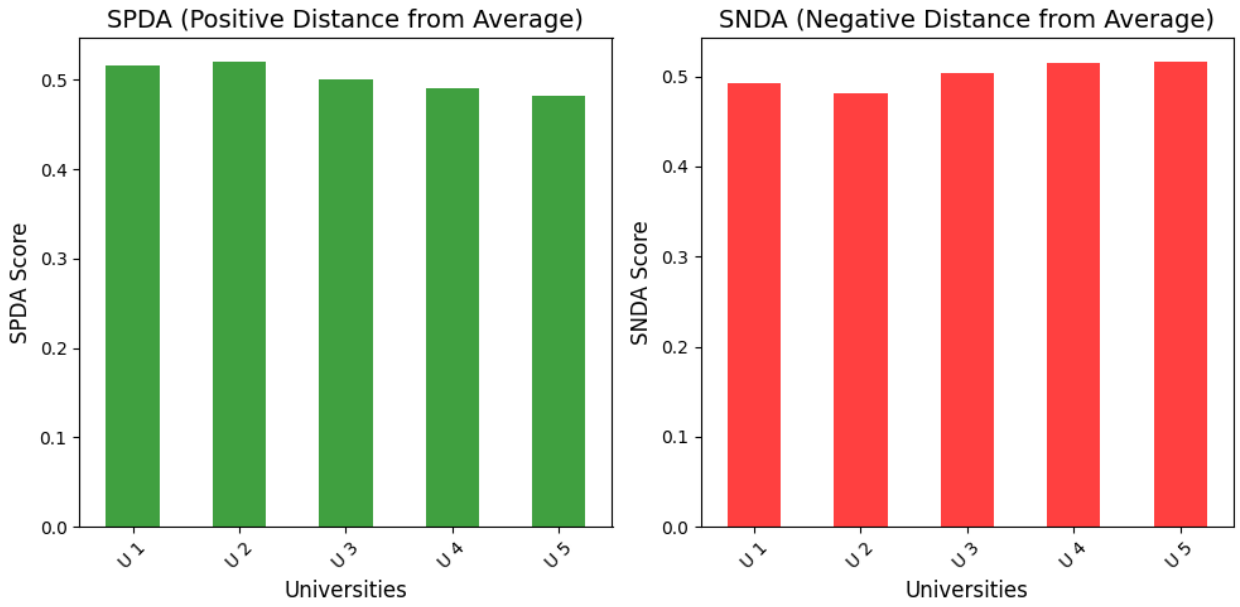

Figure 3. visualization of SPDA and SNDA scores

In Table 6, we display the performance score and the ranking of different alternatives.

Table 8. Ranking of different alternatives

Alternatives	Performance Score	Ranking
U1	**0.038700**	**2**
U2	**0.067286**	**1**
U3	**-0.014061**	**3**
U4	**-0.053327**	**4**
U5	**-.073508**	**5**

To ensure the robustness and reliability of our proposed, we conducted a sensitivity analysis by systematically varying the weights assigned to different criteria such that Scenario 1: +10%, Scenario 2: +5%, Scenario 3: 0%, Scenario 4: -5%, Scenario 5: -10%,. Then, we plot the new ranking results are displayed in Figure 4, showing the stability of the proposed model.

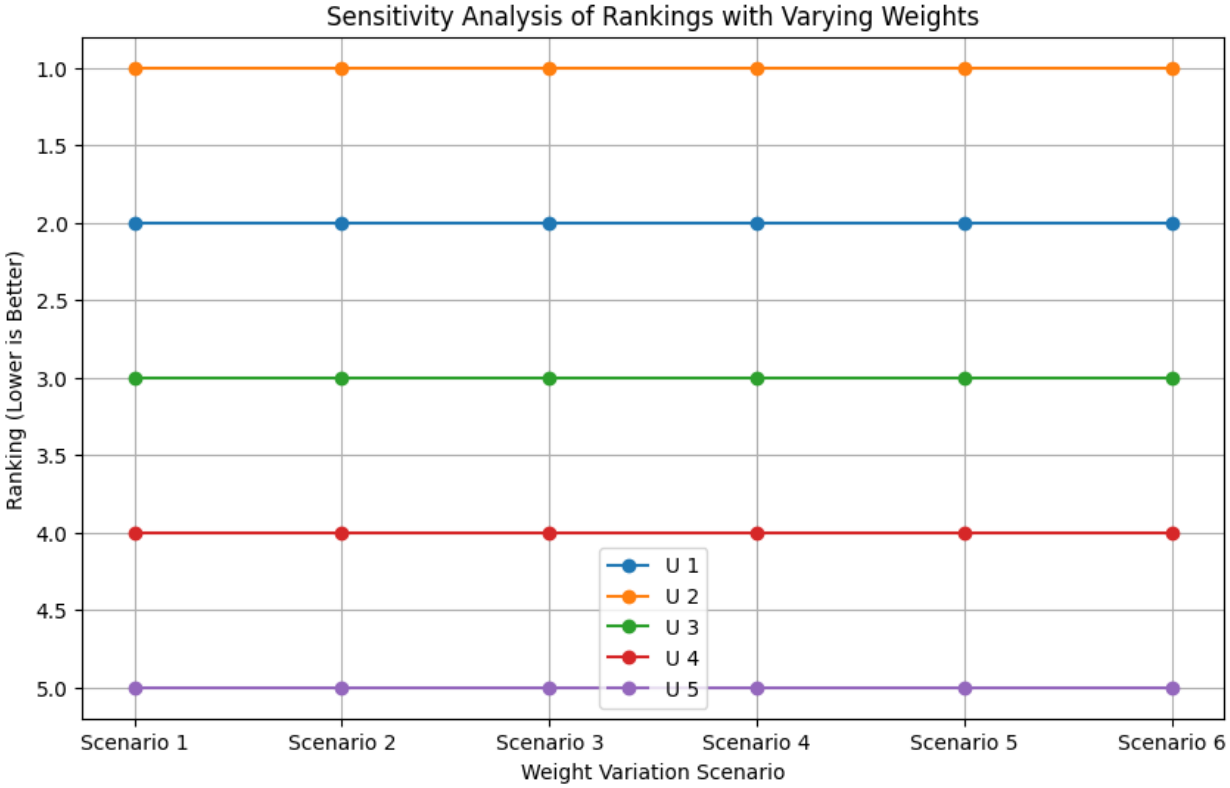

Figure 4. Sensitivity analysis of the proposed approach

6. Conclusion

This study presents an improved BNDN approach for uncertainty-based quantification of job satisfaction in higher education, addressing the limitations of traditional methods in capturing contradictory and imprecise faculty perceptions. With integration of BNSs, a bipolar entropy-based weighting method is derived to make systematically determine the importance of different criteria. Then, an BNS-based EDAS is presented to provides a more robust and comprehensive model for evaluating simultaneous satisfaction and dissatisfaction factors. We empirically validate our approach using a case study of university faculty in the Middle East, which demonstrate the effectiveness in handling uncertainty and supporting data-driven decision-making. The implications of our results offer practical insights for university administrators and policymakers to enhance faculty well-being and institutional performance.

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Study of the correlation between the behavior of work teams and knowledge management in the offices dependent on the Academic Vice-Rectorate of the National Agrarian University of La Selva: based on Plithogenic Statistics

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Abstract. In this paper, we study the statistical relationship between the behavior of work teams and knowledge management in the offices subordinate to the Academic Vice-Rectorate of the Agrarian University of La Selva in Peru. To do so, we use the Plithogenic Statistics tool. We apply this theory to data collected in a survey of 30 members of these offices. We consider that the elements we measure have different origins from the point of view of the disciplines to which they respond. In addition, the data is uncertain and indeterminate. These are the two main reasons for using Plithogenic Statistics. Neutrosophic Statistics generalizes the methods of Classical Statistics when the data or parameters appear in the form of intervals, or when the sample size or population is indeterminate. Plithogenic Statistics generalizes Neutrosophic Statistics when there is indeterminacy, not just in the form of intervals.

Keywords: Work behavior, knowledge management, public management, knowledge management processes, Plithogeny, plithogenic statistics, Single-Valued Neutrosophic Number, Plithogenic Neutrosophic Probability.

1 Introduction

Management in organizations is influenced by new forces that form the way they operate in order to meet the changing demands of the environment. One of these forces is the power of human resources, since the way people work, think, and behave dictates the direction and success of the organization. Thus, to survive the rapid pace of change, investment in people is necessary.

In the new economy, knowledge replaces capital as a critical resource and consequently, organizations are forced to generate knowledge and design processes that allow its identification, processing, distribution, storage, and maintenance. Now more than ever, consumers expect organizations to be responsive to their needs, to provide them with service and delivery quickly, and to produce high-quality goods and services at the best possible price.

Each organization must respond to the needs of its customers, to legal constraints, and economic and technological changes. To do so, they must adopt different and complementary approaches that help them achieve these objectives. Paying attention to work behavior and managing knowledge turns out to be a key strategy to meet the demands present in an organization. These are approaches that not only private companies know and have been applying, but also public entities through modernization mechanisms.

The relevance of the public entity lies in its purpose, which is to satisfy the basic needs of the population by contributing to its development. Among these, educational entities have the duty to train citizens better prepared to carry out activities that contribute to the progress of the country.

The National Agrarian University of the Jungle (UNAS in Spanish) of Peru, as a higher education institution and as a public entity, has key objectives whose achievement will depend on the management approaches it adopts

in its strategy. With work behavior and knowledge management as key action strategies, it is necessary to know the situation of both approaches to obtain an approximation of the existing gap whose narrowing becomes one of the courses of action to be adopted in the future.

Finally, the study seeks to answer the general problem, defined by the question: Is there a relationship between the behavior of work teams and knowledge management in the offices dependent on the Academic Vice-Rectorate of the National Agrarian University of La Selva?

Each of the two variables to be studied is made up of different sub-elements that belong to different domains. Behavior within work teams and knowledge management can be studied from different perspectives such as work psychology, sociology, and economics. On the other hand, the study contains uncertainty and indeterminacy that is not only random but also cognitive.

That is why, instead of applying a traditional statistical method, we chose to use the Plithogenic Statistics as a study tool [1, 2]. Firstly, the indeterminacy and uncertainty of the data can be studied within Neutrosophic Statistics, where traditional statistical methods are extended to data and parameters in the form of intervals, [3-8]. They can also be applied when the sample or population size is indeterminate. Plithogenic Statistics is even more general and allows the application of statistical methods to data with uncertainty in any form [9].

Plithogeny is the new theory created by F. Smarandache, which attempts to model concepts, ideas, phenomena, etc., in such a way that the dialectic between different elements (concepts, ideas, phenomena, etc.) is generalized, the interaction between them, their opposites and their neutrals, so that it is closer to realistic modeling of the world [10-18].

The paper surveys a group of university members to determine whether there is a relationship between the behavior of work teams and knowledge management in the offices dependent on the Academic Vice-Rectorate of the National Agrarian University of La Selva. The Spearman's Rho coefficient is applied as an indicator of correlation between the two variables, generalized to the Plithogenic theory of Statistics.

The paper is divided into a Materials and Methods section, where the basic elements of Plithogenic Statistics are explained. The details and results of the research are then presented in the next section. Conclusions are given at the end.

2 Materials and Methods

The Plithogenic Probability of an event occurring is composed of the probabilities of the event occurring for all the random variables or parameters that constitute it [6, 19]. The Plithogenic Probability based on the Plithogenic Variation Analysis, is multi-dimensional. It could be said that it is a probability of sub-probabilities, where each sub-probability refers to the behavior of a variable by assuming that the event is produced by one or more variables. Each variable is represented by a Probability Distribution (Density) Function (PDF).

According to the classification of F. Smarandache, the subclasses of Plithogenic Probability are as follows:

(1) Classical MultiVariate Probability: If all PDFs are classical.

(2) Plithogenic Neutrosophic Probability is defined when the PDF is expressed in the form of $(\mathrm{T}, \mathrm{I}, \mathrm{F})$, where T is the probability that the event occurs, I is the probability of indeterminacy that the event occurs and F is the probability that the event does not occur. Such that the following is fulfilled: $\mathrm{T}, \mathrm{I}, \mathrm{F} \in [0,1]$, $0 \leq \mathrm{T} + \mathrm{I} + \mathrm{F} \leq 3$.

(3) Plithogenic Indeterminate Probability: When all PDFs have indeterminate data or arguments.

(4) Plithogenic Intuitionistic Fuzzy Probability: When PDFs have the form (T, F) where $\mathrm{T}, \mathrm{F} \in [0,1]$, $0 \leq \mathrm{T} + \mathrm{F} \leq 1$.

(5) Plithogenic Picture Fuzzy Probability: When PDFs have the form $(\mathrm{T}, \mathrm{N}, \mathrm{F})$. $\mathrm{T}, \mathrm{N}, \mathrm{F} \in [0,1]$, $0 \leq \mathrm{T} + \mathrm{N} + \mathrm{F} \leq 1$; where T is the probability that the event occurs, N is the neutral probability that the event occurs or not, and F is the probability that the event does not occur.

(6) Plithogenic Spherical Fuzzy Probability: When PDFs have the form $(\mathrm{T}, \mathrm{H}, \mathrm{F})$. $\mathrm{T}, \mathrm{H}, \mathrm{F} \in [0,1]$, $0 \leq \mathrm{T}^2 + \mathrm{H}^2 + \mathrm{F}^2 \leq 1$; where T is the probability that the event occurs, H is the neutral probability that it occurs or not, and F is the probability that the event does not occur.

(7) Plithogenic (fuzzy-extension) Probability: when we have that all PDFs are in the form of (fuzzy-extension set) style.

(8) Plithogenic Hybrid Probability: When some PDFs are in one of the above styles and others are in other styles.

Plithogenic Statistics (PS) comprises the analysis and observations of the events studied by the Plithogenic Probability.

Plithogenic Statistics generalizes the classical Multivariate Statistics, which in turn allows an analysis of many output variables that are neutrosophic or indeterminate. It is also a multi-indeterminate statistic.

Various subclasses of Plithogenic Statistics are the following:

- Multivariate Statistics,
- Plithogenic Neutrosophic Statistics,
- Plithogenic Indeterminate Statistics,

- Plithogenic Intuitionistic Fuzzy Statistics,
- Plithogenic Picture Fuzzy Statistics,
- Plithogenic Spherical Fuzzy Statistics,
- and in general: Plithogenic (fuzzy-extension) Statistics,
- and Plithogenic Hybrid Statistics.

On the other hand, Plithogenic Refined Statistics are the most general form of statistics that studies the analysis and observations of the events described by the Plithogenic Refined Probability.

In classical inference statistics estimates the population's average of the variable from the sample's average.

When we have a classical random variable, the exact sample size is known and all elements of the sample belong 100% to the population. However, this does not reflect the dynamics of a population such as a student population, which is the example illustrated by F. Smarandache, where there is fluctuation of students within courses, in addition to the fact that the membership of each student varies depending on whether he or she is studying a course full-time, part-time or over -time [21-23].

In a Neutrosophic Population, each element has a triple probability of membership such that $0 \leq T_j + I_j + F_j \leq 3$.

If we assume we have the dataset (T_j, I_j, F_j) for $j = 1, 2, \dots, n$, where n is the sample size, then the average probability for all data in the sample is calculated by Equation 1.

$$\frac{1}{n}\sum_{j=1}^{n}(T_j, I_j, F_j) = \left(\frac{\sum_{j=1}^{n} T_j}{n}, \frac{\sum_{j=1}^{n} I_j}{n}, \frac{\sum_{j=1}^{n} F_j}{n}\right) \quad (1)$$

3 Results

In this section, we present the results obtained in our research. To do so, we rely on the methods used in [19].

Thirty administrative workers from the group of offices dependent on the Academic Vice-Rectorate of the National Agrarian University of La Selva participated in the collection of information to obtain salient characteristics that integrate the set of findings obtained in the study that are presented in the following tables.

Gender	Frequency	Percentage
Female	12	40%
Male	18	60%
Total	30	100%

Table 1. Frequency by gender of the individuals studied.

Age	Frequency	Percentage
20–30	4	13%
31–40	2	7%
41–50	6	20%
51–60	6	20%
61–70	12	40%
Total	30	100%

Table 2. Frequency by age of the individuals studied.

Educational level	Frequency	Percentage
Primary	0	0%
Secondary	12	40%
Non-university higher education	4	13%
University graduate	14	47%
Total	30	100%

Table 3. Frequency by the education level of the individuals studied.

Years in the institution	Frequency	Percentage
0-1	4	13%
>1-5	2	7%
>5-10	1	3%
>10-15	2	7%
>15-20	2	7%
>20-30	6	20%
>30-40	11	37%
>40	2	7%
Total	30	100%

Table 4. Frequency by time of service in the institution of the individuals studied.

Years in the office	Frequency	Percentage
0-1	8	27%
>1-5	8	27%
>5-10	2	7%
>10-15	2	7%
>15-20	4	13%
>20-30	5	17%
>30-40	1	3%
Total	30	100%

Table 5. Frequency by time of service in the same office of the individuals studied.

Tables 6 and 7 summarize the variables to be studied, their dimensions, and definitions.

VARIABLE	CONCEPT	DIMENSIONS	INDICATORS
The behavior of work teams	It is the study of what people do in organizations and how their behavior impacts its dynamics, focusing specifically on the behavior of the organization's groups and teams.	1. Organization	• Degree of internalization of the objectives of team organizations. • Degree of internalization of the applicable regulations. • Perception on structure organizational.
		2. Communication	• Degree of predisposition to assessment of results. • Communication frequency is open and spontaneous.
		3. Trust	• Degree of trust to carry out the work. • Perception about the disposition of mutual support. • Perception about an environment of friendship, and camaraderie.
		4. Interdependence	• Perception of interdependence in the realization of activities.
		5. Belonging	• Perception of commitment to the job. • Perception about identity with the office.
		6. Democratic leadership	• Perception about the joint decision-making. • Perception on acceptance of points of view.

Table 6. Variable to be studied "Work team behavior" (WTB), its dimensions and definitions.

VARIABLE	CONCEPT	DIMENSIONS	INDICATORS
Knowledge Management	Knowledge management is the organizational activity of creating the environment social and infrastructure so that knowledge can be accessed, shared, and created.	1. Identification	Degree of Identification of Knowledge.
		2. Generation	Degree of Generation of Knowledge.
		3. Transfer	Degree of Transfer of Knowledge.
		4. Storage	Degree of Knowledge storage.
		5. Use	Degree of use of knowledge.
		6. Measurement	Degree of Measuring knowledge.

Table 7. Variable to be studied "Knowledge management" (KM), its dimensions and definitions.

The study of the WTB variable and its respective dimensions was carried out by analyzing the opinions of workers in the everyday work context. The results obtained are presented in this section.

Classification	Frequency	Percentage
Group	14	47%
Team	16	53%
Total	30	100%

Table 8. Results of the classification of the behavior according to the individuals studied.

Classification	Frequency	Percentage
Inadequate	19	63%
Adequate	11	37%
Total	30	100%

Table 9. Description of the dimension organization according to the individuals studied.

Classification	Frequency	Percentage
Ineffective	12	40%
Effective	18	60%
Total	30	100%

Table 10. Description of the dimension communication according to the individuals studied.

Classification	Frequency	Percentage
Low	10	33%
High	20	67%
Total	30	100

Table 11. Description of the dimension of trust according to the individuals studied.

Classification	Frequency	Percentage
Weak	4	13%
Strong	26	87%
Total	30	100%

Table 12. Description of the dimension interdependence according to the individuals studied.

Classification	Frequency	Percentage
Low	11	37%
High	19	63%
Total	30	100%

Table 13. Description of the dimension belonging according to the individuals studied.

Classification	Frequency	Percentage
Authoritarian	14	47%
Democratic	16	53%
Total	30	100%

Table 14. Description of the dimension of democratic leadership according to the individuals studied.

The study of the KM variable and its dimensions was also carried out by analyzing the opinions of workers regarding the daily work environment. The results obtained are presented in the following.

Classification	Frequency	Percentage
Bad	0	0%
Deficient	5	17%
Acceptable	13	43%
Good	12	40%
Total	30	100%

Table 15. Classification of knowledge management according to the individuals studied.

Classification	Frequency	Percentage
Bad	3	10%
Deficient	2	7%
Acceptable	2	7%
Good	23	77%
Total	30	100%

Table 16. Identification dimension according to the individuals studied.

Classification	Frequency	Percentage
Bad	1	3%
Deficient	7	23%
Acceptable	11	37%
Good	11	37%
Total	30	100%

Table 17. Generation dimension according to the individuals studied.

Classification	Frequency	Percentage
Bad	1	3%
Deficient	0	0%
Acceptable	1	3%
Good	28	93%
Total	30	100%

Table 18. Transfer dimension according to the individuals studied.

Classification	Frequency	Percentage
Bad	0	0%
Deficient	7	23%
Acceptable	8	27%
Good	15	50%
Total	30	100%

Table 19. Storage dimension according to the individuals studied.

Classification	Frequency	Percentage
Bad	0	0%
Deficient	2	7%
Acceptable	2	7%
Good	26	87%
Total	30	100%

Table 20. Use dimension according to the individuals studied.

Classification	Frequency	Percentage
Bad	9	30%
Deficient	6	20%
Acceptable	1	3%
Good	14	47%
Total	30	100%

Table 21. Measurement dimension according to the individuals studied.

Each of the scales in Tables 9-21 was conveniently converted to the scale shown in Table 22.

Linguistic Value	Associated Single-Valued Neutrosophic Number
Unfavorable	(0.1,0.1,0.8)
Moderately Favorable	(0.55,0.1,0.35)
Favorable	(0.8,0.1,0.1)

Table 22. The linguistic scale and the Single-Valued Neutrosophic Numbers were used in the survey of the study carried out. Source: [19].

From the conversion, the Plithogenic Neutrosophic Probabilities were determined for each of the variables based on their dimensions using the results from the previous Tables.

These probabilities are as follows:

If x is a member of the office, we have the Plithogenic Neutrosophic Probability as follows:

$PNP_{WTB}(x) = \begin{pmatrix}(0.359, 0.1, 0.541); (0.52, 0.1, 0.38); (0.569, 0.1, 0.331); (0.709, 0.1, 0.191);\\ (0.541, 0.1, 0.359); (0.471, 0.1, 0.429)\end{pmatrix}$ is the favorable Plithogenic Probability Probability about Work Team Behavior,

$PNP_{KM}(x) = \begin{pmatrix}(0.666, 0.1, 0.234); (0.5255, 0.1, 0.3745); (0.7635, 0.099, 0.1275); (0.5715, 0.1, 0.3285);\\ (0.7415, 0.101, 0.1675); (0.4425, 0.1, 0.4575)\end{pmatrix}$

is the favorable Plithogenic Probability Probability about Knowledge Management.

Now, applying the n-norm as in [19]:

$$N(x,y) = \left(\min\left(T_x, T_y\right), \max\left(I_x, I_y\right), \max\left(F_x, F_y\right)\right) \qquad (2)$$

The aggregated WTB is $(0.359, 0.1, 0.541)$ and the aggregated KM is $(0.4425, 0.101, 0.4575)$.

In addition, the study of the correlation between both variables was as follows:

H_0: There is a correlation between the behavior of work teams and knowledge management in offices dependent on the Academic Vice-Rectorate of UNAS,

H_1: There is no correlation between the behavior of work teams and knowledge management in offices dependent on the Academic Vice-Rectorate of UNAS.

Spearman's Rho test is as follows:

Spearman's Rho test			
Variables	**Correlation Coef.**	**Sig.**	**N**
Work Team Behavior and Knowledge Management			
	0.845	0.000	30

Table 23. Application table of the Spearman's correlation coefficient.

The level of correlation between the behavior of work teams and knowledge management in the offices dependent on the Academic Vice-Rectorate of UNAS is 0.84, which indicates that it is a considerable positive correlation

The Plithogenic Probability framework not only enhances traditional probabilistic modeling by incorporating neutrosophic logic, but it also enables a more refined approach to uncertainty, contradiction, and indeterminacy in statistical analyses. The ability to account for multi-dimensional sub-probabilities makes it particularly suitable for complex decision-making environments where classical probabilistic models may fall short. Recent studies have demonstrated the effectiveness of neutrosophic-based statistical methods in diverse applications, including group dynamics and teamwork analysis, as reported by Guevara et al. [24] . Their study highlights how neutrosophic probability distributions provide a better representation of human decision-making, particularly when dealing with subjective assessments that involve varying degrees of truth, falsity, and indeterminacy. Similarly, Salama et al. [25] explored the role of neutrosophic modeling in evaluating digital transformation, emphasizing that hybrid probability models offer a more adaptive and flexible approach to measuring performance in organizations where traditional metrics struggle to capture the full spectrum of uncertainty.

Furthermore, the integration of machine learning with Plithogenic Statistics has shown promising results in predictive analytics, particularly in fields such as healthcare and risk assessment. Elhawy [26] demonstrated how statistical analysis combined with machine learning optimization improves heart attack prediction, showcasing how Plithogenic Probability allows for a more comprehensive representation of medical uncertainties in patient data. This synergy between AI-driven statistical inference and Plithogenic models opens new avenues for highly adaptive data-driven decision-making, reinforcing the need for context-aware probabilistic frameworks in research domains characterized by high variability and incomplete information. The present study, by applying Plithogenic Neutrosophic Probability to workplace dynamics, further confirms the potential of these models in organizational research, paving the way for a broader adoption of neutrosophic-based methodologies across multiple disciplines.

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Conclusion

This article was devoted to the statistical study of the correlation between two variables, "Work Team Behavior" and "Knowledge Management". The study was carried out in offices dependent on the Academic Vice-Rectorate of the National Agrarian University of La Selva, Peru. To achieve greater accuracy, a calculation was performed with the help of Neutrosophic Plithogenic Statistics. To do this, a survey was applied to 30 workers in these offices. Each of the variables above is measured by six dimensions, which are reflected in the questions in the questionnaire. The results show that there are differences between each of the dimensions. When these are aggregated together, values below Moderately Favorable are reached for both, without becoming Unfavorable. On the other hand, Spearman's Rho Coefficient is used to show that there is a notable positive correlation between the two variables. That is, the improvement of one of them will inexorably bring an improvement in the other.

The use of Plithogenic Statistics allows probabilistic calculations to be made in a multidimensional problem, where there are components of different origins. In addition, it allows the uncertainty and indeterminacy that are part of this type of phenomenon to be taken into account.

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Design Performance Measurement of Intelligent Garbage Sorting Cart using Neutrosophic Sets and SuperHyperSoft

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Abstract: With the rapid increase in urban waste generation, efficient and automated waste sorting solutions have become essential for sustainable waste management. This study evaluates the gent Garbage Sorting Cart under the decision-making methodology. We use the Quadripartitioned Single-Valued Neutrosophic Z-Numbers (QSVNZN) with four membership functions to dela with the uncertainty information. Two methods are used in this study, such as the CRITIC method to compute the criteria weights and the ARAS method to rank the alternatives. The decision-making process is used with the SuperHyperSoft set to deal with different criteria and alternatives. We used eight criteria and six alternatives. We proposed six collections of SuperHyperSoft. In each collection, we applied the ARAS methodology and ranked the alternatives. The results show stability in the ranks.

Keywords: Quadripartitioned Single-Valued Neutrosophic Z-Numbers (QSVNZN); SuperHyperSoft Set; Intelligent Garbage Sorting Cart; Intelligent Garbage Sorting Cart.

1. Introduction

With the rapid increase in urban waste generation, efficient and automated waste sorting solutions have become essential for sustainable waste management. Traditional waste segregation methods are labor-intensive and prone to human error, leading to improper disposal and environmental hazards. The introduction of intelligent garbage sorting carts, integrated with advanced artificial intelligence (AI), sensors, and autonomous navigation, has revolutionized waste management by enhancing sorting accuracy and efficiency[1], [2]. However, to ensure optimal performance and practical implementation, a systematic evaluation of the design performance of these smart carts is necessary, considering multiple functional, technical, and economic criteria.

The design performance evaluation of intelligent garbage sorting carts requires a multi-criteria assessment that considers various aspects such as sorting accuracy, sensor efficiency, energy

consumption, mobility, durability, and cost-effectiveness. These carts utilize AI-driven image recognition and sensor-based classification techniques to automatically identify and separate waste into categories like organic, recyclable, and hazardous waste[3], [4]. However, differences in design, technology integration, and operational efficiency necessitate a structured approach to compare and rank various models. A well-defined evaluation framework helps manufacturers, policymakers, and urban planners select the most effective designs for large-scale deployment.

To achieve an objective assessment, Multi-Criteria Decision-Making (MCDM) methodologies such as CRITIC, and ARAS can be applied to systematically evaluate and rank different intelligent garbage sorting cart models. These methods enable decision-makers to analyze multiple criteria simultaneously, balancing technical efficiency, cost, maintenance requirements, and user-friendliness. By leveraging MCDM techniques, stakeholders can ensure that the selected garbage sorting carts maximize waste segregation efficiency while maintaining affordability and sustainability in various operational settings, including residential areas, commercial spaces, and industrial waste management facilities.

The successful implementation of intelligent garbage sorting carts depends on a combination of technological advancements, design optimization, and real-world testing. A robust evaluation framework ensures that these smart carts meet the required performance standards while addressing practical concerns such as power consumption, maintenance costs, and ease of use. As cities move toward smart and sustainable waste management systems, evaluating the design performance of these carts is crucial to promoting efficient, eco-friendly, and automated waste sorting solutions[5]. Through continuous innovation and systematic evaluation, intelligent garbage sorting carts can significantly contribute to a cleaner and more sustainable urban environment.

This flaw in IFSs was fixed in 2005 when Smarandache proposed the idea of neutrosophic sets (NSs), which defined the indeterminacy degree as an independent component[6]. The study of neutralities, or "neutrosophy," is the field of philosophy that gave rise to Smarandache's NSs. All three components of NS theory derive their values from the non-standard unit interval, and each element is distinguished by a truth membership degree, an indeterminacy membership degree, and a falsity membership degree[7], [8]. However, NSs' actual technical, scientific, and practical uses were limited by their philosophical stance.

After that, Wang et al. [9] introduced the idea of SVNSs, which has no restrictions on the values that its constituent parts can adopt from the total of the truth, indeterminacy, and falsity degrees components falls between 0 and 3. There is still uncertainty even if the indeterminacy component in SVNSs is independent and more generic than the hesitation parameter in IFSs. It is uncertain exactly if the indeterminacy degree refers to an element's belongingness or non-belongingness. Due to this conflict of interest, Chatterjee et al. [10] developed the idea of QSVNSs, in which the indeterminacy component is further divided into "ignorance" (neither nor F) and "contradiction" (both I and F).

According to decision making theory, a model or framework that can handle the most ambiguous information effectively and ultimately provides the outcome is preferred. The descriptive power of QSVNSs can be further improved by assigning a reliability measure to each of the quadripartitioned single-valued neutrosophic components of truth, contradiction, ignorance, and falsity. QSVNSs are already an effective mathematical tool for handling imprecise or uncertain information. It may be possible to apply Zadeh's[11] Z-numbers notion in this context.

To put it simply, a Z-number is an ordered pair with fuzzy numbers as its constituents. While the second component provides an amount of assurance or dependability of information corresponding to the first component, the first component explains the limitations placed on the values that are assumed by an uncertain variable. In decision-making processes, accurate information is essential because those who rely on ambiguous or imprecise information risk financial losses, resource waste, lost time, and other consequences. Therefore, a strong instrument such as the Z-number may be used to both collect and convey the dependability of cognitive information.

According to Zadeh, the Z-number is an ordered pair of fuzzy numbers that may represent the certainty and dependability of any given piece of information. Similarly, Quadripartitioned Single-Valued Neutrosophic Sets (QSVNSs), which reflect ambiguous knowledge or facts at hand, are defined by four independent components of truth, contradiction, ignorance, and falsity degrees, based on Belnap's four-valued logic. QSVNSs are extensions of SVNSs, which SVNSs, with the indeterminacy component further divided into ignorance and contradiction. Nevertheless, QSVNSs by themselves are unable to capture the dependability metric of decision makers' allocations or preferences.

Therefore, Borah and Dutta [12] proposed a hybrid framework for the first time to improve uncertainty modeling and to integrate the knowledge of truth, contradiction, ignorance, and falsity degrees with their respective reliability qualities. Therefore, as a generalization of the Z-number and the QSVNSs, they suggested the idea of Quadripartitioned Single-Valued Neutrosophic Z-number (QSVNZN). To rank QSVNZNs, scoring functions and a few fundamental operations are also defined.

Three instances of MCDM, particularly in the context of COVID-19, are used to illustrate the applicability of our recently suggested operators and the score function. To illustrate the legitimacy, authenticity, and viability of our theoretical construction, careful comparison analysis, sensitivity analysis, runtime analysis, and the validity of our suggested methodologies are also conducted. The outcomes will be very helpful to decision-makers in handling uncertain and conflicting data effectively. Stated differently, the new framework must have adequate descriptive power from a human-cognition standpoint.

2. Basic Preliminaries

This section shows some basic preliminaries on quadripartitioned single-valued neutrosophic Z-numbers (QSVNZN)[12].

Definition 1

The QSVN can be defined by four components such as truth, $T_Q(x)$, contradiction $C_Q(x)$, ignorance $U_Q(x)$, and falsity functions $F_Q(x)$

$$0 \le T_Q(x) + C_Q(x) + U_Q(x) + F_Q(x) \le 4 \tag{1}$$

Definition 2

We can define the QSVNZN as:

$$Q_z = T_Q(R,C)(x), C_Q(R,C)(x), U_Q(R,C)(x), F_Q(R,C)(x) \tag{2}$$

each pair in the Q_z can be defined as:

$$T_Q(R,C)(x) = \left(T_Q^R(x), T_Q^C(x)\right) \tag{3}$$

$$C_Q(R,C)(x) = \left(C_Q^R(x), C_Q^C(x)\right) \tag{4}$$

$$U_Q(R,C)(x) = \left(U_Q^R(x), U_Q^C(x)\right) \tag{5}$$

$$F_Q(R,C)(x) = \left(F_Q^R(x), F_Q^C(x)\right) \tag{6}$$

$$0 \le T_Q^R(x) + C_Q^R(x) + U_Q^R(x) + F_Q^R(x) \le 4 \tag{7}$$

$$0 \le T_Q^C(x) + C_Q^C(x) + U_Q^C(x) + F_Q^C(x) \le 4 \tag{8}$$

Definition 3

Some operations of QSVNZN can be defined as:

$$Q_z^c = \left(F_Q(R,C)(x), U_Q(R,C)(x), C_Q(R,C)(x), T_Q(R,C)(x)\right) \tag{9}$$

$$Q_{Z_1} \cup Q_{Z_2} = \begin{pmatrix} \left(T_Q^{R1} \vee T_Q^{R2}, T_Q^{C1} \vee T_Q^{C2}\right), \\ \left(C_Q^{R1} \vee C_Q^{R2}, C_Q^{C1} \vee C_Q^{C2}\right), \\ \left(U_Q^{R1} \wedge U_Q^{R2}, U_Q^{C1} \wedge U_Q^{C2}\right), \\ \left(F_Q^{R1} \wedge F_Q^{R2}, F_Q^{C1} \wedge F_Q^{C2}\right) \end{pmatrix} \tag{10}$$

$$Q_{Z_1} \cap Q_{Z_2} = \begin{pmatrix} \left(T_Q^{R1} \wedge T_Q^{R2}, T_Q^{C1} \wedge T_Q^{C2}\right), \\ \left(C_Q^{R1} \wedge C_Q^{R2}, C_Q^{C1} \wedge C_Q^{C2}\right), \\ \left(U_Q^{R1} \vee U_Q^{R2}, U_Q^{C1} \vee U_Q^{C2}\right), \\ \left(F_Q^{R1} \vee F_Q^{R2}, F_Q^{C1} \vee F_Q^{C2}\right) \end{pmatrix} \tag{11}$$

$$Q_{Z_1} \oplus Q_{Z_2} = \begin{pmatrix} \left(T_Q^{R1} + T_Q^{R2} - T_Q^{R1}T_Q^{R2}, T_Q^{C1} + T_Q^{C2} - T_Q^{C1}T_Q^{C2}\right), \\ \left(C_Q^{R1} + C_Q^{R2} - C_Q^{R1}C_Q^{R2}, C_Q^{R1} + C_Q^{R2} - C_Q^{R1}C_Q^{R2}\right), \\ \left(U_Q^{R1}U_Q^{R2}, U_Q^{C1}U_Q^{C2}\right), \\ \left(F_Q^{R1}F_Q^{R2}, F_Q^{C1}F_Q^{C2}\right) \end{pmatrix} \tag{12}$$

$$Q_{Z_1} \otimes Q_{Z_2} = \begin{pmatrix} \left(T_Q^{R1}T_Q^{R2}, T_Q^{C1}T_Q^{C2}\right), \\ \left(C_Q^{R1}C_Q^{R2}, C_Q^{C1}C_Q^{C2}\right), \\ \left(U_Q^{R1} + U_Q^{R2} - U_Q^{R1}U_Q^{R2}, U_Q^{C1} + U_Q^{C2} - U_Q^{C1}U_Q^{C2}\right), \\ \left(F_Q^{R1} + F_Q^{R2} - F_Q^{R1}F_Q^{R2}, F_Q^{C1} + F_Q^{C2} - F_Q^{C1}F_Q^{C2}\right) \end{pmatrix} \tag{13}$$

3. QSVNZN-CRITIC-ARAS Method

This section presents the steps of the proposed approach. We used the CRITIC method to compute the criteria weights and the ARAS method to rank the alternatives. These methods are used under the QSVNZN environment to deal with uncertainty information.

QSVNZN-CRITIC

We compute the criteria weights by the CRITIC method.

Build the decision matrix.

We used the QSVNZN to evaluate the criteria and alternative and build the decision matrix such as:

$$Q = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{m1} & \cdots & q_{mn} \end{bmatrix}_{m \times n} ; i = 1, \dots, m; j = 1, \dots, n \tag{14}$$

Normalize the decision matrix

The decision matrix is normalized for positive and cost criterion

$$u_{ij} = \frac{q_{ij} - \min q_{ij}}{\max q_{ij} - \min q_{ij}} \tag{15}$$

$$u_{ij} = \frac{q_{ij} - \max q_{ij}}{\min q_{ij} - \max q_{ij}} \tag{16}$$

Obtain the correlation matrix between the criteria O_{jk}

Compute the C index

$$C_j = f_j \sum_{k=1}^{n} \left(1 - O_{jk}\right) \tag{17}$$

Where f_j refers to the standard deviation

Compute the criteria weights

$$W_j = \frac{C_j}{\sum_{j=1}^{n} C_j} \tag{18}$$

QSVNZN-ARAS

We use the ARAS method to rank the alternatives.

Normalize the decision matrix

$$y_{ij} = \frac{q_{ij}}{\sum_{i=0}^{m} q_{ij}} \tag{19}$$

Compute the weighted normalized decision matrix

$$T_{ij} = W_j y_{ij} \tag{20}$$

Compute the optimality function

$$L_i = \sum_{j=1}^{n} T_{ij} \tag{21}$$

Compute the utility degree

$$E_i = \frac{L_i}{\max L_i} \tag{22}$$

SuperHyperSoft (SHS)

SHS is used to deal with the standards and sub standards with different values. It is defined based on the HyperSoft set[13], [14]. Let the universe set $U = \{K_1, K_2, \ldots . K_n\}$. The power set of U is a $P(U)$ and V_1, V_2, V_3 are select as a criteria. $P(V_1) \times P(V_2)$ and $P(V_3)$ are $powersets$ of V_1, V_2, V_3

Let $F: P(V_1) \times P(V_2) \to P(V)$ this called SHS over V.

$$P(V_1) \times P(V_2) \times P(V_3) = \left\{ \begin{matrix} \{V_{11}\}, \{V_{12}\}, \{V_{11}, V_{12}\} \times \\ \{V_{21}\}, \{V_{22}\}, \{V_{21}, V_{22}\} \times \\ \{V_{31}\}, \{V_{32}\}, \{V_{33}\}, \{V_{31}, V_{32}\}, \{V_{31}, V_{33}\}, \\ \{V_{32}, V_{33}\}, \{V_{31}, V_{32}, V_{33}\} \end{matrix} \right\} \tag{23}$$

4. An application

This section shows the results of the proposed approach. Three experts are invited to evaluate the criteria and alternatives. Eight criteria and six alternatives are used in this study to select the best one based on a set of criteria.

The criteria and sub values are:

- ✓ User-Friendliness {Complex, Moderate, Simple}
- ✓ Maintenance Requirements {Frequent, Moderate, Minimal}
- ✓ Energy Consumption {High, Moderate, Low}
- ✓ Mobility & Navigation {Limited, Advanced}
- ✓ Sensor Efficiency {Inefficient, Moderate, Efficient}
- ✓ Cost-Effectiveness {Expensive, Moderate, Affordable}
- ✓ Durability & Build Quality {Weak, Moderate, Strong}
- ✓ Sorting Accuracy {Low, Moderate, High}

The alternatives of this study are:

- ✓ Basic AI-Sensor Cart
- ✓ Solar-Powered Smart Cart
- ✓ Autonomous Navigation Cart
- ✓ Heavy-Duty Industrial Cart
- ✓ Compact Household Cart
- ✓ AI-Enhanced Precision Cart

We used the QSVNZN to build the decision matrix as shown in Tables 1-4.

Then we normalize the decision matrix using Eq. (15) as shown in Table 5

Then we obtain the correlation matrix between the criteria O_{jk}.

Then we compute the C index using Eq. (17)

Then we compute the criteria weights using Eq. (18). The criteria weights can be represented as: C1= 0.114925131, C2= 0.102561851, C3= 0.13194509, C4= 0.108780779, C5= 0.165034631, C6= 0.089873444, C7= 0.120758257, C8= 0.166120817

Table 1. The first values of QSVNZN.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))
A_2	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))
A_3	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))
A_4	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))
A_5	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))
A_6	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))

Table 2. The second values of QSVNZN.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))
A_2	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))
A_3	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))
A_4	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))
A_5	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))
A_6	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))

Table 3. The third values of QSVNZN.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))
A_2	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))
A_3	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))

A_4	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))
A_5	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))
A_6	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))

Table 4. The fourth values of QSVNZN.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))
A_2	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))
A_3	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))
A_4	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))
A_5	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.5,0.6),(0.6,0.7),(0.4,0.5),(0.2,0.1))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.7,0.7),(0.6,0.7),(0.4,0.5),(0.3,0.2))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.3,0.2),(0.2,0.3),(0.6,0.5),(0.7,0.9))
A_6	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.8,0.9),(0.8,0.8),(0.2,0.3),(0.3,0.4))	((0.1,0.2),(0.2,0.3),(0.5,0.7),(0.6,0.8))	((0.1,0.1),(0.2,0.2),(0.7,0.9),(0.8,0.9))	((0.5,0.5),(0.6,0.4),(0.3,0.4),(0.3,0.2))

Table 5. The normalized values of QSVNZN by CRITIC.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	1	0.664615	0.853293	0	0	0.445367	0	1
A_2	0.336449	0.146154	0.847305	0.30491	0.052061	0.697095	1	0
A_3	0.623832	1	0	0.397933	0.78308	0.780083	0.713392	0.284211
A_4	0.614486	0.618462	1	0.742894	0.893709	1	0.732165	0.64386
A_5	0.158879	0.563077	0.663174	1	0.958785	0.387275	0.684606	0.231579
A_6	0	0	0.158683	0.102067	1	0	0.440551	0.9

QSVNZN-ARAS

This study uses the SuperHyperSoft set to treat various criteria and values. This study divided the values of criteria into several HyperSoft sets. Then we ranked the alternatives based on these divisions. The values of criteria are selected as: {Simple}, {Moderate}, {High}, {Limited, Advanced}, Inefficient, Moderate, Efficient}, {Affordable}, {Strong}, {High}

- **Collection 1:** {Simple}, {Moderate}, {High}, {Limited}, {Inefficient}, {Affordable}, {Strong}, {High}
- **Collection 2:** {Simple}, {Moderate}, {High}, {Limited}, {Moderate}, {Affordable}, {Strong}, {High}
- **Collection 3:** {Simple}, {Moderate}, {High}, {Limited}, {Efficient}, {Affordable}, {Strong}, {High}
- **Collection 4:** {Simple}, {Moderate}, {High}, {Advanced}, {Inefficient}, {Affordable}, {Strong}, {High}
- **Collection 5:** {Simple}, {Moderate}, {High}, {Advanced}, {Moderate}, {Affordable}, {Strong}, {High}

- **Collection 6:** {Simple}, {Moderate}, {High}, {Advanced}, {Efficient}, {Affordable}, {Strong}, {High}

Ranking the alternatives based on collection 1

Eq. (19) is used to normalize the decision matrix as shown in Fig 1.

Eq. (20) is used to compute the weighted normalized decision matrix as shown in Fig 2.

Eq. (21) is used to compute the optimality function.

Eq. (22) is used to compute the utility degree.

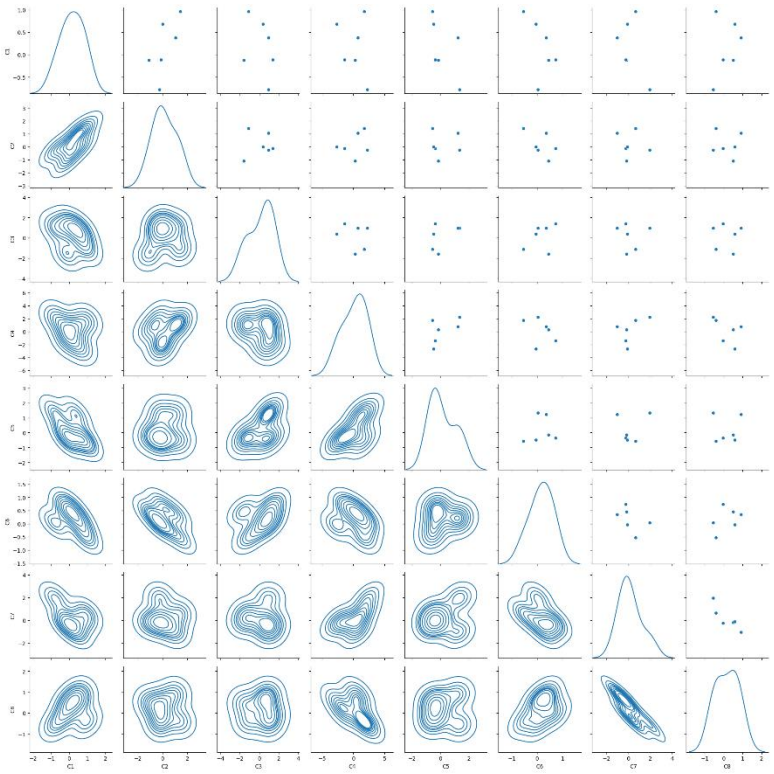

Fig 1. The normalization values.

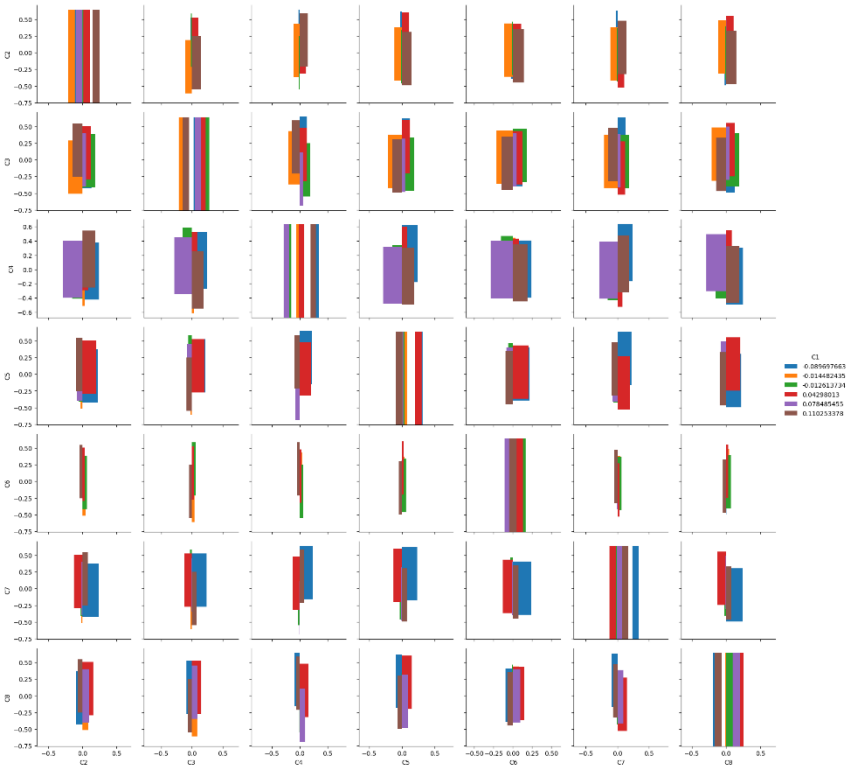

Fig 2. The weighted normalization values.

Ranking the alternatives based on collection 2

We normalize the decision matrix as shown in Fig 3.

We compute the weighted normalized decision matrix as shown in Fig 4.

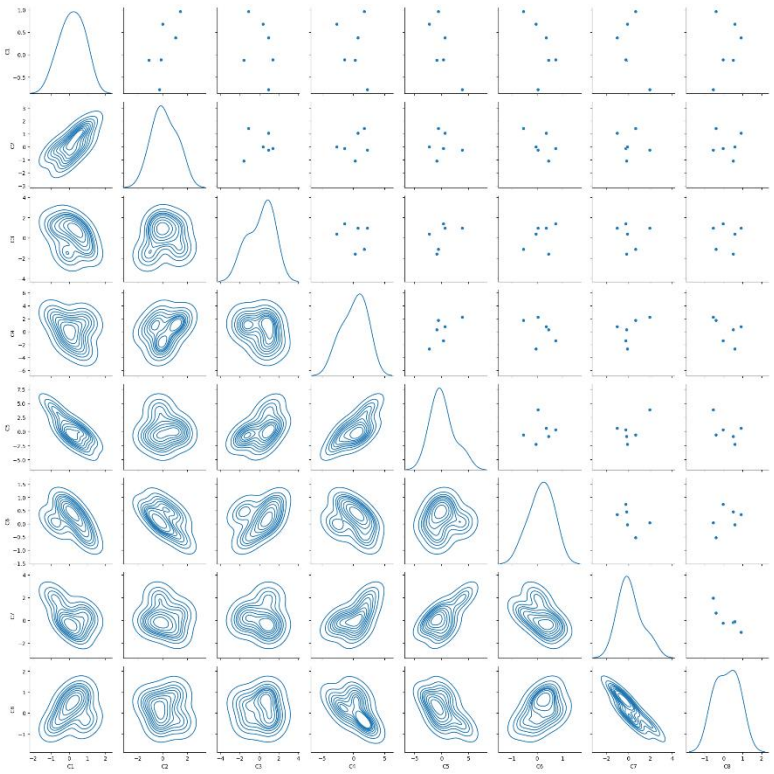

Fig 3. The normalization values.

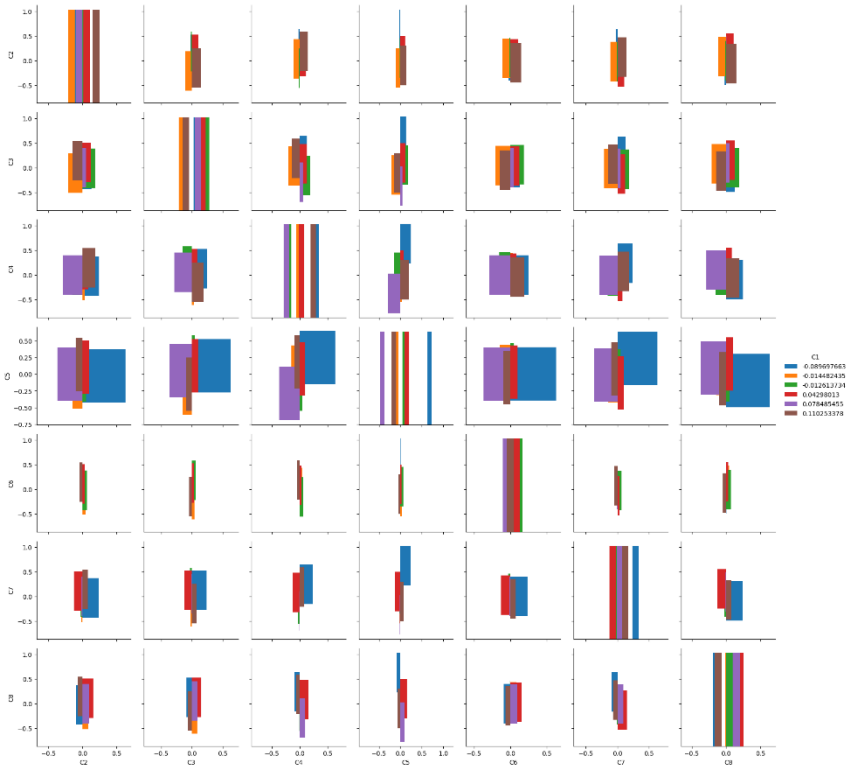

Fig 4. The weighted normalization values.

Ranking the alternatives based on collection 3

We normalize the decision matrix as shown in Fig 5.

We compute the weighted normalized decision matrix as shown in Fig 6.

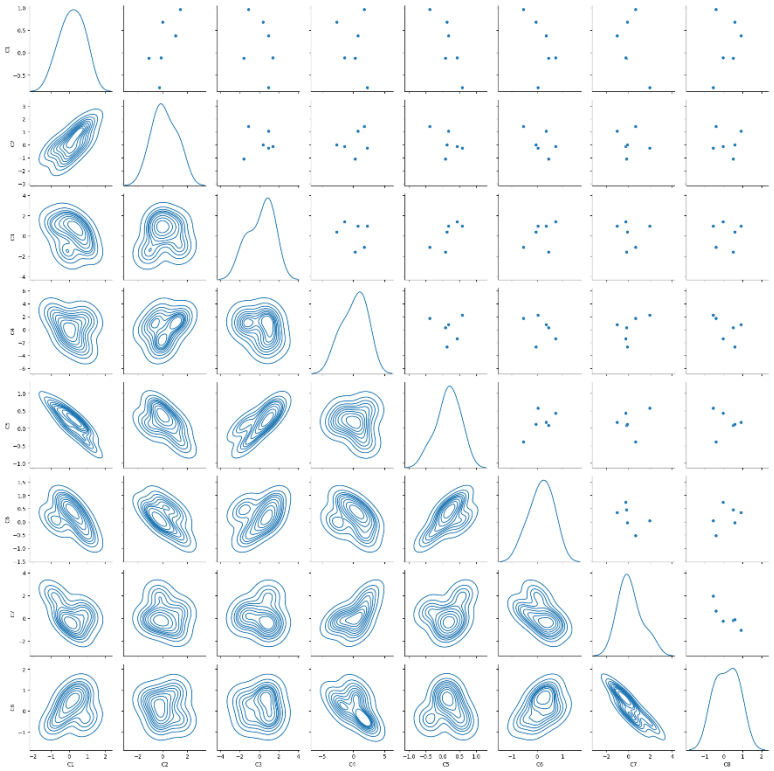

Fig 5. The normalization values.

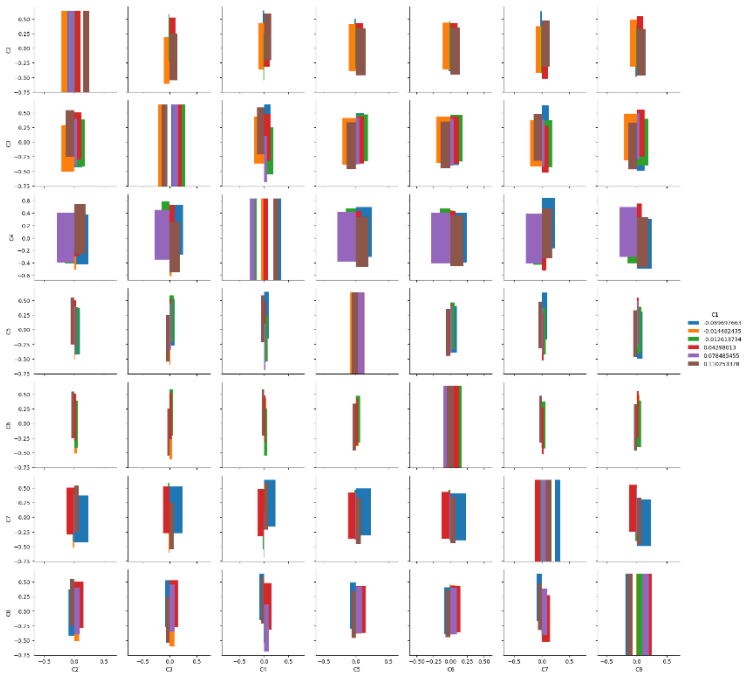

Fig 6. The weighted normalization values.

Ranking the alternatives based on collection 4

We normalize the decision matrix as shown in Fig 7.

We compute the weighted normalized decision matrix as shown in Fig 8.

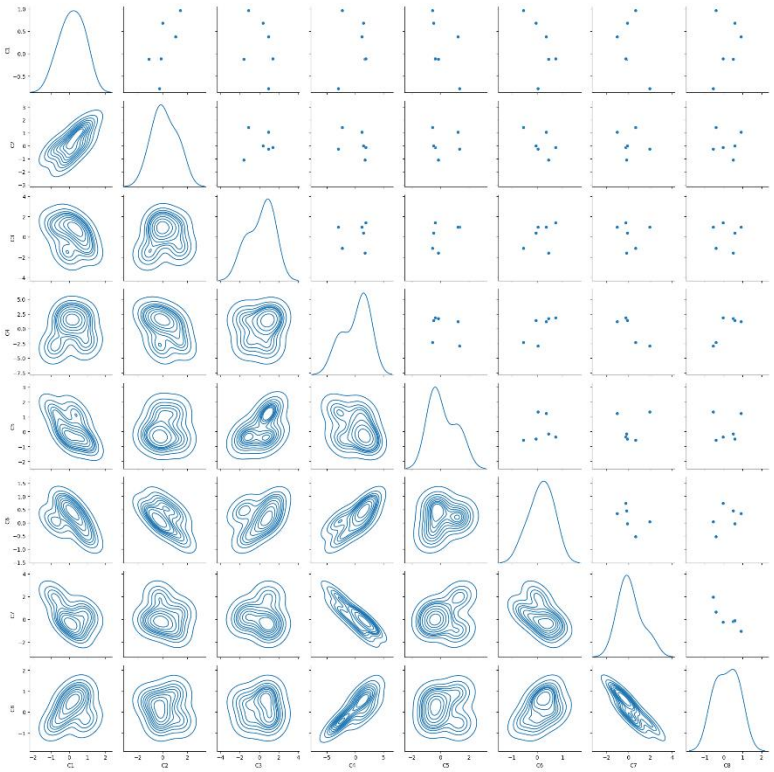

Fig 7 The normalization values.

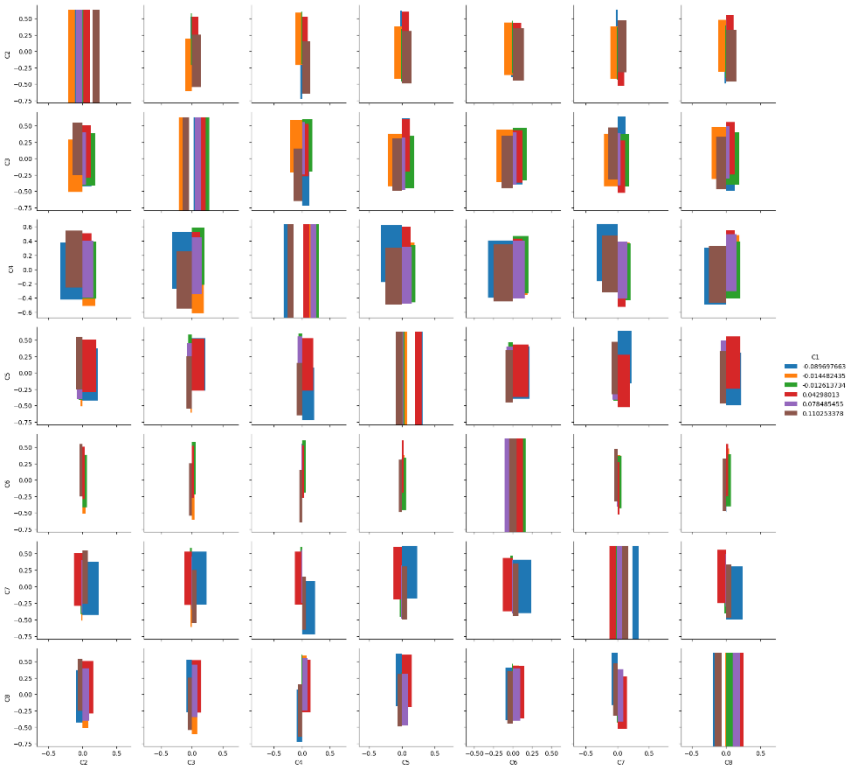

Fig 8. The weighted normalization values.

Ranking the alternatives based on collection 5

We normalize the decision matrix as shown in Fig 9.

We compute the weighted normalized decision matrix as shown in Fig 10.

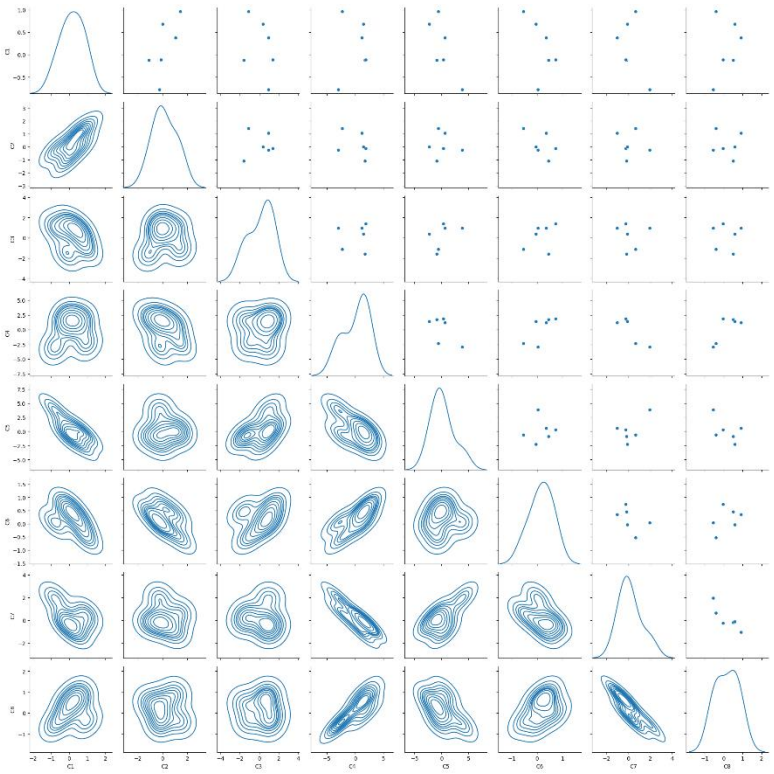

Fig 9. The normalization values.

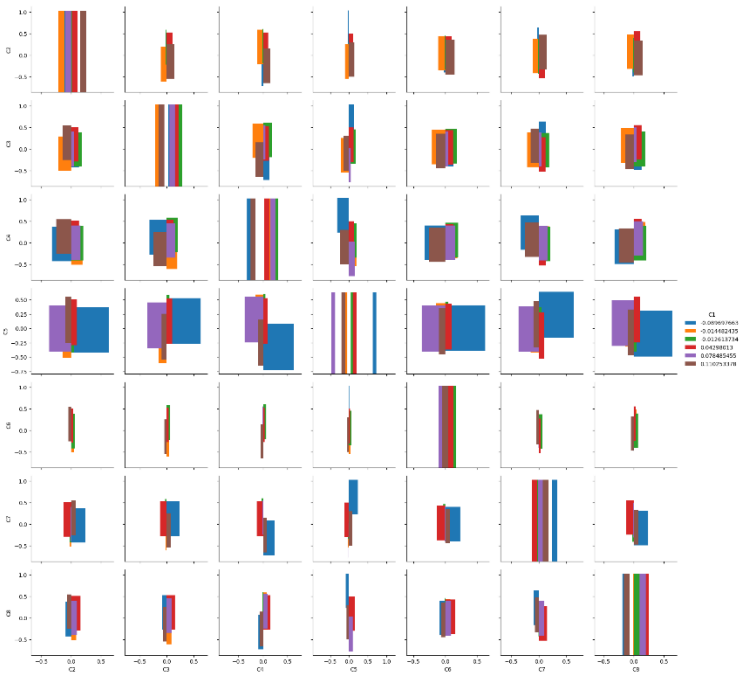

Fig 10. The weighted normalization values.

Ranking the alternatives based on collection 6

We normalize the decision matrix as shown in Fig 11.

We compute the weighted normalized decision matrix as shown in Fig 12.

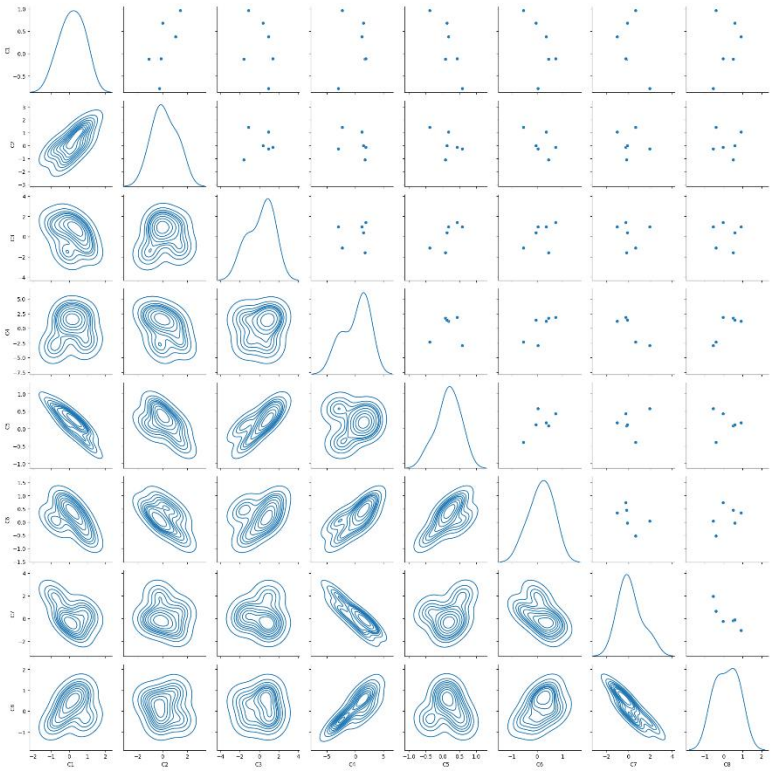

Fig 11. The normalization values.

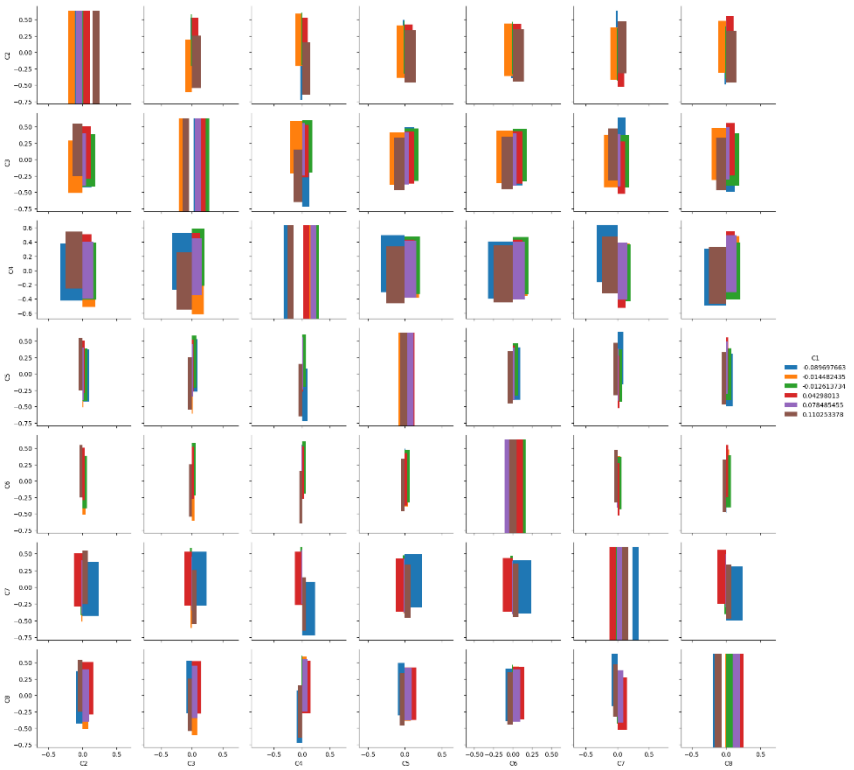

Fig 12. The weighted normalization values.

Finally, we compute the utility degree of each collection as shown in Fig 13. Then we ranked the alternatives in each collection as shown in Fig 14.

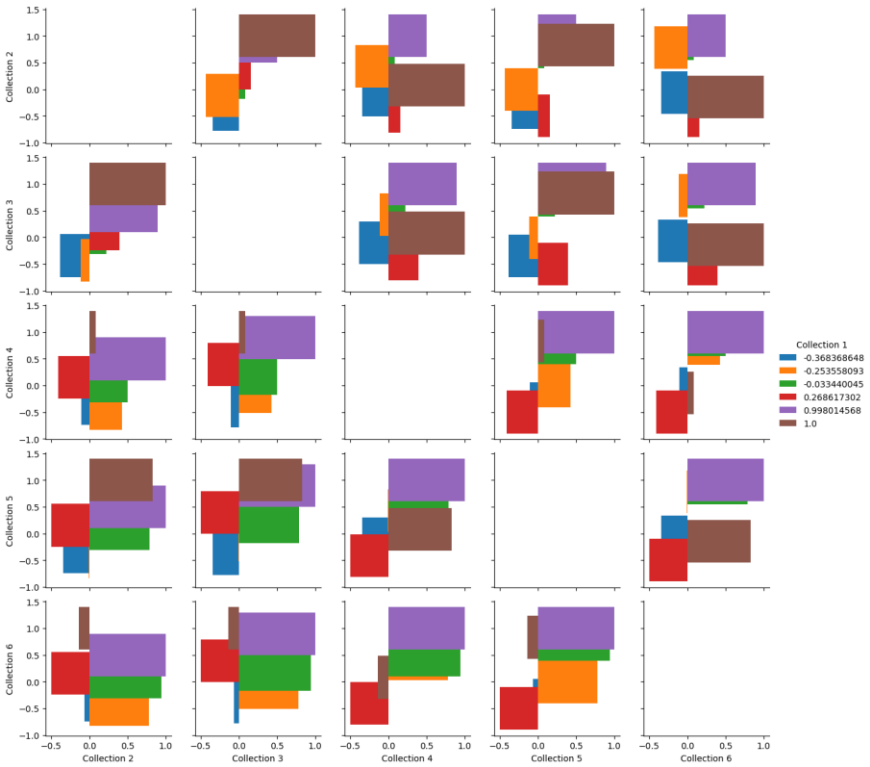

Fig 13. The utility degree of each collection.

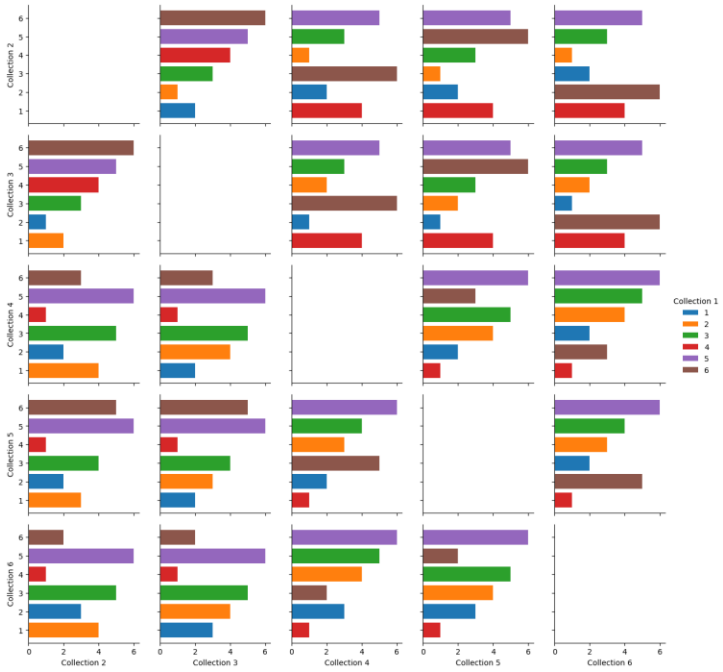

Fig 14. The ranks of each collection.

5. Conclusions

There is uncertainty everywhere. Most real-world issues that are often articulated in natural language are predicated on information that is vague, ambiguous, and/or only partially trustworthy. People naturally make judgments in a variety of settings, from straightforward to intricate. However, the effectiveness of people's decisions may be impacted if the information is unreliable. This paper addresses the current flaws of missing reliability measurements in quadripartitioned neutrosophic information by proposing a hybrid idea of QSVNZN set that conveys cognitive information with its appropriate reliability measures. A more manageable and efficient method of modeling the system and assisting with human decision-making is to use QSVNZNs, which indicate the selection-makers' level of confidence. We integrated the SuperHyperSoft set with the neutrosophic sets to dela with the different criteria and alternatives. we proposed six collections in the SuperHyperSoft set. Then we applied the ARAS method on these collections and then ranked the alternatives. The results show stability in the ranks of the alternatives under the SuperHyperSoft set and neutrosophic sets.

Acknowledgment

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Neutrosophic Wald Distribution with Applications to Reliability Analysis

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Abstract: In this article, we develop neutrosophic extension of the Wald (Inverse Gaussian) distribution to present more realistic modelling for real data by introducing uncertainty in its parameters. We derive fundamental statistical properties such as the probability density function (PDF), cumulative distribution function (CDF) and quantile function, and compare it with the classical model. This comparison shows the versatility and great robustness of the neutrosophic model against the imprecise data. Considering that the Wald distribution plays a significant role in the theory of reliability, we extend some key reliability functions into a neutrosophic framework. Under neutrosophic uncertainty, we derive and study the survival function, the reliability function and the hazard function which results in a more generalized and pragmatic approach for modeling reliability. These functions provide an improved decision-making process for situations in which classical models are unable to capture the inbuilt uncertainties of systems. To make it even more applicable, we propose an approach to generate random samples from neutrosophic Wald distribution using quantile function so that neutrosophic Wald distribution can be simulated and empirically validated. In addition, we also develop an estimation procedure through the method of moments (mom), which shows a simple way of estimating the parameters.

Keywords: Neutrosophic distribution; neutrosophic probability; estimation; simulation

1. Introduction

Statistical distributions are commonly used tools for modeling, analyzing, and predicting real-world phenomena across various domains, but especially in reliability engineering and survival analysis [1]. Reliability theory describes the time-to-failure behavior of systems, components, and machinery through statistical distributions, allowing engineers to assess product lifespan, failure rates, maintenance schedules, and product reliability (i.e. estimates of the probability of survival within a certain time) [2]. The exponential, Weibull, gamma, and inverse Gaussian (Wald)

distributions are commonly used distributions in reliability analysis, each suitable for different failure patterns [3]. The exponential distribution, for example, represents fixed failure rates, and the Weibull, with its parameters, is more flexible to capture failure rates that rise or fall as time progresses [4]. Likewise, the inverse Gaussian (Wald) distribution has been applied extensively to a variety of first-passage time problems, so it is useful for modeling system lifetimes when they are exposed to random shocks [5]. Statistical distributions are pivotal in survival analysis for modeling lifespan data in the fields of medical and biological studies [6]. The Kaplan-Meier estimator, Cox proportional hazards model, and parametric distributions (i.e. log-normal, gamma and inverse Gaussian distributions) are well established methods to estimate survival probabilities and hazard rates [7]. These distributions are used to derive the hazard function, which describes the probability of an event occurring at a specific point in time, playing a key role in medical decision-making, risk assessment, and insurance studies [8]. Additionally, statistical distributions are utilized for quality control, warranty predictions, and risk analysis, all of which are necessary to ensure the product is designed optimally and is inexpensive to maintain over its lifetime in industrial applications. Statistical distributions are widely used in reliability and survival analysis due to their significance in providing some quantitative insights, predicting future outcomes, and helping in optimization of some of the decision-making processes [9]. The Wald distribution is one of the very importance distributions in statistical applications [10]. The Wald distribution (also known as the Inverse Gaussian distribution) is a basic probability model in statistics particularly in first-passage time problems and in reliability problems and provides better model results for stochastic processes [11]. It refers to the amount of time until a certain Brownian motion with positive drift hits a given threshold level and hence is well-suited for use in sequential analysis, financial growth approximations, and similar fields [12]. For similar reasons, the Wald distribution is of interest to reliability engineering, when we wish to model the lifetime of systems and components that are subject to random wear and degradation with time. Commonly used in biostatistics and in the survival analysis, it is used to model waiting or survival time, namely a situation when early failures as well as long-term survival probabilities vary greatly. Two of the most used characteristics of the survival functions are the location and shape parameters. In econometrics and finance, it is used to model log stock prices, default times in credit risk analysis, time-to-event processes, etc. In decision processes, the Wald distribution is instrumental; specifically, in sequential probability ratio tests (SPRT), it aids in optimal stopping rules based on cumulative evidence over time.

Although classical probability theory is the most widely used method for modeling uncertainty and is based on well-defined probability measures, it requires crisp, well-defined and complete information about events. But in a large class of real-word problems, the uncertainties are not purely stochastic but also linguistic, ambiguous, or imprecise, which makes classical probability models inadequate [13]. This also highlights one of the main limitations of classical probability theory: that it does not deal well with subjectivity, uncertainty or vagueness at the level of decision-making, medical diagnosis, financial forecasting, and risk assessment [14]. In contrast, traditional probability models require an rigorously defined sample space and velvet probabilities that are often impossible to have in complex systems where uncertainty arises from human perception, incomplete data, or simply lack of precision measuring devices [15]. For instance, in medical prognosis, determining a patient's recovery period may not always be expressed in terms of crisp probabilities since it relies on many uncertain factors such as lifestyle choices, genetic compositions, treatment efficacy, etc. To overcome these limitations, the use of fuzzy set theory, fuzzy probability theory, and fuzzy distributions have been proposed to deal with situations with uncertain, vague, or natural language type uncertainties [16]. In fuzzy probability theory, classical probability is generalized to where membership functions specify the likelihood or degree of belief concerning an event. They basically represent probability values as fuzzy numbers rather than precise values; therefore, they are

appropriate for problems in which uncertainty is not purely random but also imprecise or subjective. This paradigm is central to artificial intelligence, machine learning, decision support systems, climate modeling and human-centered processes where data is often incomplete, unclear or inconsistent [17]. Through fuzzy possibility decision making, prediction, delay, and uncertainty analysis, fuzzy probability can expand the application of fuzzy mathematics and help solve problems in many fields.

Although fuzzy set theory is ineffective in reconstructing higher levels of vagueness, conflict, and undetermined data [18]. Fuzzy logic attaches a membership in the range of 0 to 1, yet it does not distinguish between lack of information and contradictory data. Fuzzy set theory and fuzzy logic cannot completely model uncertainty that results in the real- world problems, especially in the decision, in the medical diagnosis, and artificial intelligence where uncertainty originates from partial, inconsistent, and contradictory information [19]. Aiming to solve these problems, neutrosophic logic was proposed by Smarandache, where each of the entities inside an uncertain statement, truth (T), indeterminacy (I) and falsity (F), is treated as a query that works independently from the others, and each of these entities takes values in the [0,1] domain [20-23]. Neutrosophic logic, as compared to fuzzy logic which can only accommodate partial truth, explicitly models uncertainty and contradiction [24]. This is particularly helpful in the areas of decision making in very complex situations, machine learning, medical diagnosis, and disparate uncertain data analysis, where uncertainty and counter-intuitive data pair [25-27]. Since classical probability fails to take into consideration real-world uncertainty, neutrosophic probability and neutrosophic distributions have emerged as important and applicable areas in many engineering and scientific fields. Neutrosophic theory is also helpful to model uncertainty because it adds indeterminacy [28].

In this work, we present the neutrosophic version of the Wald model for better modelling the real-life problems. Neutrosophic extension of Wald distribution provides additional power in dealing with ambiguous and incomplete information, rendering the models more suitable for practical scenarios.

The work is structured as follows. In section 2 the statistical model of the proposed model with uncertainty parameters are presented. Section 3 data generating model using the quantile function is discussed. Section 4 provides estimation procedure of the proposed model. Finally, Section 5 concludes the major findings of the work.

2. Proposed Model

In this section, we derive and present the statistical properties of the Wald distribution with essential functions used in the reliability analysis. The proposed model, as the classical one, is a continuous probability distribution on the positive real axis, whose parameters, through the use of the neutrosophic logic, can take the values of indeterminate and uncertain events. By assuming the imprecision in the distributional parameters, proposed model can be defined as:

$$f_N(t;\mu_N,\lambda_N)=\sqrt{\frac{\lambda_N}{2\pi t^3}}\exp\left(-\frac{\lambda_N(t-\mu_N)^2}{2\mu_N^2 t}\right),\ t,\lambda_N,\mu_N>0 \tag{1}$$

where $\mu_N=[\mu_l,\mu_u]$ and $\lambda_N=[\lambda_l,\lambda_u]$ are vague parameters of the proposed model.

The proposed model generalizes the traditional Wald distribution, as the mean and shape parameters are crisp values in the traditional Wald distribution, while in the neutrosophic Wald distribution, they can be imprecise, vague, indeterminate, or interval-based, leading to a broader application in real-world where exact values are not available. It finds application especially in reliability analysis, survival studies, and stochastic modeling when uncertainty originates from measurement errors, human judgment, or random environmental factors. For certain values of μ_N and λ_N the PDF of the proposed distribution is shown in Figure 1.

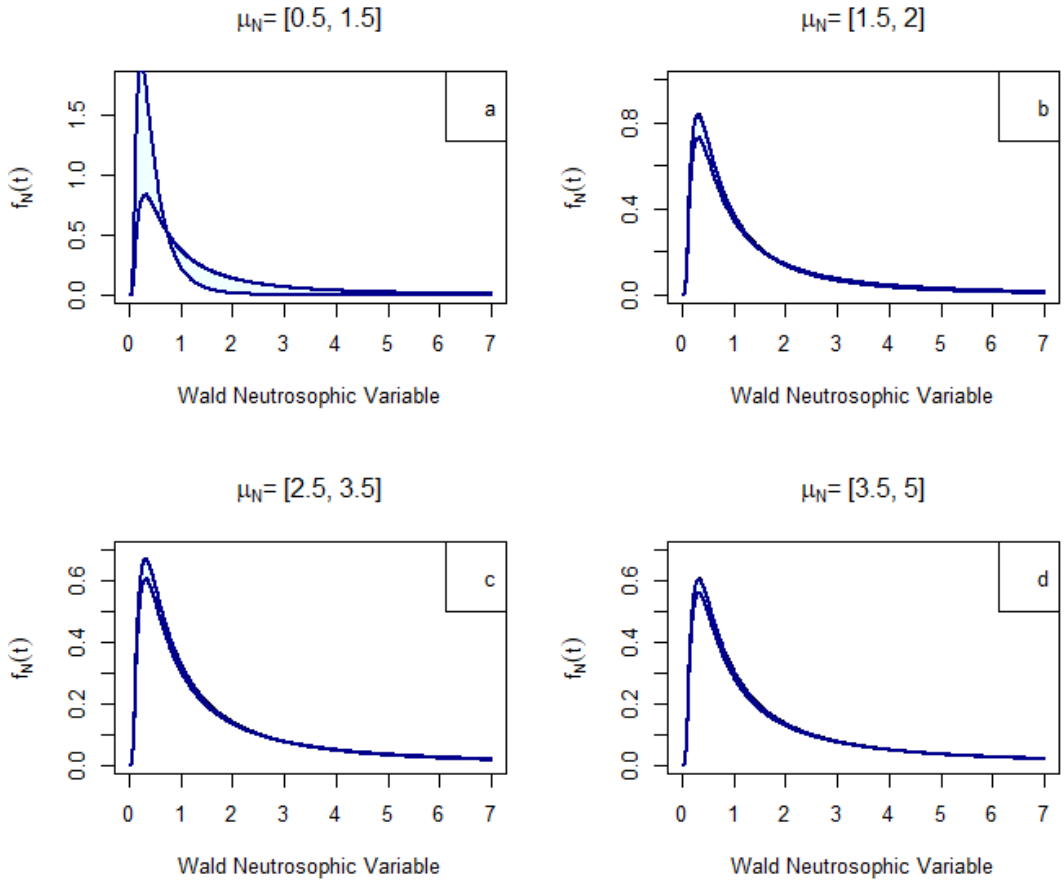

Figure 1 The PDF of the proposed distribution with imprecise location parameter values

The PDF of neutrosophic Wald distribution for different values of μ_N is displayed in Figure 1, and explains the behaviour of the distribution under indeterminacy. Each panel in the Figure 1 corresponds to a different interval of the parameter μ_N, and encompasses a range of density curves rather than a single deterministic function. The shaded areas between the two curves in both panels show the uncertainty in the PDF as a consequence of the neutrosophic representation for a, meaning that the actual value of the parameter is unknown (it may take any value within a defined interval).The distribution in the first panel μ_N=[0.5, 1.5] is highly right-skewed, it has a higher peak and a slower decay, which indicates that higher probability density points towards to further left towards of the Wald neutrosophic variable. The initial left panel is consistent with a very exaggerated deformation of periodicity as the mean interval shifts upward, and it serves as a representation of complete degradation towards random noise with a steadily increasing scandals in the mean interval as μ_N =[1.5,2], μ_N=[2.5,3.5], μ_N=[3.5,5] panels, as the peak moves to the right and the peak appears flat, showing an increasing dispersion and lower skewness.

Similarly, the CDF of the proposed model is given by:

$$F_N(t;\mu_N,\lambda_N) = \Phi\left(\sqrt{\frac{\lambda_N}{t}}\left(\frac{t}{\mu_N}-1\right)\right) + \exp\left(\frac{2\lambda_N}{\mu_N}\right)\Phi\left(-\sqrt{\frac{\lambda_N}{t}}\left(\frac{t}{\mu_N}+1\right)\right) \quad (2)$$

By assuming different values of μ_N and λ_N the CDF curves of the proposed model is shown in Figure 2.

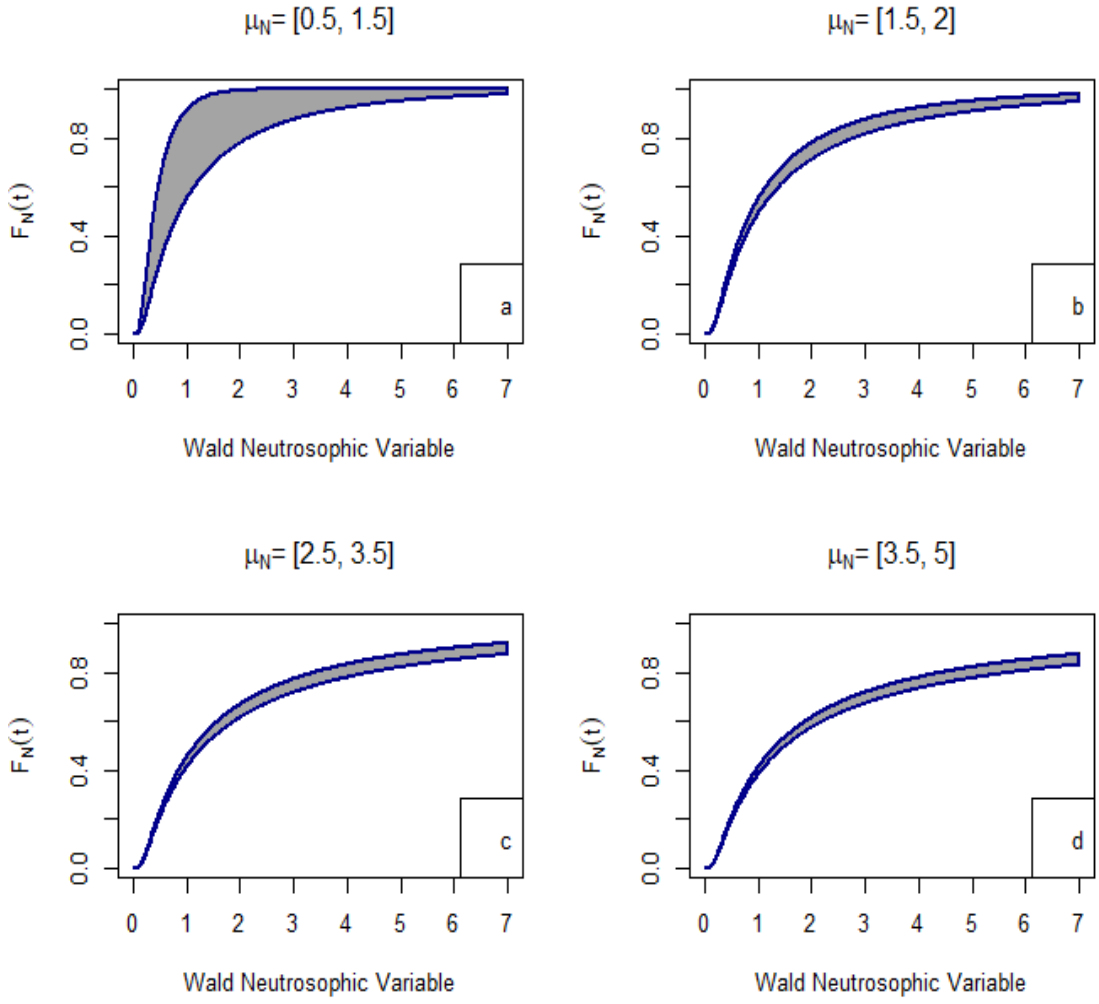

Figure 2 The CDF plot of the proposed model with different values of location parameter

The effect of uncertainty in shape parameter on the cumulative probability of the Wald neutrosophic variable is shown in Figure 2, which depicts the cumulative distribution function (CDF) of Neutrosophic Wald distribution. This means that each subpanel shows its own intervals of the parameter μ_N , describing the possible CDF curves instead of a single deterministic function. In each panel, the shaded areas representing the space between the two curves illustrate the area of uncertainty or indeterminacy associated with the neutrosophic parameter, for which the true value of μ_N is not known exactly, but for which the value belongs to the interval of the range. As we can see in the first panel (μ_N =[0.5,1.5]), the CDF increases more steeply than the previous case, indicating that the probability of finding smaller values of the Wald variable is higher since it is strongly right-skewed. In the other rows, which represent the same CDFs but for larger values of a in the mean parameter (μ_N =[1.5,2], μ_N =[2.5,3.5], and μ_N = [3.5,5]), we notice that as the value of a increases across a, the CDF curves shift to the rightmost parts of the plot, meaning that the distribution becomes wider, and larger values of the Wald variable become more likely. As μ_N becomes μ_N larger, the shaded regions become the wider, representing greater uncertainty regarding cumulative probabilities. Figure 2 illustrates how neutrosophic uncertainty on the shape parameter modifies the cumulative probability shape, allowing the Neutrosophic Wald distribution to be very flexible to different types of uncertain, imprecise, and incomplete information.

A program in R is written that use the structure of "*neutrostat* " package [29] to draw the PDF and CDF values of the proposed model. The PDF and CDF values at specified values with μ_N =[1, 2] and λ_N =[0.5,1] are given in Table 1.

Table 1 PDF and CDF values of the proposed model using neutrostat R package

Z values	PDF values	CDF values
0.5	[0.643,0.967]	[0.232, 0.249]
1	[0.352, 0.564]	[0.490, 0.627]
1.5	[0.212, 0.259]	[0.627, 0.824]
2	[0.120, 0.141]	[0.713, 0.915]
2.5	[0.058,0.099]	[0.773, 0.957]
3	[0.028, 0.073]	[0.815, 0.978]
3.5	[0.014, 0.056]	[0.848, 0.988]
4	[0.007, 0.044]	[0.873, 0.994]

Results in Table 1 show at each value of neutrosophic random variable Z we have interval value of PDF and CDF. This is because of indeterminacy that we have assumed in the neutrosophic distributional parameters.

The survival function is another important function of the proposed model. For uncertain parameters, survival function of the proposed model is given by:

$$S_N(t;\mu_N,\lambda_N) = 1 - F_N(t;\mu_N,\lambda_N) \quad (3)$$

The survival function indicates the probability of subject or system survival past time. Mathematically T is a random variable representing the time to occurrence of an event of interest (e.g., failure, death, or any type of endpoint of interest). In reliability study, clinical studies, and actuarial science, the survival function provides important information about the longevity or durability of a system or population. It is a non-increasing function. For some specific values of μ_N and λ_N,survival function of the proposed model is shown in Figure 3.

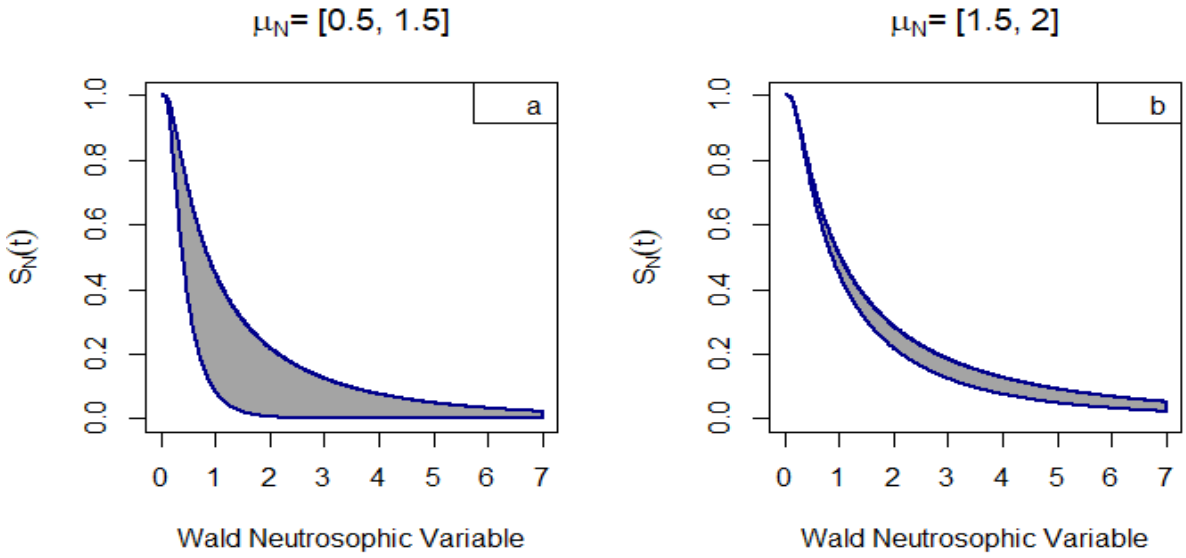

Figure 3 Survival function of the proposed model with imprecise parameter values.

Interval values of the neutrosophic mean parameters of the survival function of the Neutrosophic Wald Distribution are shown in this Figure 3. The survival function $S_N(t;\mu_N,\lambda_N)$ corresponds to the probability that the Wald neutrosophic variable T > t. It plots the survival functions corresponding to two edge values of the neutrosophic location parameter, namely μ_N=0.5 and μ_N=1.5 (left panel). The area in gray symbolizes the uncertainty in the neutrosophic extension, as each realization would result in a possible survival probability ranging from a minimum to a maximum value. In the right panel on the left is plotted survival function for the interval μ_N=1.5 and μ_N=2. Survival probabilities decrease as t increases, and the shaded area once more emphasizes the fluctuations of possible values fueled by neutrosophic uncertainty. The results show how the uncertainty of the location parameter impacts the survival probabilities, and confirming the proposed distribution's ability to model a survival problem where the values of the parameters are not exact. Neutrosophic interval effect is visually depicted by the upper and lower boundaries of the shaded region.

The other function related to survival is hazard function which is defined below:

$$H_N(t;\mu_N,\lambda_N) = \frac{f_N(t;\mu_N,\lambda_N)}{S_N(t;\mu_N,\lambda_N)} \quad (4)$$

The exact expression of Eq (4) is quite complicated, how visual depiction of hazard function is given in Figure 4.

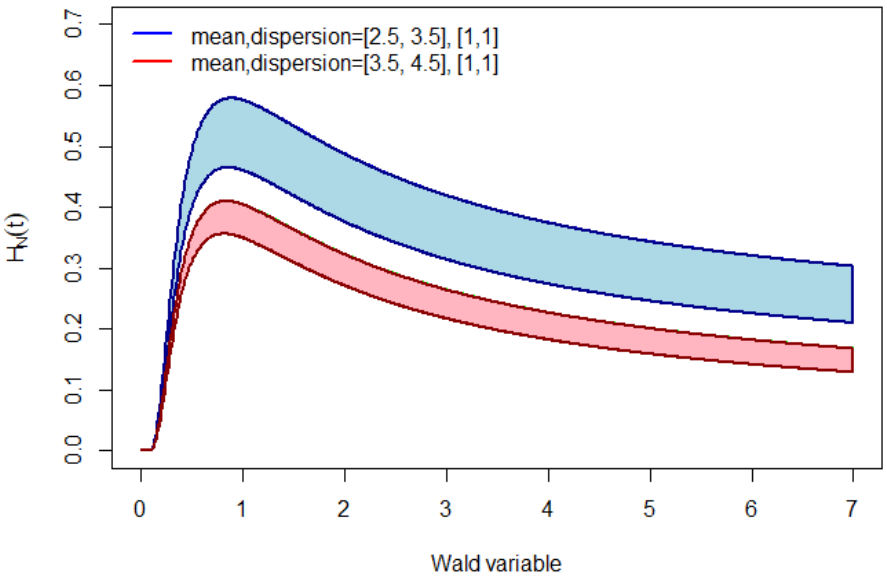

Figure 4 Hazard function of the proposed model with different inexact distribution parameters

In survival analysis and reliability studies, the hazard function describes the instantaneous failure rate of a subject or system at a given time, subject to prior survival. It gives insights on how failure of components changes over its life and is widely used in reliability modeling, reliability evaluation, medicine, engineering applications, etc.

The cumulative hazard function for the proposed model can be derived by :

$$H_N(t;\mu_N,\lambda_N) = -\ln S_N(t;\mu_N,\lambda_N) \quad (5)$$

The $H_N(t;\mu_N,\lambda_N)$ is has an important role in survival and reliability data modelling. It provides a measure of the expected number of failures occurring up to a specific time in survival and reliability analysis. One of the primary reliability metrics that must be computed is the Mean Time to Failure (MTTF), which represents the average failure time for a system or component that cannot be repaired. The average time to failure (MTTF) is the average time that the failure occurs or is equal to the mean of the failure time distribution, which is often broadly applied in engineering, manufacturing, and survival analysis to estimate product life, calculate optimal maintenance approaches, increase system reliability, and reduce incidence of surprises.

The MTTF for the proposed model is defined by:

$$\text{MTTF}_N = E[T] = \int_0^\infty t\, f_N(t;\mu_N,\lambda_N)\, dt$$

$$E[T] = \int_0^\infty t \sqrt{\frac{\lambda_N}{2\pi t^3}} \exp\left(-\frac{\lambda_N(t-\mu_N)^2}{2\mu_N^2 t}\right) dt \quad (6)$$

Solving (6) further yielded:

$$E[T] = \mu_N$$

The mean residual life (MRL) function can also derived as:

$$\text{MRL}_N(t;\mu_N,\lambda_N) = \frac{\int_t^\infty S_N(u;\mu_N,\lambda_N)du}{S_N(t;\mu_N,\lambda_N)} \quad (7)$$

The MRL function characterizes the expected remaining functional lifetime of an item (system) given that it has survived up to the instant t. It is used in reliability analysis, survival studies, and maintenance planning. Ultimately, MRL assists in assessing longevity, predictions on failures, and optimization of replacement strategies, both for engineering and health purposes.

The variance of the model can be derived as follows:

$$\text{Var}(T) = E[T^2] - (E[T])^2 \quad (8)$$

Now

$$E[T^2] = \int_0^\infty t^2 f_N(t;\mu_N,\lambda_N)\ dt \quad (9)$$

$$E[T^2] = \mu_N^2 + \frac{\mu_N^3}{\lambda_N} \quad (10)$$

$$E[T] = \mu_N \quad (11)$$

Using Eq (10) and Eq (11) in Eq (9), we get:

$$\text{Var}(T) = \frac{\mu_N^3}{\lambda_N} \quad (12)$$

Likewise, mode of the distribution can be obtained as:

$$\frac{d}{dT} f_N(T;\mu_N,\lambda_N) = 0 \quad (13)$$

$$f_N(T) = \sqrt{\frac{\lambda_N}{2\pi T^3}} \exp\left(-\frac{\lambda_N(T-\mu_N)^2}{2\mu_N^2 T}\right)$$

Solving for T, mode is given by:

$$\frac{d}{dt} f_N(t) = f_N(t)\left[\frac{3}{2t} - \frac{\lambda_N}{\mu_N^2}\left(1-\frac{\mu_N}{t}\right)\right]$$

$$\frac{3}{2t} - \frac{\lambda_N}{\mu_N^2}\left(1-\frac{\mu_N}{t}\right) = 0$$

Solving for T, mode is given by:

$$\text{Mode}(T) = \mu_N\left(\sqrt{1+\frac{9\mu_N}{4\lambda_N}} - \frac{3\mu_N}{2\lambda_N}\right) \quad (14)$$

The skewness can be determined by solving the expression:

$$\gamma_1 = \frac{E\left[(T-E[T])^3\right]}{\text{Var}(T)^{3/2}} \quad (15)$$

$$\text{Var}(T) = \frac{\mu_N^3}{\lambda_N}$$

and $E[T] = \mu_N$

Using these expressions in Eq (15), yielded:

$$\gamma_{1N} = \frac{\frac{3\mu_N^5}{\lambda_N}}{\left(\frac{\mu_N^3}{\lambda_N}\right)^{3/2}}$$

$$\gamma_{1N} = \frac{3\mu_N}{\sqrt{\lambda_N}} \quad (16)$$

We can derive the expression for kurtosis coefficient as:

$$\gamma_2 = \frac{E\left[(T-E[T])^4\right]}{\mathrm{Var}(T)^2} \tag{17}$$

Eq (17) further can be simplified as:

$$E[(T-\mu_N)^4] = \frac{15\mu_N^7}{\lambda_N^3}$$

$$\gamma_{2N} = \frac{\frac{15\mu_N^7}{\lambda_N^3}}{\left(\frac{\mu_N^3}{\lambda_N}\right)^2}$$

$$\gamma_{2N} = \frac{15\mu_N^7}{\lambda_N^3} \times \frac{\lambda_N^2}{\mu_N^6}$$

$$\gamma_{2N} = \frac{15\mu_N}{\lambda_N} \tag{18}$$

Skewness and kurtosis coefficients explain probability distribution shape. Skewness tells us about the asymmetry of a distribution a positive number means the distribution is right-skewed (has a longer right tail), and a negative number means that it is left-skewed. On other hand kurtosis refers to the tails of a distribution; high kurtosis means a lot of extreme outliers, low kurtosis means a flatter distribution.

3. Random Sample Generation

The inverse cumulative function method, also referred to as the inverse transform sampling technique [30], is the basic technique for generating random samples from a given probability distribution. Uniform random numbers is then generated according to the standard uniform distribution (a simple approach is that R use runif(.), or Python use random.uniform(.) etc). Using the inverse of the cumulative distribution function (CDF) of the desired distribution, these uniform values are then transformed. The CDF gives us the probability that a random variable will be less than or equal to a value at a given point. The inverse function is then applied to map the uniform values based on values of the target distribution. This technique is popular, since, provided that appropriate sampling techniques are followed, the produced samples will adhere exactly to the theoretical distribution without any error of approximation. This is valid for several distributions. The inverse cumulative function method is especially beneficial in Monte Carlo simulations, statistical modeling, and stochastic processes, where random sample generation is critical for decision-making and predictive analysis.
The inverse cumulative function or quantile function of the proposed model is defined as:

$$F_N(t;\mu_N,\lambda_N) = \Phi\left(\sqrt{\frac{\lambda_N}{t}}\left(\frac{t}{\mu_N}-1\right)\right) + \exp\left(\frac{2\lambda_N}{\mu_N}\right)\Phi\left(-\sqrt{\frac{\lambda_N}{t}}\left(\frac{t}{\mu_N}+1\right)\right)$$

Now quantile function is obtained by solving the expression:

$$p = F_N(t;\mu_N,\lambda_N)$$

$$Q_N(p) = \frac{\mu_N}{2\lambda_N}\left(2\lambda_N\Phi^{-1}(p) + \sqrt{4\lambda_N^2\left(\Phi^{-1}(p)\right)^2 + 4\mu_N\lambda_N}\right) \tag{19}$$

To generate random samples from the neutrosophic Wald distribution, we follow the simulation procedure described below. The neutrosophic parameters (e.g., mean and shape factor) become involved by defining them as interval or fuzzy values (in other words, uncertainty is taken into account). We assume that $\mu_N = [0.5, 2]$ and $\lambda_N = [1,1]$. Then, uniform random numbers from a uniform distribution are generated to use as transformation inputs. The output is directive adjustment based on the accepted neutrosophic likelihood parameters and will return neutrosophic Wald distributed samples. In cases when the parameters are represented by intervals, lower and upper bounds are computed separately, yielding an interval-valued dataset. This gives an opportunity to model on

uncertain and imprecise data and versatility in reliability analysis, survival studies and uncertain decision-making environments. A program in R is used to generate random samples from the proposed model. At specific seed setting first forty random samples values using Eq (19) are given in Table 2.

Table 2 Random samples generated from the proposed with imprecise parameter values

Random Data				
[1.072, 1.893]	[0.954, 1.603]	[1.346, 1.939]	[1.87, 2.462]	[1.284, 1.462]
[1.186, 1.958]	[0.619, 1.377]	[1.296, 1.976]	[0.851, 0.865]	[1.563, 1.564]
[1.769, 2.199]	[1.722, 1.949]	[1.146, 1.945]	[0.957, 1.224]	[1.087, 1.762]
[0.684, 1.367]	[1.362, 1.487]	[1.125, 1.352]	[1.353, 2.287]	[1.504, 2.388]
[0.731, 1.748]	[1.135, 1.33]	[1.001, 2.233]	[1.466, 2.471]	[1.539, 1.726]
[0.741, 2.153]	[1.59, 1.706]	[1.108, 1.186]	[1.334, 1.863]	[1.521, 2.251]
[1.672, 1.839]	[1.75, 1.849]	[0.666, 2.185]	[0.987, 2.027]	[0.888, 1.645]
[0.691, 2.271]	[1.754, 1.947]	[0.532, 1.689]	[0.818, 1.614]	[0.806, 1.891]

Table 2 shows the interval random sample values generated from the proposed model with parameter setting $\mu_N = [1, 2]$ and $\lambda_N = [0.5,1]$. Each interval value indicates imprecise because of imprecise neutrosophic parameters.

4 Estimation Method

Estimation methods are statistical modeling techniques used to estimate the unknown parameters of a probability distribution based on observed neutrosophic data. Some common estimation methods in literature include the maximum likelihood estimation (MLE), method of moments (mom), and least squared method (LSM). Despite its asymptotic efficiency, which makes MLE popular, computing it involves solving complicated optimization problems (especially for non-standard distributions). However sometime, it is difficult to find the closed form likelihood expressions. In contrast, the method of moments is simpler and more intuitive, as it works by equating theoretical based moments to the sample based moments to estimate parameters. The inverse Wald distribution can be more easily fitted using the method of moments than by any of the other methods. This is due to the complex likelihood function of the inverse Wald distribution, which involves relatively computationally intensive MLE process that could even sometimes require iterative numerical methods. The method of moments, in contrast, features simple estimates where parameters can be obtained directly from sample-based properties (mean and variance). It is not subject to complex optimization routines and allows for faster estimation, making it especially useful for practical applications, especially when memory is limited or when the datasets are large.

Based on MoM approach the equating moments can be achieved as follows:

$$\widehat{\mu_N} = \bar{T} \tag{19}$$

$$\widehat{\lambda_N} = \frac{\overline{T^3}}{S^2} \tag{20}$$

If we assume the random samples generated from the proposed model with $\mu_N = [0.5, 2]$ and $\lambda_N = [1,1]$ then estimated values with other statistical characteristics are shown in Table 3.

Table 3 Estimated values of proposed model using moments method

Characteristics	Estimated values
$\widehat{\mu_N}$	[1.817, 1.187]
$\widehat{\lambda_N}$	[11.98, 42.63]
Mean	[1.187, 1.817]
Median	[1.038,1.750]
Variance	[0.139, 0.141]
Mode	[1.191,1.817]

Table 3 shows the estimated values of the proposed model using simulated data given in Table 2. Results indicate that due to imprecision in the parameters $\mu_N = [1, 2]$ and $\lambda_N = [0.5,1]$, each estimated quantity involves uncertainty. Moreover, if we assume that indeterminacy is zero then results of the classical model match with the neutrosophic version of the Wald distribution.

5 Conclusions

In this work, we proposed the neutrosophic Wald distribution which broadens the applicability of this model in practical situations by allowing uncertainty of its parameters. Utilizing this extension we also obtained key statistical properties such as the probability density function, cumulative distribution function and quantile function, which shows how flexible and robust the model is in comparison with the classical Wald distribution. The proposed addressed in our study provides a pragmatic and significant tool for fitting data with fuzzy and vague characteristics in the count of failures and analysis of reliability. Specifically, we examined some elementary reliability metrics like survival function, reliability function, and hazard function in the neutrosophic framework, thus extending definite reliability functions into a neutrosophic context. In addition, we presented a quantile-driven simulation method for the generation of random samples from the proposed model for the purposes of empirical evidence. We also proposed the moments estimation procedure that is simple and computationally inexpensive.

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Tree Soft Set and Decision-Making Approaches for Tourist Satisfaction Assessment of Tourist Attractions

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Abstract: One of the most precise decision-making techniques is multi-criteria decision-making (MCDM), which has been dubbed a revolution in the area. The act of choosing alternatives by considering several criteria to ascertain which is optimal is known as multicriteria decision-making, or MCDM. The techniques and tools developed by MCDM have a wide range of applications in engineering, design, and finance. To address the ambiguity of possibilities in application-oriented issues with many criteria, Smarandache developed Tree soft sets, which are an extension of hypersoft sets. To determine the optimal desalination technique utilizing one of the MCDM approaches, we will examine the real-world application-oriented issue of Tourist Satisfaction Evaluation of Tourist Attractions in the context of tree soft sets in this work.

Keywords: Tree Soft Set; Decision making; Tourist Satisfaction; Tourist Attractions; MCDM Methodology.

1. Introduction

Tourist satisfaction plays a crucial role in the competitiveness and sustainability of tourist destinations worldwide. As travelers seek memorable and enriching experiences, their perceptions of a destination are shaped by various factors such as scenic beauty, accessibility, service quality, available activities, and cost-effectiveness. A positive tourism experience enhances the reputation of an attraction, leading to increased visitor numbers, economic growth, and long-term destination success. Conversely, dissatisfaction due to poor service, inadequate facilities, or high costs can result in negative reviews and declining tourist interest[1], [2]. Therefore, evaluating tourist satisfaction is essential for tourism operators and policymakers to enhance destination management and improve overall visitor experience. To comprehensively evaluate tourist satisfaction, a multi-criteria assessment approach is required. Factors such as natural and cultural appeal, ease of transportation, quality of facilities, variety of recreational activities, and affordability significantly impact visitor experiences. Different types of attractions—ranging from natural landscapes offer diverse experiences, necessitating tailored

evaluation frameworks. By systematically analyzing these criteria, tourism planners can identify strengths and weaknesses, allowing for strategic improvements that align with visitor expectations and industry trends[3], [4]. Modern evaluation techniques, particularly multi-criteria decision-making (MCDM) methods, provide structured approaches to ranking tourist attractions based on performance indicators. These methods help in quantifying subjective experiences by incorporating expert opinions and visitor feedback, resulting in objective and data-driven rankings. Additionally, with the rise of digital platforms and social media, analyzing tourist reviews and sentiment analysis further refine satisfaction assessments. By leveraging these evaluation models, decision-makers can prioritize infrastructure enhancements, optimize tourism marketing strategies, and improve overall service delivery to maximize visitor satisfaction. As global tourism continues to evolve, driven by technological advancements and changing traveler preferences, the importance of systematic tourist satisfaction evaluation will only grow. Sustainable tourism development requires a balance between preserving cultural and natural heritage while ensuring high-quality visitor experiences. By continuously assessing and improving tourism attractions based on visitor feedback and performance evaluations, stakeholders can create more inclusive, enjoyable, and sustainable tourism destinations that cater to the expectations of modern travelers[5], [6]. Making decisions is a complex intellectual process that entails resolving issues while accounting for several variables to reach a desired result. Both explicit and implicit assumptions may be used in this process, which might be illogical or reasonable and influenced by social, cultural, physiological, and biological variables. These elements, along with the degree of risk and power involved, affect how complicated decisions are. Using mathematical equations, various statistics, mathematics, and economic theories, computer technology may help automatically calculate and estimate answers to decision-making difficulties in the modern world. By considering several factors throughout the selection process, MCDM seeks to identify the optimal choice.

Numerous MCDM tools and methods may be used in a variety of fields, including robotics, engineering, and finance. Regardless of the extent of missing data, periodic mathematics (PM), fuzzy set theory (FST), probabilistic set theory (PST), and rough set theory (RST) are thought to be the scientific methods for handling incomplete data. Zadeh initially introduced fuzzy sets to deal with ambiguity, which made them beneficial for a variety of applications like pattern recognition, medical diagnosis, and decision-making. Numerous fuzzy set generalizations, including rough sets, soft sets, intuitionistic fuzzy sets, bipolar fuzzy sets, bipolar soft sets, m-polar soft sets, hypersoft sets, and many more, have been created because of their popularity[7], [8]. At the maximum level of incompleteness, soft sets (Ss) are an interesting extension of fuzzy sets with further ambiguity and vagueness characteristics. Soft sets handle ambiguous, imprecise, and vague components. The function of parameters is important in all decision-making situations, but professionals must assess them when they are ambiguous. A multi-argument approximation function was employed by the hypersoft sets, a soft set extension, to address the drawbacks of the existing structures for attribute-valued disjoint sets. Eventually, to handle ambiguous situations in real-world problems, MultiSoft sets were introduced to soft sets[9].

Additionally, tree soft sets—which are quite like hypersoft sets—have recently been developed as an expansion of MultiSoft sets. Hypersoft sets deal with criteria and sub- criteria, whereas tree soft sets deal with criteria, sub- criteria, sub- criteria, and so on. The capacity of tree soft sets to have many sub- criteria, represented as level 0, level 1, level 2,…, level n, is one of its distinctive features. Using tree soft sets will significantly reduce uncertainty in real-world applications[10], [11].

2. Tree Soft Set (TSS)

This section shows the definitions of the TSS. Let U as a universal set and $P(U)$ is a power set of U. Let a set of criteria such as $c_1, c_2, \dots c_n and\ n \geq 1\ for\ the\ first\ level$ and intersection of these criteria are non-empty set. Every criterion $C_i, 1 \leq 1 \leq n$, is formed by sub criteria[12], [13]:

$$\begin{aligned} C_1 &= \{C_{11}, C_{12}, C_{13}, C_{14}, C_{15}\} \\ C_2 &= \{C_{21}, C_{22}, C_{23}, C_{24}, C_{25}\} \\ C_3 &= \{C_{31}, C_{32}, C_{33}, C_{34}, C_{35}\} \\ C_4 &= \{C_{41}, C_{42}, C_{43}, C_{44}, C_{45}\} \\ C_5 &= \{C_{51}, C_{52}, C_{53}, C_{54}, C_{55}\} \\ C_6 &= \{C_{61}, C_{62}, C_{63}, C_{64}, C_{65}\} \\ C_7 &= \{C_{71}, C_{72}, C_{73}, C_{74}, C_{75}\} \\ &\vdots \\ C_n &= \{C_{n1}, C_{n2}, C_{n3}, C_{n4}, C_{n5}\} \end{aligned} \tag{1}$$

Where C_{ij} are sub criteria and these criteria are formed as level in tree. The first level presents the root nodes, the second level presents the main criteria, and the third level presents the sub criteria values. Fig 1 shows the tree nodes.

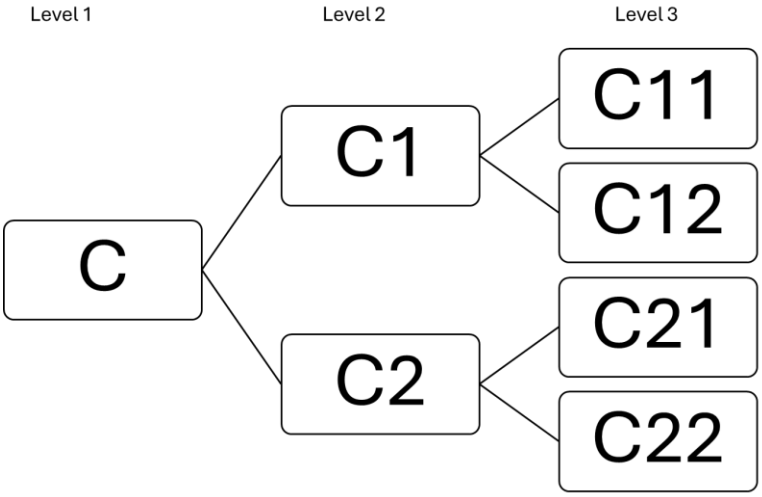

Fig 1. The tree nodes.

The TSS can be formed as:

$$F: P\big(Tree(A)\big) \rightarrow P(H) \tag{2}$$

Example:

The criteria of Tourist Satisfaction can be considered as a TSS, and the node of this problem is shown as levels of the tree as shown in Fig 2.

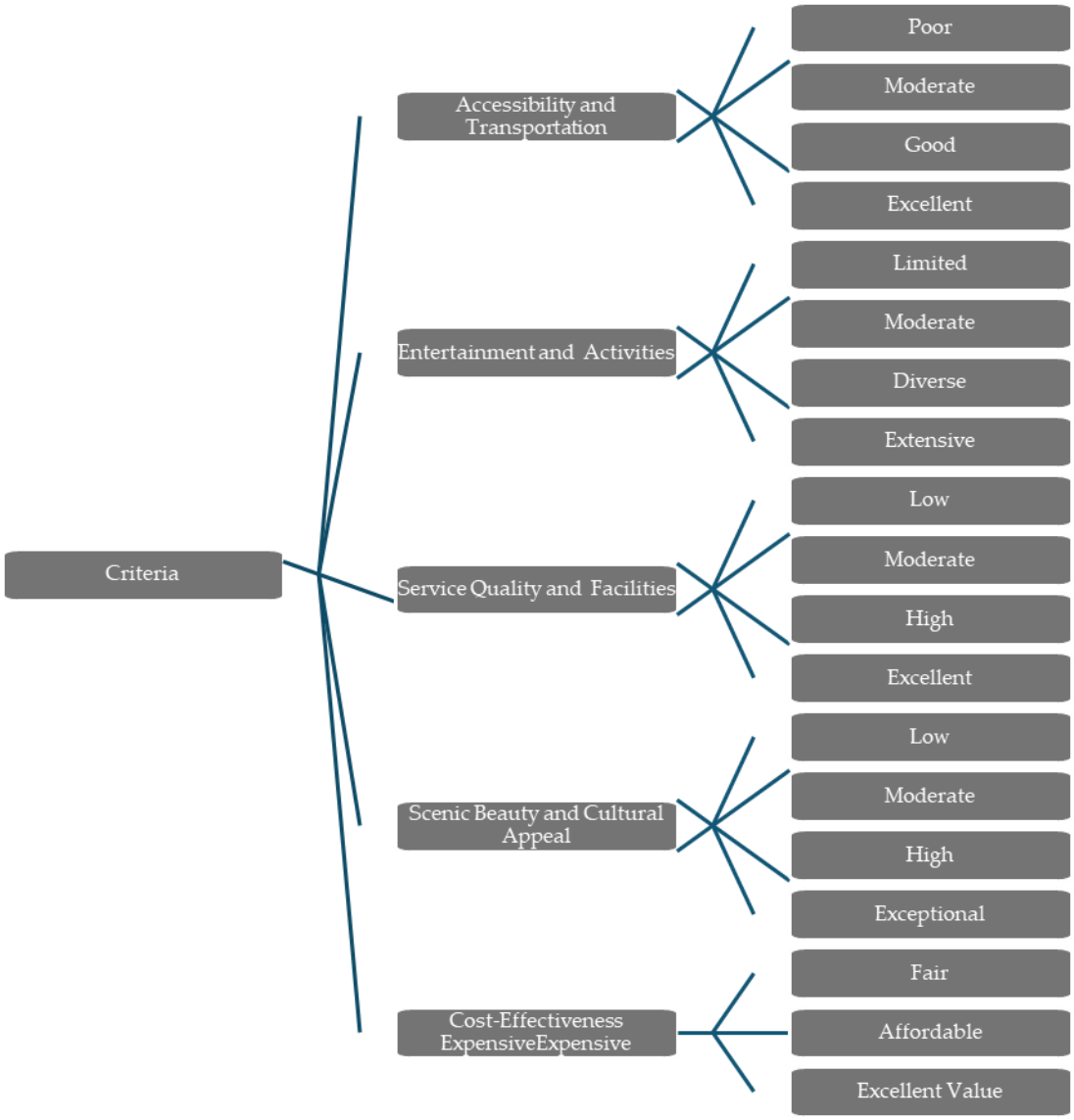

Fig 2. The Tourist Satisfaction tree.

3. MCDM Approach

This section shows the steps of the proposed approach. We use two MCDM methods in this study. We use the ITARA methodology to compute the criteria weights [14], [15]and the MABAC methodology to rank the alternatives.

ITARA Methodology

Build the decision matrix.

The decision matrix is built using the opinions of the experts. Then we combine these values into one matrix.

Set the indifference threshold h_j.

The experts formulate the indifference threshold

Compute the normalized matrix

$$n_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \tag{3}$$

Sort the elements in the decision matrix

Compute the dispersion degree of the adjacent elements

$$u_{ij} = n_{i+1,j} - n_{ij} \tag{4}$$

Compute the distance between the $u_{ij}\ and\ h_j$.

$$p_{ij} = \begin{cases} u_{ij} - Nh_j, u_{ij} > Nh_j \\ 0, \qquad\quad u_{ij} \le Nh_j \end{cases} \tag{5}$$

Compute the criteria weights

$$w_j = \frac{\left(\sum_{i=1}^{m-1} P_{ij}^{b}\right)^{\frac{1}{b}}}{\sum_{i=1}^{n}\left(\sum_{i=1}^{m-1} P_{ij}^{b}\right)^{\frac{1}{b}}} \tag{6}$$

MABAC Methodology

This section shows the steps of the MABAC Methodology to rank the alternatives[16], [17].

Normalize the decision matrix

$$t_{ij} = \frac{x_{ij} - \min x_{ij}}{\max x_{ij} - \min x_{ij}} \tag{7}$$

$$t_{ij} = \frac{x_{ij} - \max x_{ij}}{\min x_{ij} - \max x_{ij}} \tag{8}$$

Compute the weighted decision matrix

$$o_{ij} = w_j + w_j t_{ij} \tag{9}$$

Compute the border approximation area matrix

$$q_j = \left(\prod_{i=1}^{m} o_{ij}\right)^{\frac{1}{m}} \tag{10}$$

Compute the distance from the border q_j

$$e_{ij} = o_{ij} - q_j \tag{11}$$

Compute the total distance

$$s_i = \sum_{j=1}^{n} e_{ij} \tag{12}$$

Rank the alternatives.

4. Results

This section shows the results of two MCDM methods. We are obtaining the criteria weights and ranking the alternatives. Three experts have evaluated the criteria and alternatives. They used linguistic terms as Very High, High, Medium High, Medium, Medium Low, Low, Very Low by using values between 0.1 to 0.9. We used the concept of TSS by dividing the criteria and sub criteria into three levels as shown in Fig 2. We select the values of each criterion as excellent, moderate, excellent, high, and fair.

Build the decision matrix.

Three experts are building the decision matrix as shown in Table 1.

Then we set the indifference threshold h_j.

Then we compute the normalized matrix as shown in Fig 3 using Eq. (3).

Then we sort the elements in the decision matrix

Then we compute the dispersion degree of the adjacent elements using Eq. (4).

Then we compute the distance between the $u_{ij}\ and\ h_j$ using Eq. (5).

Then we compute the criteria weights using Eq. (6) as shown in Fig 4.

Table 1. Linguistic terms by experts.

	C_1	C_2	C_3	C_4	C_5
A_1	High	Medium High	Medium	Medium Low	Low
A_2	Very Low	Very High	High	Medium High	Medium
A_3	Low	Very High	Medium	Medium Low	Low
A_4	Medium Low	High	Medium High	High	Very High
A_5	Medium	Very High	Medium Low	Low	Very Low
A_6	Medium High	Medium Low	Medium	High	Very Low
A_7	Very Low	Medium	Medium Low	Very High	Medium High
A_8	Very High	Medium High	Very High	Medium Low	Medium
A_9	High	Very Low	Very Low	Very Low	Medium Low
A_{10}	Low	Medium Low	Medium	Medium High	Very Low
	C_1	C_2	C_3	C_4	C_5
A_1	Very High	Medium High	Medium	Medium Low	Low
A_2	High	Low	High	Medium High	Medium
A_3	Medium High	Low	Medium	Medium Low	Low
A_4	Medium	High	Medium High	Very Low	Very High

A_5	Medium Low	Low	Medium Low	Very High	High
A_6	Medium High	Very Low	Very High	High	Medium High
A_7	Very Low	Medium High	High	Medium High	Medium
A_8	Very High	Medium	Medium High	Medium	Medium Low
A_9	High	Medium Low	Medium	Medium Low	Medium Low
A_{10}	Low	Medium Low	Medium Low	Medium High	High
	C_1	C_2	C_3	C_4	C_5
A_1	Very Low	Medium High	Medium	Medium Low	Medium Low
A_2	Very Low	Medium Low	High	Medium High	Medium
A_3	Very High	Medium	Medium	Medium Low	Medium High
A_4	Medium Low	Medium High	Medium Low	High	High
A_5	Medium	Very Low	Medium	Medium Low	Medium Low
A_6	Medium High	Medium Low	Medium High	Medium	Medium
A_7	High	Medium	High	Medium High	Medium High
A_8	Low	Medium High	Low	Very Low	Very Low
A_9	High	High	High	High	Medium Low
A_{10}	Very High	Medium Low	Medium	Medium High	High

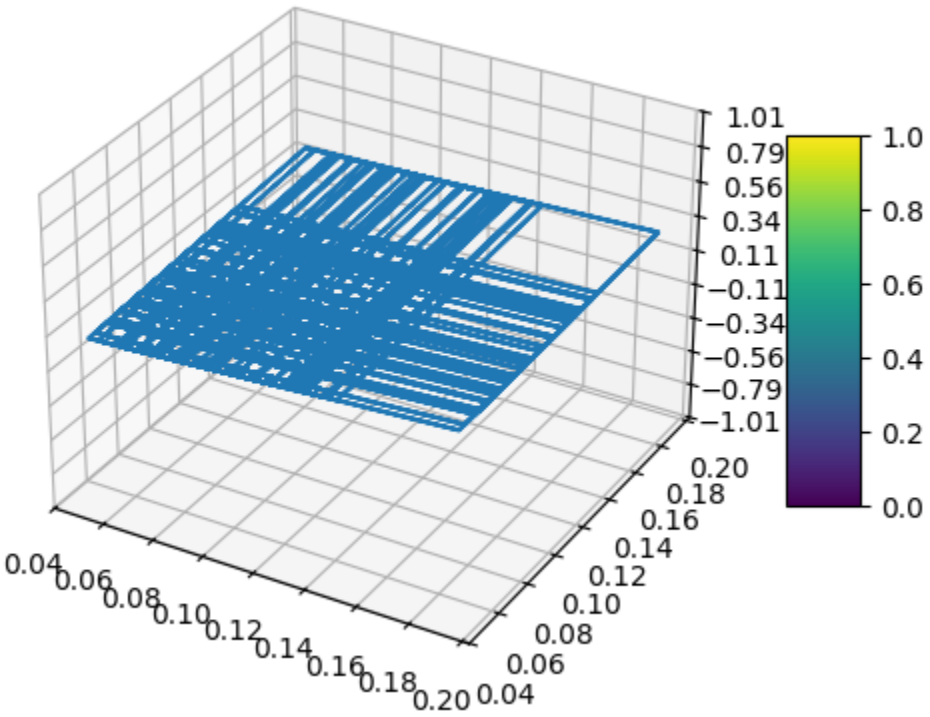

Fig 3. The normalization matrix by weighting method.

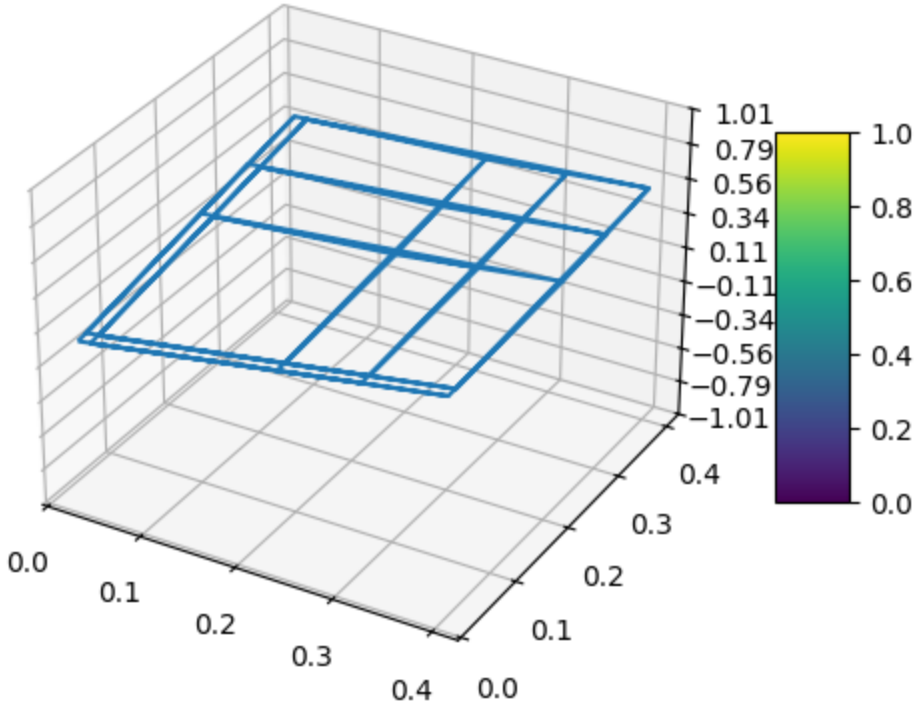

Fig 4. The criteria weights.

MABAC Methodology

We used Eq. (7) to normalize the decision matrix as shown in Fig 5.

Then we compute the weighted decision matrix as shown in Fig 6.

Then we compute the border approximation area matrix using Eq. (10) as shown in Fig 7.

Then we compute the distance from the border q_j using Eq. (11) as shown in Fig 8.

Then we compute the total distance using Eq. (12) as shown in Fig 9.

Finally, we ranked the alternatives as shown in Fig 10.

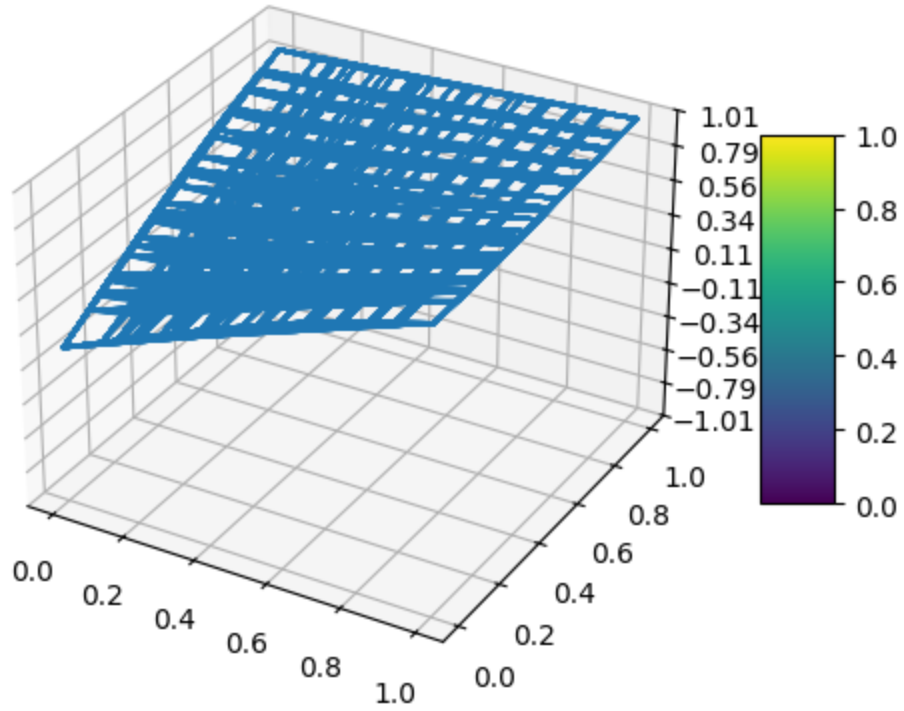

Fig 5. The normalization values are by the MABAC method.

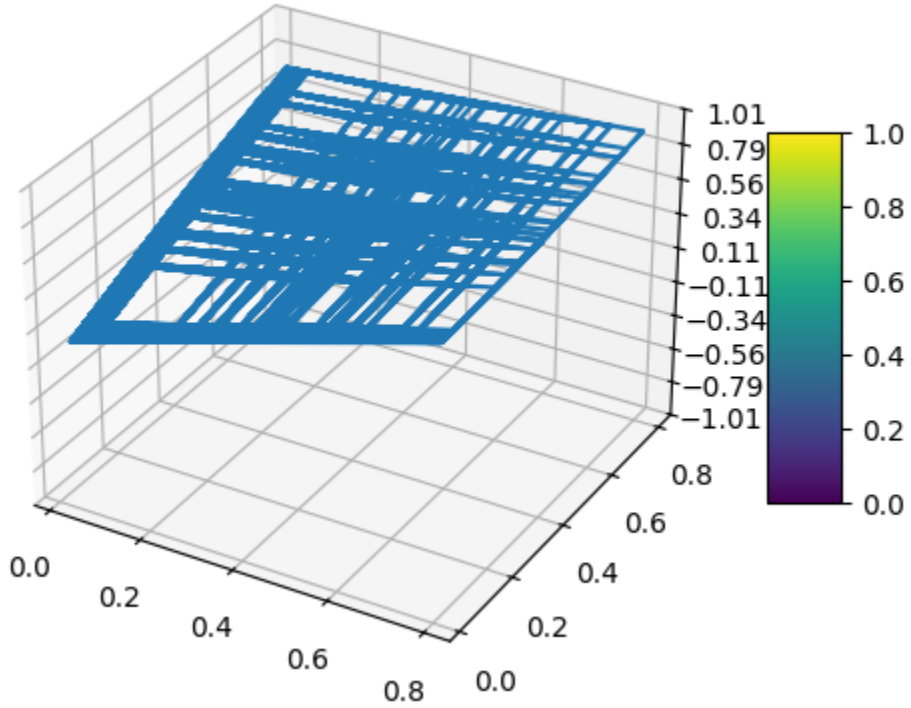

Fig 6. The weighted decision values.

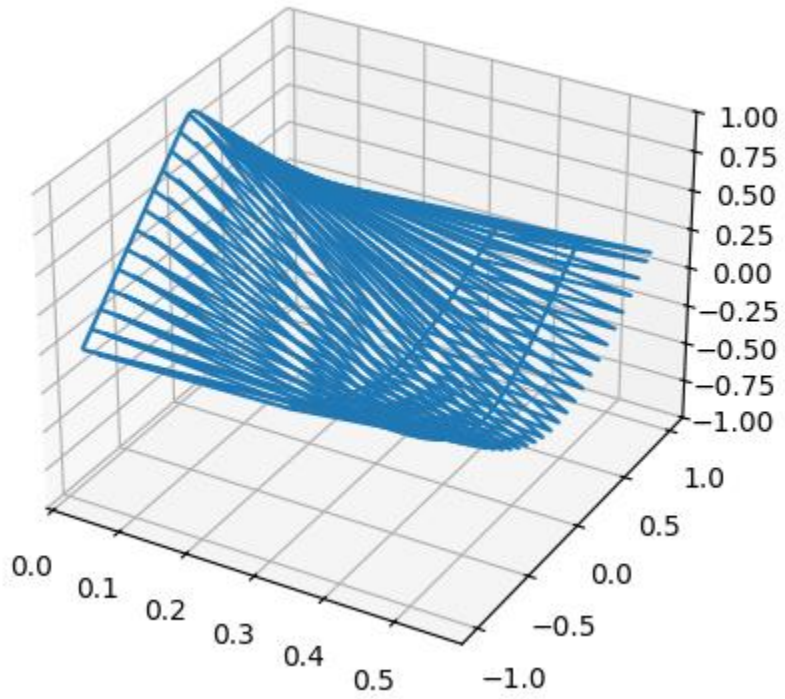

Fig 7. The border approximation values.

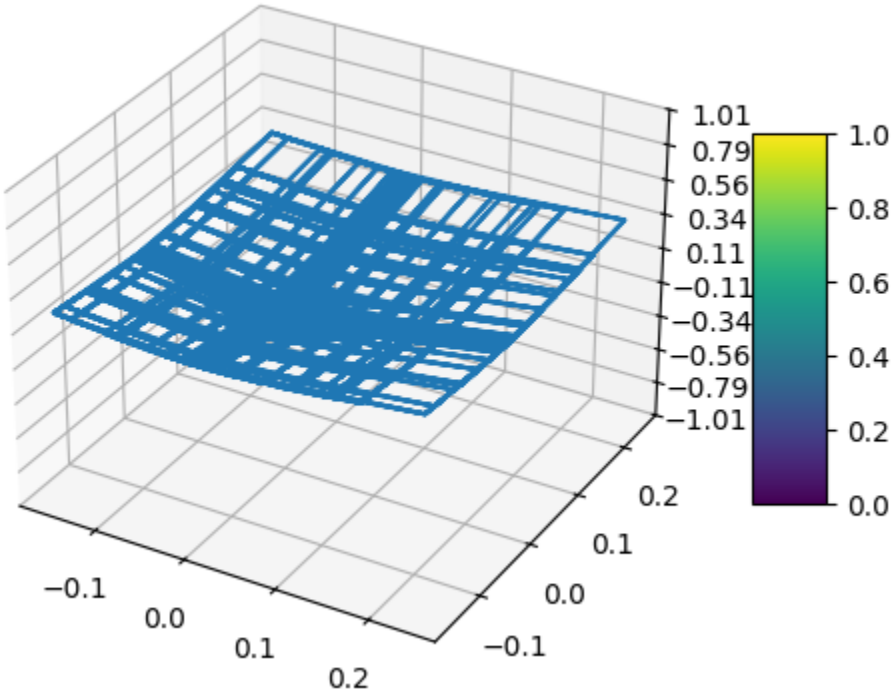

Fig 8. The values of distances.

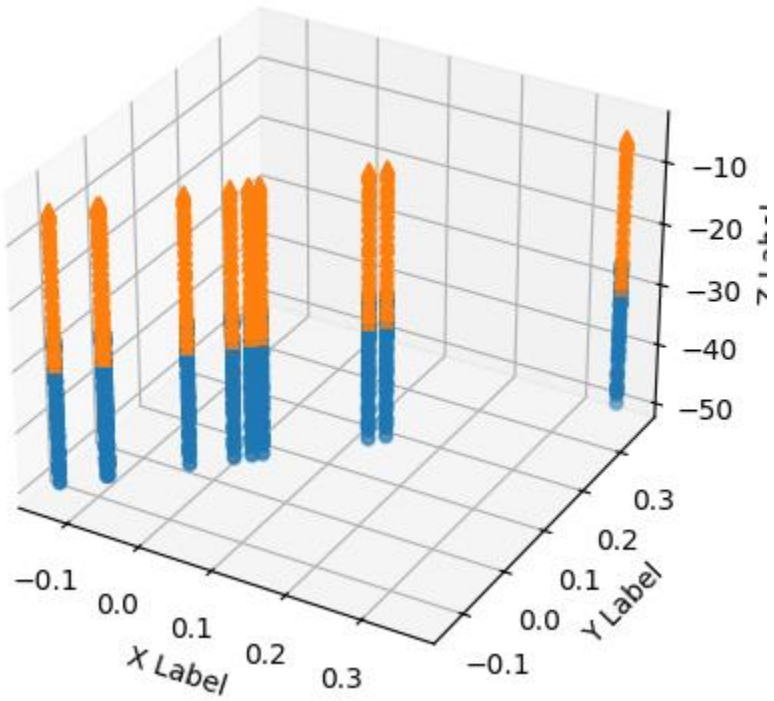

Fig 9. The total distances.

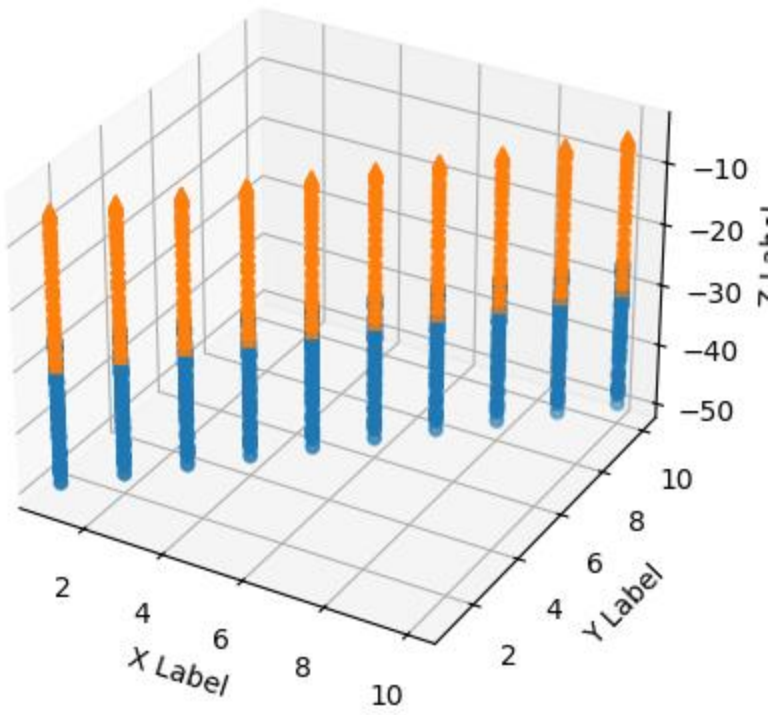

Fig 10. The rank of alternatives.

Comparative analysis

We compare the proposed approach by different MCDM methods to show the effectiveness of the proposed approach. We compared the proposed approach by four different MCDM methods. All methods agree with the best alternative and worst alternative. We show alternative 6 is the best and alternative 5 is the worst in all methods. Fig 11 shows the results of comparative analysis. The results show the proposed approach is effective compared to other MCDM methods.

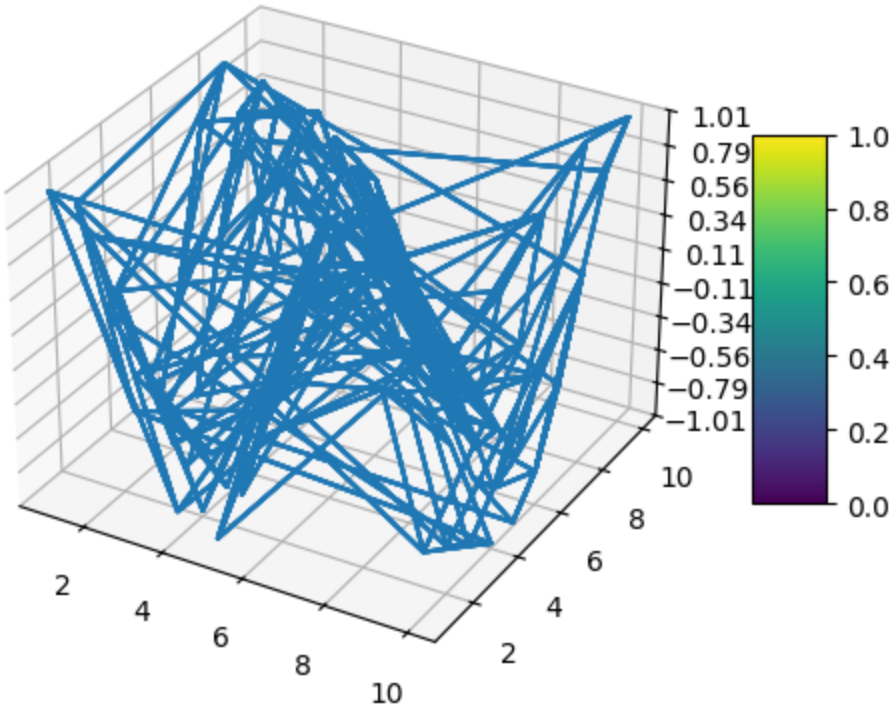

Fig 11. The ranks of different MCDM methods.

5. Conclusions

This study tackles a real-world problem of Tourist Satisfaction Evaluation of Tourist Attractions method using one of the MCDM methodologies to talk about the Tree soft set environment. The associated properties are likewise prioritized while selecting the procedure. Performance has already been improved in issues involving multiple criterion outranking using the MABAC technique. Additionally, by using Tree soft sets, we may resolve more complex and multi-attribute issues. In subsequent studies, tree soft sets may be applied to all other fuzzy settings, including intuitionistic, pythagorean, neurosophic, and others.

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Prediction of sleep disorders using Novel decision support neutrosophic based machine learning models

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Abstract: Sleep disorders significantly impact human health, productivity, and overall quality of life. Early diagnosis and prediction of these disorders are crucial for effective treatment. This study introduces a novel decision support system utilizing a neutrosophic machine learning prediction model to enhance the accuracy and reliability of sleep disorder diagnosis. Unlike traditional machine learning approaches, our model integrates neutrosophic logic, which considers three values—truth, indeterminacy, and falsity—to effectively handle uncertainty and inconsistencies in sleep-related data. The proposed model processes diverse patient information, including demographics, clinical parameters, and polysomnography details, ensuring comprehensive analysis. Experimental results demonstrate that our approach surpasses conventional machine learning methods in predictive accuracy, robustness, and interpretability. Furthermore, this research provides an advanced framework

for clinicians to assess potential sleep disorders while accommodating inherent uncertainties in medical data. The study highlights the impact of neutrosophic machine learning in healthcare decision support systems and outlines potential avenues for future research.

Keywords: Neutrosophic sets, Machine Learning, Uncertainty handling, Sleep disorder, Classification

1. Introduction

Sleep disorders represent a significant public health issue impacting large sections of the global population and simultaneously posing immense risks for respiratory, cardiovascular, metabolic, psychiatric, cognitive, and neurologic disorders [1,2]. Yet, traditional diagnostic modalities for sleep disorders are heavily reliant on subjective self-report measures and polysomnography, a resource intensive, costly, and time-consuming diagnostic test. Over the last couple of years, machine learning has been recognized as a game changer in medical diagnostics technology due to its ability to rapidly and accurately predict disease states after analyzing complex data [3, 4]. Traditional ML models may not have a clear promise in working against the uncertain or ambiguous information that is a natural part of the medical lexicon, but uncertainty is even higher after a sleep study is carried out.

Neutrosophy is a branch of mathematics that aims to add to and improve classical and fuzzy set theories. It has become an important tool for dealing with vagueness, indeterminacy, and uncertainty in many areas. In this situation, neutrosophic sets and their more complex forms, such as Fermatean neutrosophic sets, have gotten a lot of attention because they are good at showing uncertainty and making decisions easier. Neutrosophic sets have the basic parts of truth, uncertainty, and falsity, which makes them a better way to show complicated events. Fermateanneutrosophic sets, as an extension, provide more flexibility and accuracy, thereby expanding their use in complex decision-making and computational challenges.

A multitude of research articles have been published examining the theoretical underpinnings, attributes, and practical applications of these sets, underscoring their significance in areas such as multicriteria decision-making, optimization, and artificial intelligence [5,6,7,8,9,10,11,12,13,14,15,16,17,23]. These papers focus on how neutrosophic and Fermatean neutrosophic sets can be used to solve real-world problems where there is uncertainty and not enough information. The growing body of research shows that these

mathematical constructions can be used in a wide range of fields. This highlights how important they are to the progress of modern mathematics and its practical applications in science, engineering, and other areas.

which is deal when data is un that studies the properties and relations of negations, has been introduced into ML as well; By including the neutrosophic sets into the ML algorithms, one can, therefore, model data with different truth, falsity, and indeterminacy levels, and therefore identify underlying patterns more richly [18]. Neutrosophic-based ML models can improve results because they enable to deal with noisy, incomplete or even contradictory information more effectively than classical ML approaches can [19]. This prevents from biased predictions due to uncertainties in data whilst enabling richer feature extraction and classification. In predicting sleep disorders, the neutrosophic models are able to fine-grain within clinical and observational data, where individuals variate, which can be ignored, but leads to a greater higher prediction accuracies reliability. Further, the flexibility of neutrosophic theory allows a better adaptation to the variation of individual characteristics in sleep patterns, fostering personalized diagnostics.

This research discovers the application of neutrosophic-based machine learning models to predict sleep disorders by adapting clinical and observational sleep data into neutrosophic sets. By leveraging this hybrid approach, the study aims to bridge the gap between complex data uncertainties and the need for accurate diagnostic tools, thus contributing to advancements in personalized and efficient sleep disorder management. Through this, we aim to highlight the transformative potential of neutrosophic-based methods in enhancing predictive accuracy, reducing diagnostic errors, and promoting a deeper understanding of sleep disorders.

This research article is organized into the following sections: Section-2 explains the main contributions of this research paper. Section-3 presents the materials and methods of the proposed work. Section-4 describes the machine learning model. Section-5 discusses the results. Section-6 provides the discussion, and Section-7 presents the conclusion.

2. The main contributions of this study are:

1. Integrating the concept of neutrosophic sets into machine learning algorithms.
2. Providing a robust neutrosophic-based machine learning model for sleep disorder classification.
3. Comparison of the current model with the existing model.

3. Materials and Methods (proposed work with more details)

Step 1: Load the Data

Step 2: Preprocess the Data

Step 3: Convert Data into Neutrosophic Sets

Step 4: Split the Data into Training and Testing Sets

Step 5: Train a Machine Learning Model

Step 6: Make Predictions and Evaluate the Model

3.1. Proposed methodology

A structured framework for sleep disorder classification used Neutrosophic based machine learning models in this study. The data pre-processing was performed to fill missing values and convert the parameters into Neutrosophic sets, which represent truth, indeterminate, and false statements. Using evaluation metrics, including accuracy, sensitivity, specificity, F1 score, MCC, and precision, machine learning models like Random Forest, SVM, LR, KNN, and Naive Bayes were trained and validated with the processed dataset. A decision-making framework was established to maximize the selection and performance of models and to adapt to heterogeneous data in sleep disorders. In order to determine the best model for the precise and trustworthy classification of sleep disorders, the findings were finally examined.

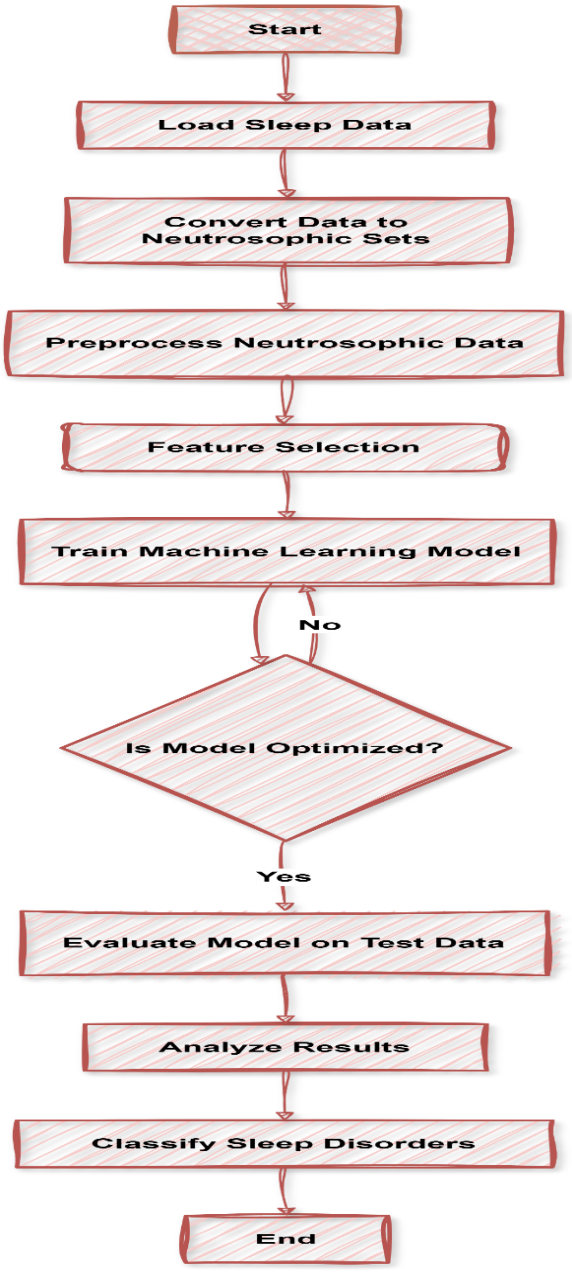

Figure 1 Proposed methodology for Neutrosophic-based machine learning models

3.2. Dataset description

This Kaggle.com dataset provides holistic insights into factors influencing sleep health, such as gender, age, occupation, sleep duration, quality of sleep, level of physical activity, stress levels, BMI category, blood pressure, heart rate, daily steps, and associated sleep disorders [5]. The data captures a diverse range of people with different lifestyles and health profiles, which foretells the interrelationship between sleep patterns and other physiological or behavioral factors. As illustrated in the dataset, people with higher stress levels and lower levels of physical activity are more likely to have a sleep disorder, such as insomnia or sleep apnea, whereas people with normal body mass index values and active lifestyles report better sleep quality and longer duration. Thus, from the obtained data, one could clearly categorize

trends in terms of relationships between occupation type and stress levels, the influence of physical activity on sleep disorders, and so on. These findings could provide credible guidance for developing personalized health interventions, managing sleep disorders, and promoting a healthy lifestyle.

3.3. Preprocessing of data

The sleep dataset was preprocessed to ensure effective machine-learning analysis. The dataset contained sleep duration, quality, physical activity, stress level, BMI category, and a categorical target variable for different sleep disorders. Target variable encoding was done with numerical values for better model understanding. Missing values were imputed while duplicate rows were removed from the dataset to improve quality. Continuous features such as sleep duration, physical activity, and daily steps were normalized using Min-Max scaling to result in uniformity while improving the performance of the models on this particular task. All these preprocessing steps provided more consistency in the dataset, which optimized it for precise sleep disorder prediction.

3.4. Neutrosophic Sets

Neutrosophic sets are an extension of classical sets, fuzzy sets, and intuitionistic fuzzy sets, introduced by Florentin Smarandache in 1995. They are used to represent imprecise, inconsistent, and incomplete information, particularly in decision-making and problem-solving scenarios.It is defined using three independent membership functions that quantify the degree of truth ($\breve{T}$), indeterminacy ($\breve{I}$), and falsity ($\breve{F}$) for an element in the set.

For every element $\breve{x} \in U$, the condition $0 \leq \breve{T}_{\breve{A}}(\breve{x}) + \breve{I}_{\breve{A}}(\breve{x}) + \breve{F}_{\breve{A}}(\breve{x}) \leq 3$ must be satisfied.

A Neutrosophic set A is defined as $A = (\breve{x}, \left(\breve{T}_{\breve{A}}(\breve{x}), \breve{I}_{\breve{A}}(\breve{x}), \breve{F}_{\breve{A}}(\breve{x})\right) : \breve{x} \in \breve{X}.$

3.5. Neutrosophic dataset formation

The code transforms the dataset into neutrosophic numbers (Truth, Indeterminacy, and Falsehood) using the following steps:

1. For each value in a column, the Truth value (t) is computed based on its normalized position between the column's minimum and maximum values.
2. Indeterminacy (i) is derived as 1 - t, and Falsehood (f) is calculated as 1 - t - i.
3. These values are combined into a tuple (t, i, f) for each data point, creating a neutrosophic representation for the entire dataset.

The resulting transformed dataset retains the original structure while each value is replaced by its neutrosophic representation.

```
In [4]: # Define function to convert each value into a neutrosophic set (Truth, Indeterminacy, Falsehood)
def to_neutrosophic_set(value, column_min, column_max):
    t = (value - column_min) / (column_max - column_min) if column_max > column_min else 0  # Truth
    i = 1 - t  # Indeterminacy
    f = 1 - t  # Falsehood
    return (t, i, f)

# Convert the dataset into neutrosophic sets
X_neutrosophic = X.copy()  # Create a copy to retain original structure
for col in X.columns:
    col_min = X[col].min()
    col_max = X[col].max()
    X_neutrosophic[col] = X[col].apply(lambda val: to_neutrosophic_set(val, col_min, col_max))

# Verify the neutrosophic transformation
print("Neutrosophic set representation (column-wise):")
print(X_neutrosophic.head())
```

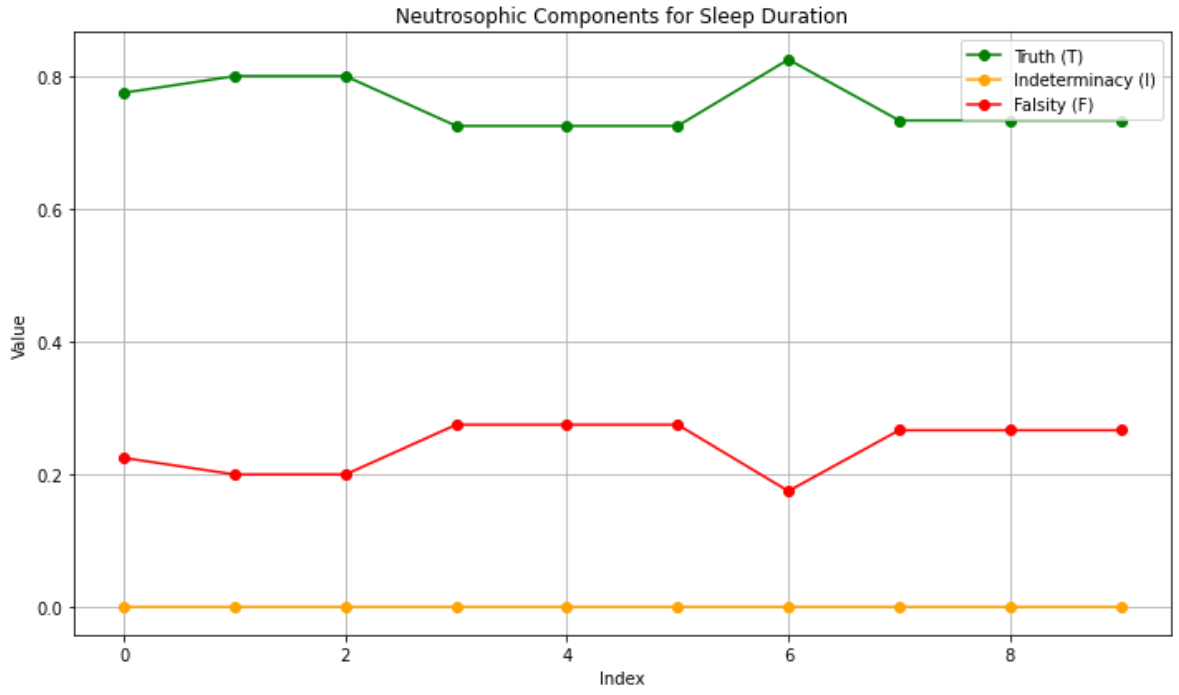

Figure 2 Neutrosophic components for sleep duration

4. Machine learning models

Machine learning models are necessary when it comes to prediction and classification of data and there are plethora of algorithms that fall into this category. We can consider Naive Bayes algorithm, K-Nearest Neighbors (KNN), Support Vector Machine (SVM), Logistic Regression and Random Forest, each excelling in different contexts [6,7]. Naive Bayes is a probabilistic based algorithm that relies on the Bayes theorem while also assuming that all features are independent of one another. Due to its nature, the model is fairly quick as it mostly includes low amount of computations and is best suited for smaller datasets which is why it is extensively used in textual classification and even spam filtering. K-Nearest Neighbors (KNN) is a non-parametrical and an instance-based language that is centered towards algorithms. It targets the missing data by considering the k nearest data points and

getting a vote among them to classify the data. KNN is easy to use and implements but is limited to small datasets, especially since it involves calculations related to distance. Support Vector machines are better thought of as supervised learning as they join two classes in feature space with a hyperplane. Both Linear and Non-Linear data supports SVM which makes use of kernel tricks and it is especially effective with high dimension datasets, however it requires more resources when working with large datasets.

5. Results

The results obtained exhibited a strong statistical significance over the several sleep disorder groups for most of the parameters (all $p < 0.001$). Concerning age, the mean ± SD was highest for sleep apnoea (49.71 ± 8.99) and lowest for the No Disorder group (39.04 ± 7.83). Sleep disorder group participants who had no disorder slept the longest (7.36 ± 0.73) while insomnia patients had the shortest sleeping hours (6.59 ± 0.39). Patients without a disorder had the highest quality of sleep (7.63 ± 0.98) while the lowest was for insomnia patients (6.53 ± 0.80). Sleep disorder group with Insomnia reported the lowest physical activity (46.82 ± 11.75) while those who suffered from sleep apnoea had the highest physical activity levels (74.79 ± 17.93). Sleep Apnoea patients exhibited the highest systolic (137.77 ± 5.14) and diastolic (92.72 ± 4.49) blood pressures with patients suffering no disorder at the lowest end (124.05 ± 5.73 and 81.00 ± 3.99, respectively). On the other hand, participants with sleep apnoea had this greatest value (7619.23 ± 2168.19) while people with insomnia recorded the lowest value (5901.3 ± 1000.33). Lastly, stress levels were highest for people with insomnia (5.87 ± 1.46) and lowest for people without the disorder (5.11 ± 1.59).

Table 1 Statistical analysis of sleep disorders.

	Sleep Disorder	**N**	**Mean**	**SD**	**SE**	**P Value**
Age	Insomnia	77	43.52	4.808	0.548	
	None	219	39.04	7.828	0.529	
	Sleep Apnea	78	49.71	8.991	1.018	<0.001
Sleep Duration	Insomnia	77	6.59	0.387	0.0441	
	None	219	7.36	0.732	0.0495	
	Sleep Apnea	78	7.03	0.975	0.1104	<0.001
Quality of Sleep	Insomnia	77	6.53	0.804	0.0917	
	None	219	7.63	0.975	0.0659	

	Sleep Apnea	78	7.21	1.646	0.1864	<0.001
Physical Activity Level	Insomnia	77	46.82	11.752	1.3392	
	None	219	57.95	20.93	1.4143	
	Sleep Apnea	78	74.79	17.927	2.0298	<0.001
Systolic blood pressure	Insomnia	77	132.04	3.935	0.4485	
	None	219	124.05	5.735	0.3875	
	Sleep Apnea	78	137.77	5.142	0.5822	<0.001
Diastolic blood pressure	Insomnia	77	86.86	3.178	0.3621	
	None	219	81	3.991	0.2697	
	Sleep Apnea	78	92.72	4.489	0.5083	<0.001
Daily Steps	Insomnia	77	5901.3	1000.328	113.998	
	None	219	6852.97	1393.474	94.1622	
	Sleep Apnea	78	7619.23	2168.191	245.499	<0.001
Stress Level	Insomnia	77	5.87	1.463	0.1667	
	None	219	5.11	1.591	0.1075	
	Sleep Apnea	78	5.67	2.334	0.2642	<0.001

Table 2 shows the evaluation metrics based on Neutrosophic set for all the machine learning models, and their strengths to predict sleep disorders. Random Forest stands out as the best model with the estimated accuracy of 96% (95%Confidence Interval:0.94-0.98), specificity of 97% (95%Confidence Interval:0.95-0.99) and precision of 96% (95%Confidence Interval: 0.94-0.98). Also, its sensitivity at 94% (95% CI: 0.92–0.96), F1 score at 95% (95% CI: 0.93–

0.97) and Matthews Correlation Coefficient (MCC) of 0.94(95% CI:0.92-0.96) bear sufficient testimony to its accuracy in performing both the positive and negative classifications and such characteristics makes the model reliable in the context of complex datasets. The accuracy recorded by SVM was slightly lower than RF at 94% (95%CI:0.92-0.96) and showed great sensitivity and specificity at 92% (95% CI: 0.89-0.95) and 96% (95%CI:0.94-0.98) respectively making it moderately powerful. KNN is not superior in any of those metrics but shows convergence in range as well, attaining 93% (95%Confidence Interval: 0.91-0.95) accuracy, 91% (95%Confidence Interval: 0.88-0.94) sensitivity and 92% (95%Confidence Interval: 0.89-0.95) in F1 score which is promising in the recognition of local patterns. However, Naive Bayes performed slightly poor in sensitivity and only recorded an accuracy of 92% (95% CI:0.90-0.94).Overall, the results indicate that Random Forest and SVM are the most reliable models for accurate predictions

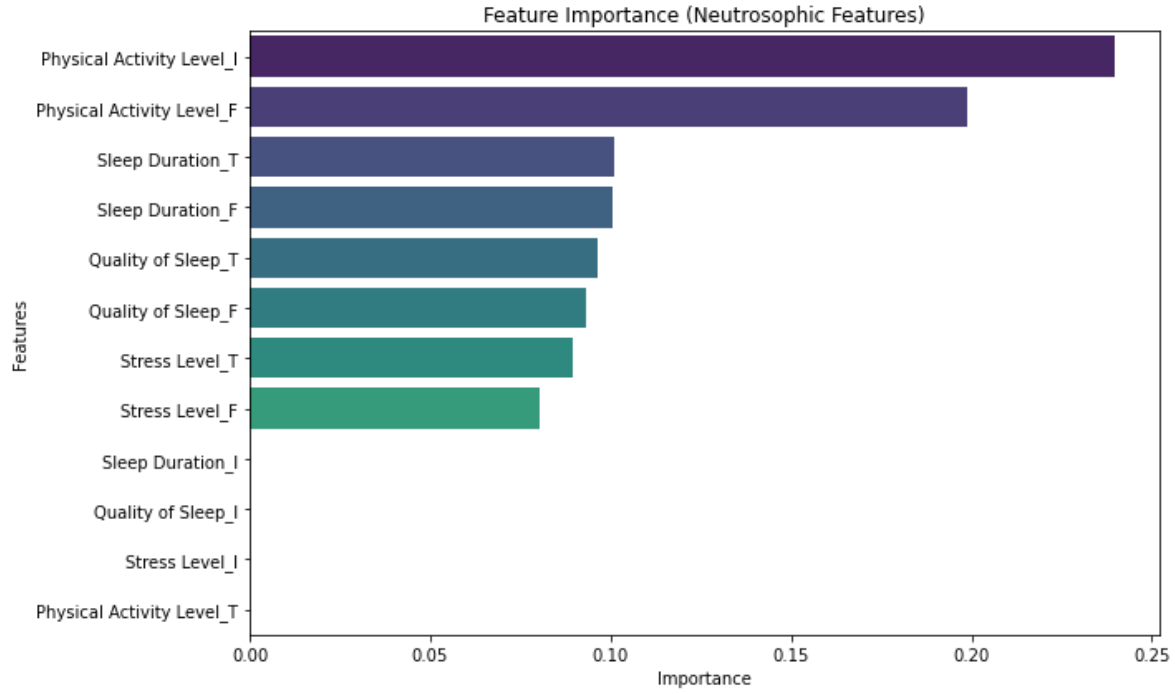

Figure 3 Neutrosophic feature importance

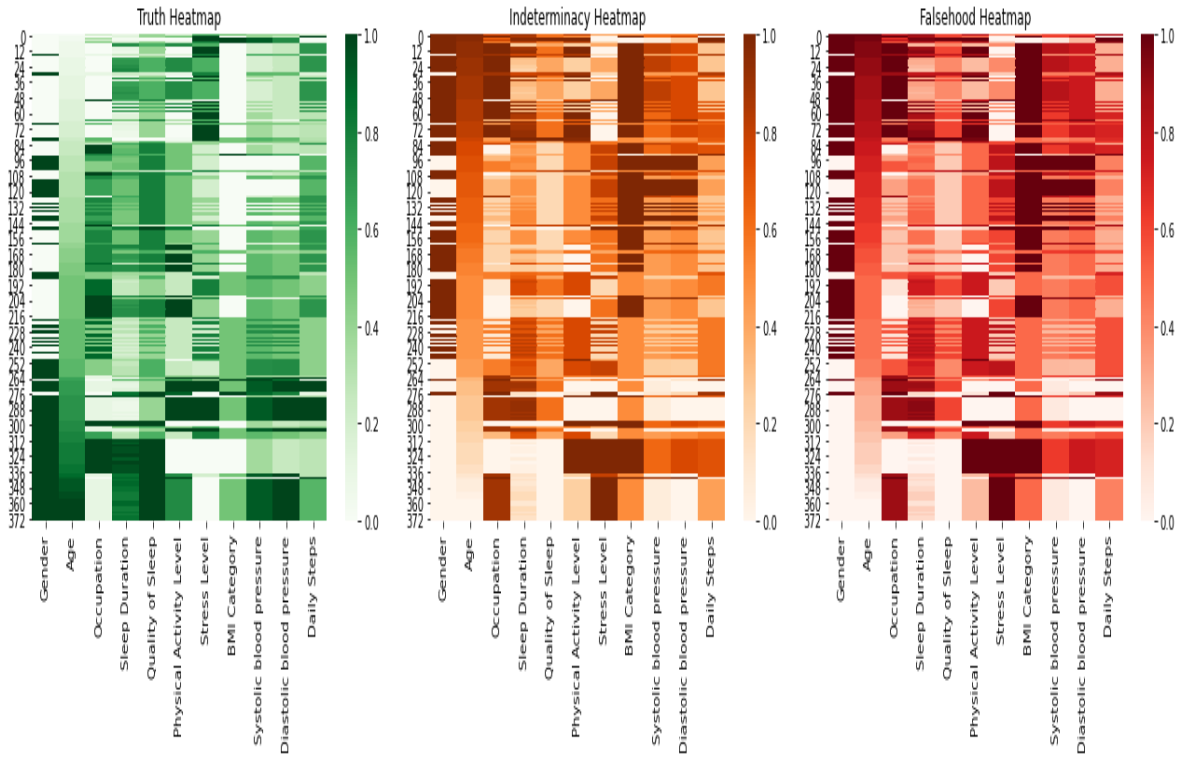

Figure 4 Truth, indeterminacy and falsehood heatmap of all features

Table 2 Neutrosophic Based evaluation matrices for machine learning models for sleep disorder classification.

Metric	Naive Bayes	KNN	SVM	Logistic Regression	Random Forest
Accuracy (95% CI)	0.92 (0.90–0.94)	0.93 (0.91–0.95)	0.94 (0.92–0.96)	0.91 (0.89–0.93)	0.96 (0.94–0.98)
Sensitivity (95% CI)	0.89 (0.86–0.92)	0.91 (0.88–0.94)	0.92 (0.89–0.95)	0.88 (0.85–0.91)	0.94 (0.92–0.96)
Specificity (95% CI)	0.94 (0.92–0.96)	0.95 (0.93–0.97)	0.96 (0.94–0.98)	0.93 (0.91–0.95)	0.97 (0.95–0.99)
F1 Score (95% CI)	0.90 (0.87–0.93)	0.92 (0.89–0.95)	0.93 (0.90–0.96)	0.89 (0.86–0.92)	0.95 (0.93–0.97)
PPV (Precision) (95% CI)	0.91 (0.88–0.94)	0.93 (0.90–0.96)	0.94 (0.91–0.97)	0.90 (0.87–0.93)	0.96 (0.94–0.98)

NPV (95% CI)	0.93 (0.90–0.95)	0.94 (0.91–0.96)	0.95 (0.92–0.97)	0.92 (0.89–0.94)	0.97 (0.95–0.99)
MCC (95% CI)	0.88 (0.85–0.91)	0.90 (0.87–0.93)	0.92 (0.89–0.95)	0.87 (0.84–0.90)	0.94 (0.92–0.96)

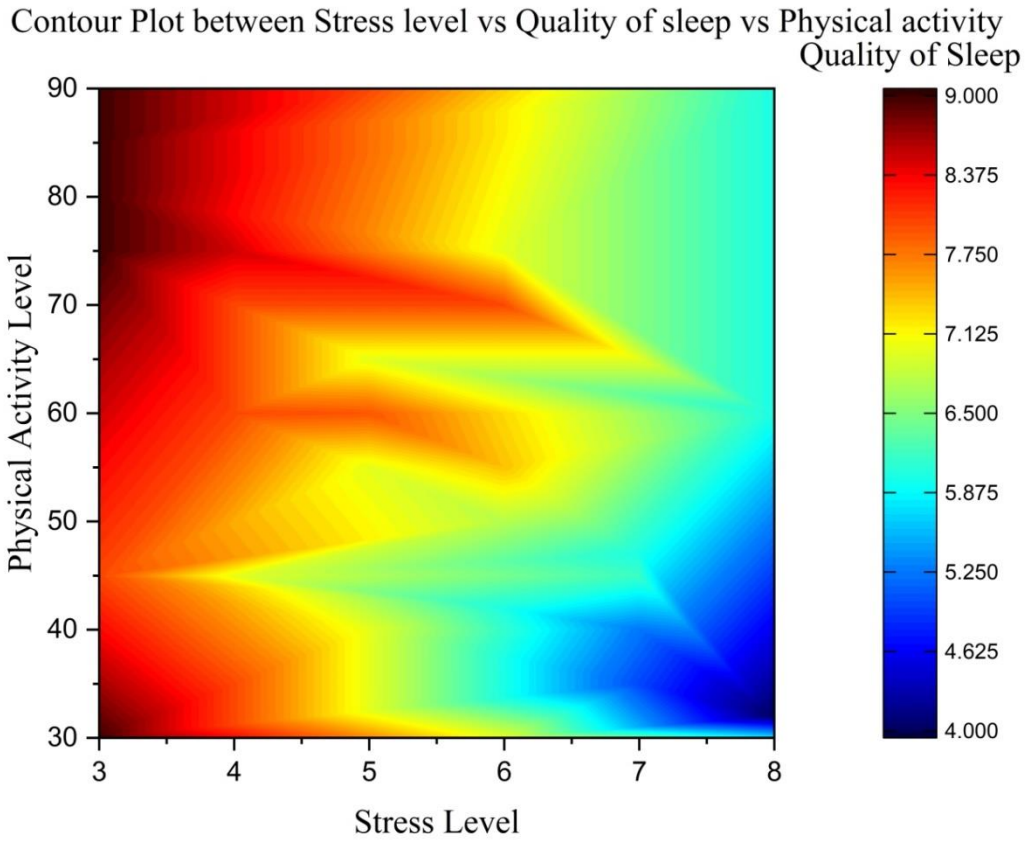

Figure 5 Contour plot between stress level, quality of sleep &physical activity

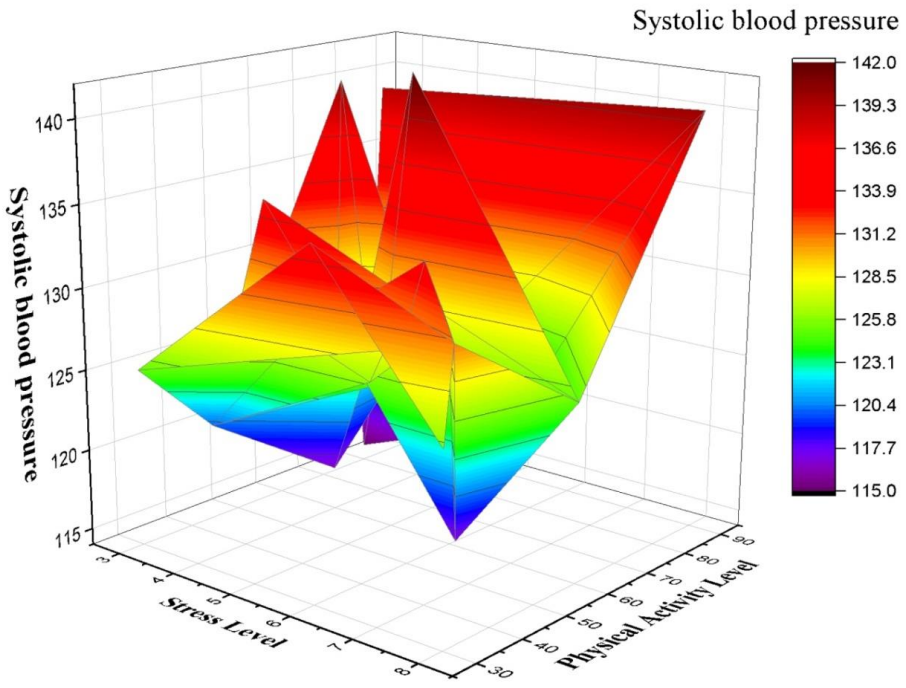

Figure 6Surface plot between stress level, systolic blood pressure physical activity

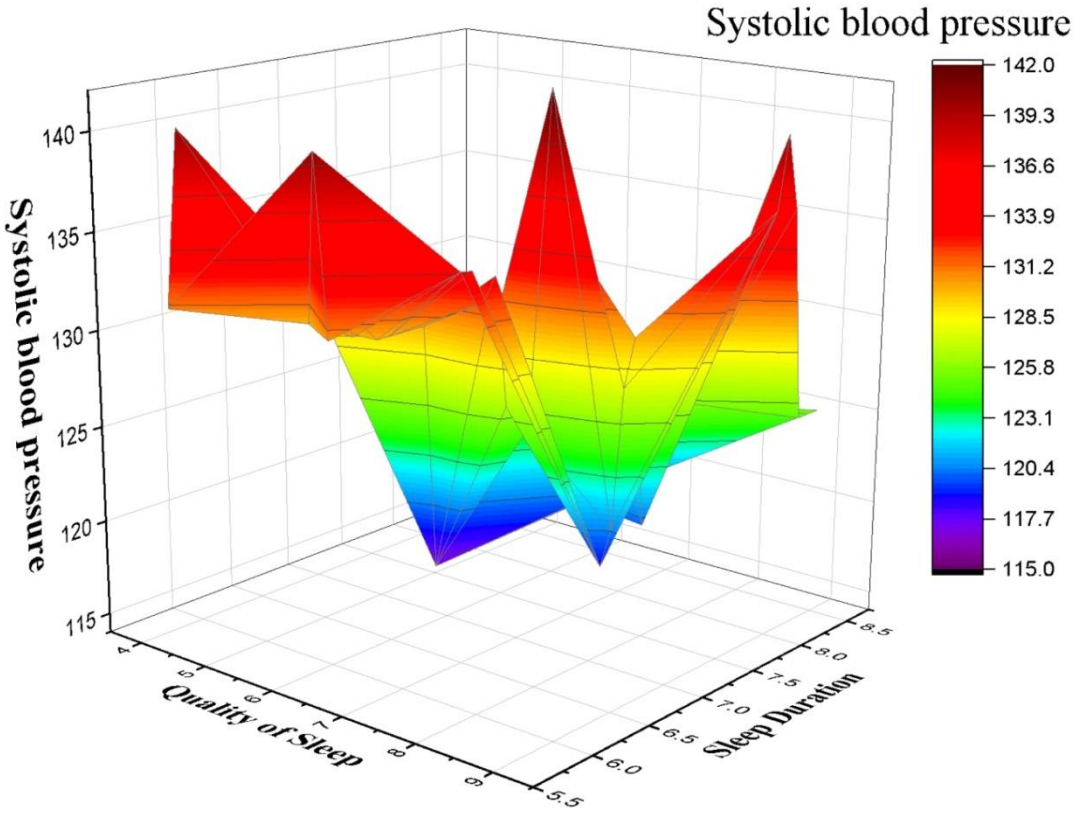

Figure 7Surface plot between quality of sleep, systolic blood pressure& sleep duration

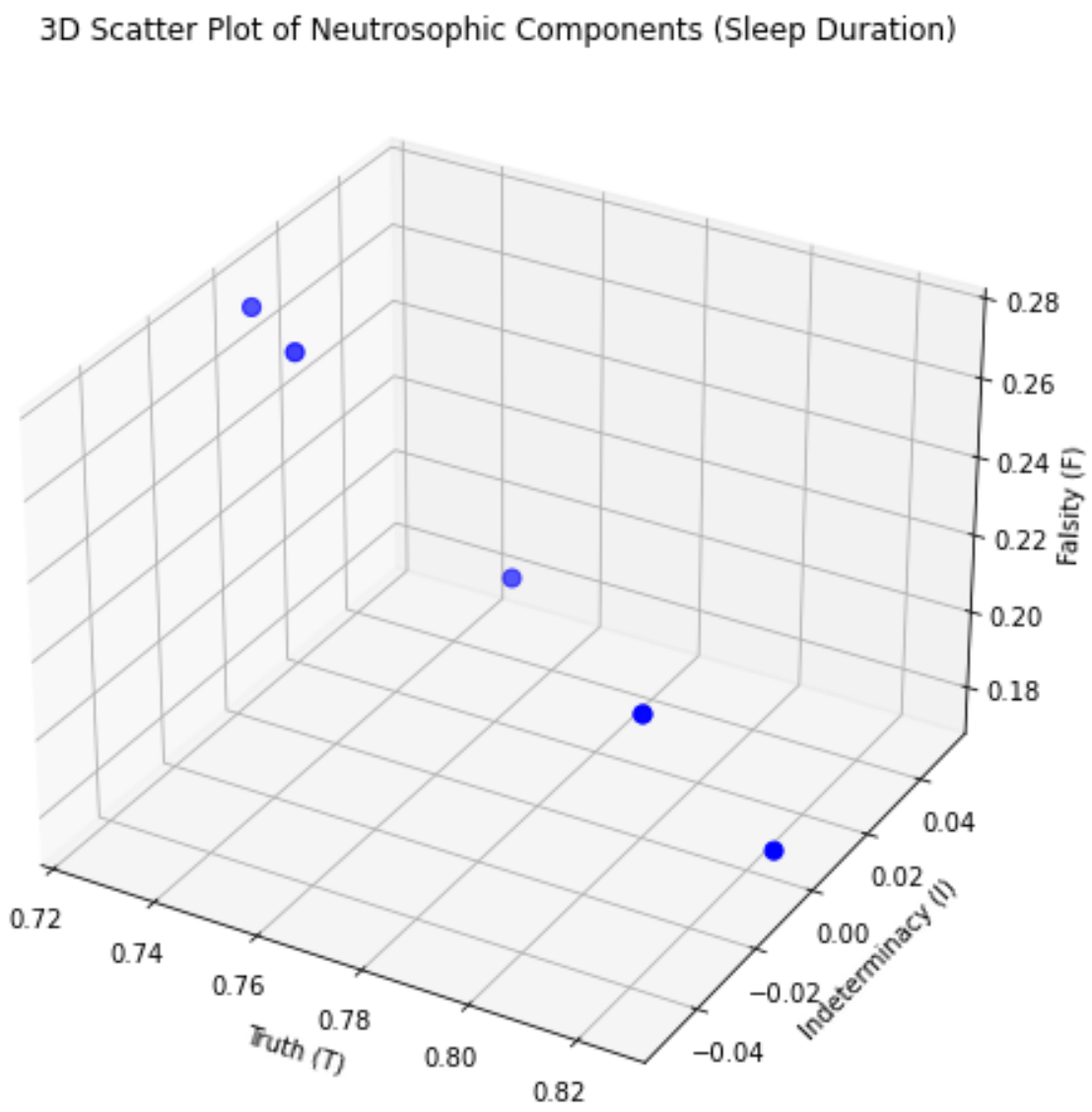

Figure 8 3D scatter plot of Neutrosophic components (Sleep duration)

6. Discussion

Neutrosophic logic in machine learning models has recently shown a remarkably efficient use towards sleep disorder classifications and proved superiority over others handling uncertainty, imprecision, or incompleteness commonly associated in medical datasets. The models capture and process uncertainty better than the traditional approaches because sleep-related parameters such as age, sleep duration, quality of sleep, level of physical activity, blood pressure, daily steps, and stress levels are transformed into Neutrosophic sets. For example, the performance metrics of Random Forest and SVM showed higher accuracy with 96% and 94% sensitivity for robust classification of sleep disorders such as insomnia and sleep apnea on complex and noisy datasets. Neutrosophic-based models allow more interpretability of probabilistic classifications as in Naive Bayes but with high precision and

specificity, which are important in reducing false positives. These models excel in balancing sensitivity and specificity so that positive and negative classifications are highly reliable. Further, the improved MCC and F1 scores of all the models show that they can be consistent in their performance despite the variability in the data. Neutrosophic approach also facilitates adaptive learning; thus, these models are very apt for personalized medicine where the patient data may vary significantly. In classifying sleep disorders, where subtle and overlapping symptoms can complicate diagnosis, Neutrosophic-based ML models provide a reliable, flexible, and precise method for predictive analytics, enabling healthcare practitioners to make more informed and confident clinical decisions. A study by zhang et al., develops machine learning models to predict severe sleep disturbances in adolescents, addressing the lack of reliable tools. It evaluates various algorithms, with XGBM achieving the highest AUC (0.872), demonstrating superior predictive performance. Statistical insights reveal a 5.28% incidence rate, enhancing understanding of sleep disturbances [24]. Another study by El-kenawy et al. achieved 95% accuracy in classifying sleep disorders using Logistic Regression, with the Dipper Throated Optimization Algorithm proving effective for feature selection. These results support early detection and personalized treatment of sleep disorders [25].

7. Conclusion

Our research presents an effective approach for predicting sleep disorders by integrating neutrosophic logic with machine learning models. The results demonstrate that this combination significantly enhances accuracy, making it a valuable tool for clinicians dealing with uncertain medical data. The ability of neutrosophic models to handle complex and ambiguous datasets ensures more reliable predictions, which can improve diagnosis and treatment decisions.One of the key contributions of our study is the application of neutrosophic logic in clinical research, addressing the limitations of traditional machine learning models when managing uncertainty. This approach strengthens model robustness and provides greater interpretability in medical decision-making. Despite its advantages, our research has some limitations. The dataset, though comprehensive, requires further validation with diverse populations to ensure generalizability. Additionally, the computational complexity of neutrosophic models may pose challenges in real-time clinical applications, necessitating optimization for practical use.

Future research will focus on enhancing the scalability and efficiency of these models for broader healthcare applications. We also aim to explore deep learning integration to further improve predictive capabilities. This work lays the foundation for advanced decision-support systems that can revolutionize sleep disorder diagnosis and treatment.

Conflicts of Interest: The authors declare no conflict of interest.

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Performance Evaluation of Human Resource Management in Power Enterprises using Multi-Criteria Decision-Making Methodology and TreeSoft Set

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Abstract: Human Resource Management (HRM) plays a crucial role in the operational success and sustainability of power enterprises. So, this study proposes a multi-criteria decision-making approach (MCDM) for Performance Evaluation of Human Resource Management in Power Enterprises. Two MCDM methods are used in this study such as CIMAS to compute the criteria weights and COPRAS method to rank the alternatives. We used the TreeSoft set with the MCDM approach to deal with various criteria and sub criteria. We divide the problem into three levels such as first level present root node, second level presents the criteria, and third level presents the sub criteria. Then we propose a case study to show the validation of the proposed approach. We applied comparative analysis to show the effectiveness of the proposed approach.

Keywords: TreeSoft Set; MCDM Method; Human Resource Management; Power Enterprises; COPRAS.

1. Introduction

Human Resource Management (HRM) plays a crucial role in the operational success and sustainability of power enterprises. As the energy sector continues to evolve with technological advancements, regulatory changes, and workforce transformations, the ability of HRM to effectively manage talent, optimize productivity, and foster employee engagement has become a key determinant of organizational performance. Unlike other industries, power enterprises require highly skilled professionals to handle complex systems, ensure safety compliance, and drive innovation in energy generation, distribution, and maintenance[1], [2]. Therefore, evaluating the performance of HRM in this sector is essential for ensuring long-term competitiveness and efficiency. The performance evaluation of HRM in power enterprises involves multiple key criteria, including employee productivity and efficiency, talent development and training, employee satisfaction and retention, and HR technology digitalization. These factors influence the overall effectiveness of workforce management,

determining how well employees adapt to industry changes, acquire new skills, and remain committed to organizational goals. High employee satisfaction and retention rates lead to a stable and motivated workforce, while advanced HR digitalization streamlines recruitment, training, and performance assessment. By assessing these criteria, power enterprises can identify HRM strengths and areas needing improvement, ultimately enhancing their ability to attract and retain top talent. To ensure comprehensive HRM evaluation, multi-criteria decision-making (MCDM) methodologies can be utilized. These analytical tools provide a structured approach to ranking and comparing HRM strategies across different power enterprises. By integrating qualitative and quantitative assessments, decision-makers can make data-driven improvements to HR policies, training programs, and employee engagement strategies. Additionally, the incorporation of artificial intelligence and HR analytics enables enterprises to track workforce performance in real time, predict future HR trends, and optimize resource allocation for better decision-making[3], [4]. As power enterprises continue to expand and integrate renewable energy sources, HRM strategies must adapt to evolving industry needs. The demand for specialized skills in smart grid technology, sustainable energy management, and automation requires continuous learning and workforce development initiatives. By systematically evaluating HRM performance, power enterprises can enhance employee capabilities, drive operational efficiency, and ensure long-term sustainability in an increasingly competitive energy market. A well-structured HRM evaluation framework serves as a strategic tool for achieving organizational excellence, ensuring that human capital remains a key driver of success in the power sector[5], [6].

1.1 TreeSoft Set

Decision-making in real-world applications often involves complex, uncertain, and hierarchical data structures, requiring advanced mathematical models for effective analysis. The TreeSoft Set is designed to handle hierarchical and multi-level relationships between parameters and objects. Unlike conventional soft sets, where parameters are considered independent, TreeSoft sets organize parameters in a tree-like structure, allowing for a more structured and granular representation of decision-making problems[7], [8]. This hierarchical approach is particularly useful in domains such as MCDM, hierarchical classification, medical diagnosis, and engineering assessments. The fundamental concept of the TreeSoft Set provides a flexible framework for handling vagueness and uncertainty in data. In a TreeSoft set, parameters are arranged in a parent-child relationship, forming a structured tree where each node represents a specific attribute, and its sub-nodes refine the attribute further. This structure enables decision-makers to evaluate both general and detailed aspects of a problem simultaneously[9], [10]. For example, in an employee performance evaluation, the top-level parameter could be "Work Efficiency," with sub-parameters like "Task Completion Time" and "Error Rate" branching out to provide more detailed insights. One of the main advantages of the TreeSoft Set is its ability to model hierarchical decision systems while maintaining computational efficiency. It allows for better organization of criteria dependencies, reducing redundancy and improving interpretability in complex decision-making scenarios. This structured approach is particularly beneficial in engineering, medical

diagnostics, environmental assessment, and intelligent systems, where multiple interdependent factors must be considered[11], [12]. Additionally, the TreeSoft set framework can be integrated with other decision-making methods, to enhance decision accuracy and ranking efficiency. As research in soft computing and decision analysis advances, the TreeSoft Set continues to gain attention for its ability to handle uncertainty in structured data environments. By incorporating a hierarchical decision-making perspective, it provides a robust solution for problems where parameters cannot be treated in isolation[13], [14]. Future developments in this field may focus on fuzzy TreeSoft sets, hybrid models with artificial intelligence, and real-world applications in big data analytics, further expanding its potential in solving complex, multi-layered decision problems.

2. Proposed Model

This section shows the steps of the MCDM methodology to compute the criteria weights and ranking the alternatives.

CIMAS Methodology

This methodology is used to compute the criteria weights[15].

Build the decision matrix

Experts build the decision matrix between the criteria and the alternatives. They used scale between 0.9 to 0.1 to evaluate the criteria and alternatives.

Combine the decision matrix

Accomplish the normalization decision matrix

$$r_{ij} = \frac{x_{ij}}{\sigma_j} \tag{1}$$

Where σ_j refers to the standard deviation

Establish the expert-weighted criteria decision matrix.

$$E_{ij} = r_{ij} q_{ij} \tag{2}$$

Where q_{ij} refers to the wight of experts

Compute the maximum and minimum value of positive and cost criteria.

$$y_{ij} = \max_i E_{ij} \tag{3}$$

$$y_{ij} = \min_i E_{ij} \tag{4}$$

Calculate the matrix disparities between the positive and cost criteria.

$$L_j = y_{ij}^{max} - y_{ij}^{min} \tag{5}$$

Compute the criteria weights.

$$W_j = \frac{L_j}{\sum_{j=1}^{n} L_j} \tag{6}$$

COPRAS Method

This methodology is used to rank the alternatives[16], [17].

Normalize the decision matrix

$$s_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \tag{7}$$

Compute the weighted decision matrix.

$$d_{ij} = s_{ij} w_j \tag{8}$$

Compute the max and min indexes for positive and cost criteria

$$K_{+i} = \sum_{j=1}^{g} d_{ij} \tag{9}$$

$$K_{-i} = \sum_{j=g+1}^{n} d_{ij} \tag{10}$$

Where g refers to the number of positive criteria and n-g refers to the number of negative criteria

Compute the relative value

$$H_o = K_{+i} + \frac{\sum_{i=1}^{m} K_{-i}}{K_{-i} \sum_{i=1}^{m} \frac{1}{K_{-i}}} \tag{11}$$

Rank the alternatives.

TreeSoft Set (TSS)

Let U as a universal set and $P(U)$ is a power set of U. Let a set of criteria such as $R_1, R_2, \dots R_n and\ n \geq 1\ for\ the\ first\ level$. Every criterion $R_i, 1 \leq 1 \leq n$, is formed by sub criteria:

$$\begin{aligned} R_1 &= \{R_{11}, R_{12}, R_{13}, R_{14}, R_{15}\} \\ R_2 &= \{R_{21}, R_{22}, R_{23}, R_{24}, R_{25}\} \\ R_3 &= \{R_{31}, R_{32}, R_{33}, R_{34}, R_{35}\} \\ R_4 &= \{R_{41}, R_{42}, R_{43}, R_{44}, R_{45}\} \\ R_5 &= \{R_{51}, R_{52}, R_{53}, R_{54}, R_{55}\} \\ &\vdots \\ R_n &= \{R_{n1}, R_{n2}, R_{n3}, R_{n4}, R_{n5}\} \end{aligned} \tag{12}$$

Where R_{ij} are sub criteria and these criteria are formed as level in tree. The first level presents the root nodes, the second level presents the main criteria, and the third level presents the sub criteria values. Fig 1 shows the tree nodes.

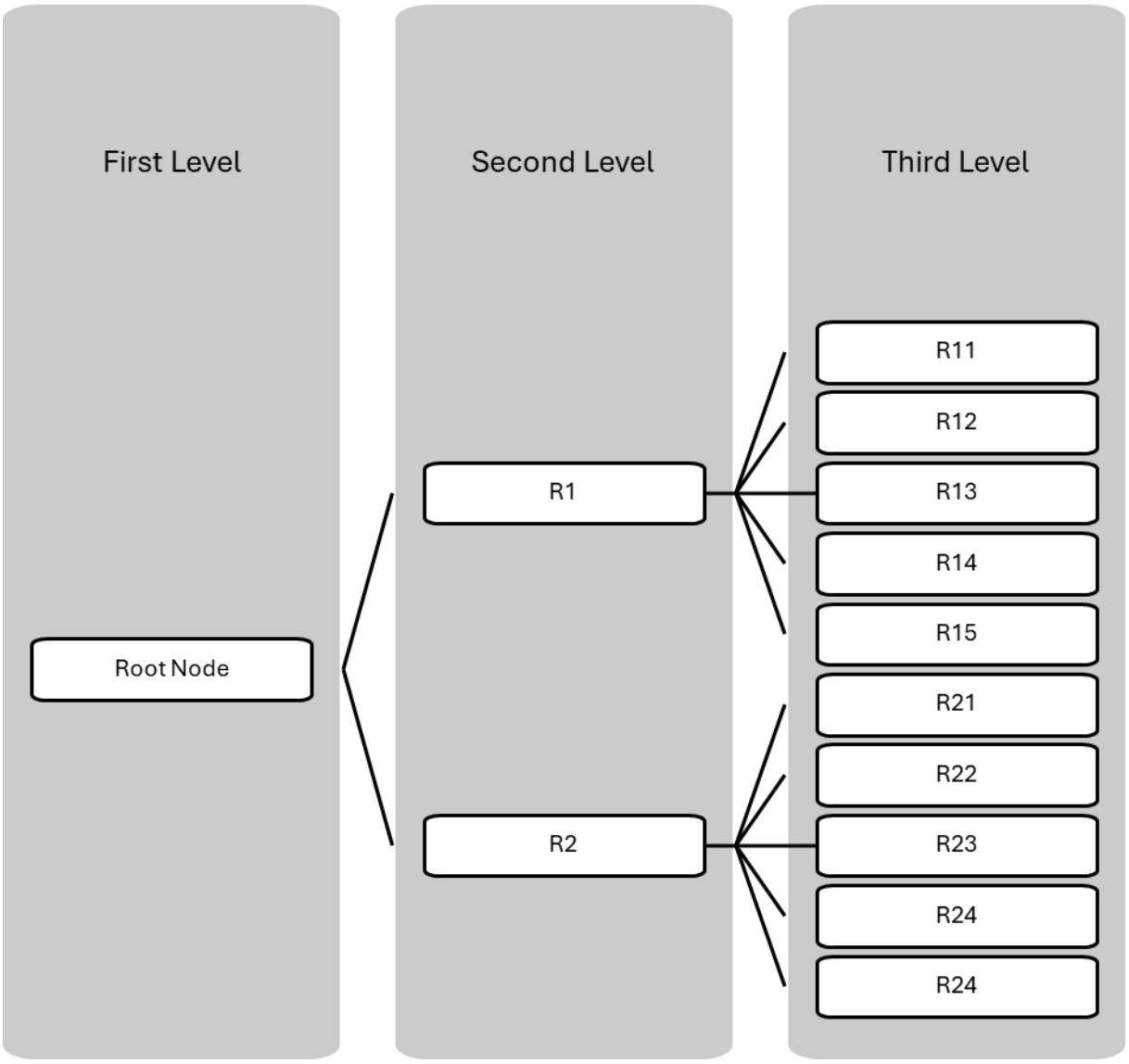

Fig 1. The nodes in TreeSoft set.

The TSS can be formed as:

$$F: P\big(Tree(A)\big) \to P(H) \tag{13}$$

3. Results and Discussion

We applied the steps of two MCDM methods to compute the criteria weights and ranking the alternatives. Four experts are evaluated the criteria and alternatives. They used scale between 0.1 and 0.9. This study uses four criteria and seven alternatives to be evaluated. We used the TreeSoft set to deal with various criteria. Fig 2 shows the different levels of TreeSoft set.

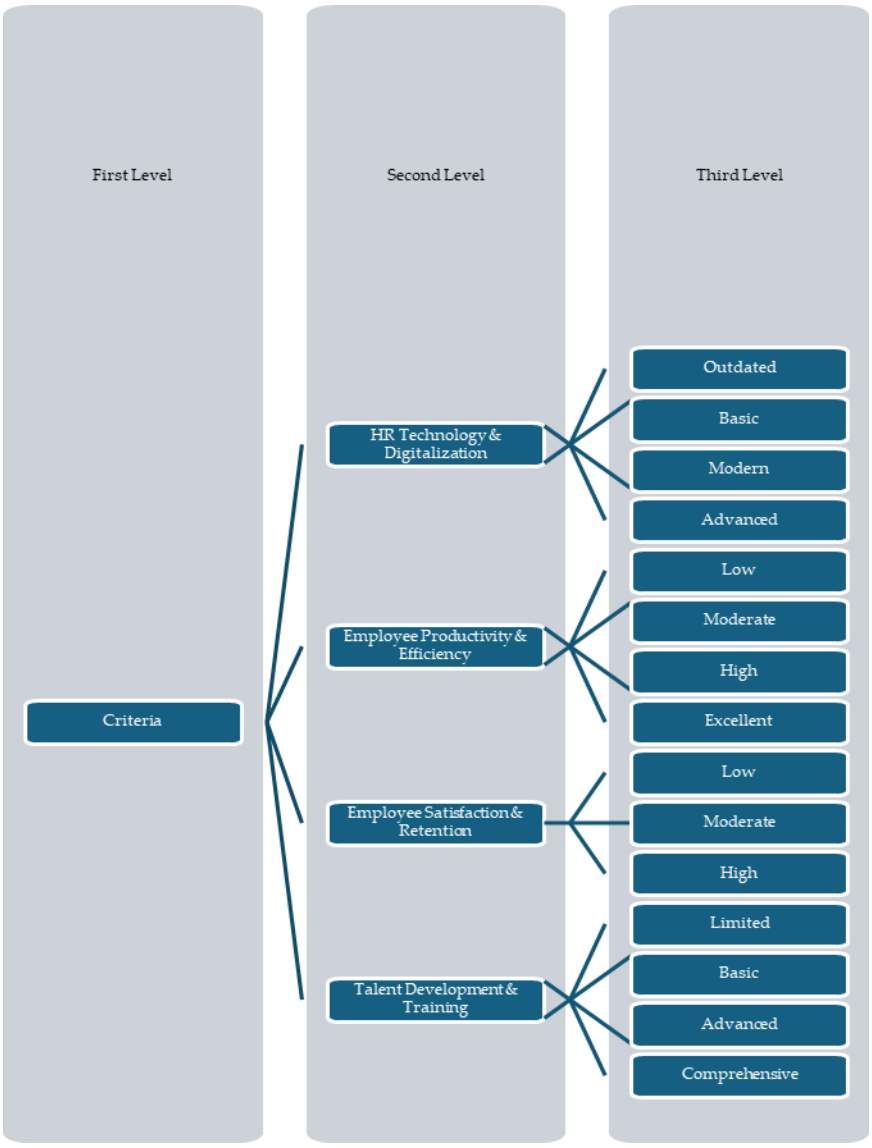

Fig 2. Three different TreeSoft set in this study.

We used the best values in the third level. Then we have four criteria and seven alternatives. Then we applied the CIMAS and COPRAS methods.

CIMAS Methodology Results

We build the decision matrix between the criteria and alternatives using scale between 0.9 to 0.1 as shown in Table 1.

Then we combine the decision matrix

Then we accomplish the normalization decision matrix using eq. (1) as shown in Fig 3.

Then we establish the expert-weighted criteria decision matrix using eq. (2) as shown in Fig 4.

Then we compute the maximum and minimum value of positive and cost criteria.

Then we calculate the matrix disparities between the positive and cost criteria using eq. (5).

Then we compute the criteria weights using eq. (6) as shown in Fig 5.

Table 1. The decision matrix.

	R_1	R_2	R_3	R_4
A_1	0.8	0.7	0.6	0.5
A_2	0.8	0.1	0.2	0.4
A_3	0.7	0.6	0.5	0.4
A_4	0.5	0.6	0.7	0.8
A_5	0.4	0.5	0.6	0.7
A_6	0.4	0.4	0.5	0.4
A_7	0.2	0.2	0.6	0.2
	R_1	R_2	R_3	R_4
A_1	0.5	0.7	0.6	0.5
A_2	0.6	0.1	0.2	0.5
A_3	0.7	0.6	0.5	0.6
A_4	0.8	0.6	0.7	0.7
A_5	0.1	0.5	0.6	0.8
A_6	0.5	0.5	0.5	0.1
A_7	0.6	0.6	0.6	0.2
	R_1	R_2	R_3	R_4
A_1	0.8	0.7	0.6	0.5
A_2	0.1	0.1	0.2	0.4
A_3	0.2	0.8	0.5	0.4
A_4	0.4	0.1	0.8	0.8
A_5	0.5	0.2	0.1	0.8
A_6	0.6	0.4	0.2	0.1
A_7	0.7	0.5	0.4	0.2
	R_1	R_2	R_3	R_4
A_1	0.2	0.1	0.7	0.1
A_2	0.1	0.8	0.8	0.8
A_3	0.1	0.7	0.1	0.7
A_4	0.7	0.7	0.7	0.4
A_5	0.8	0.8	0.8	0.8
A_6	0.1	0.1	0.1	0.7
A_7	0.8	0.6	0.5	0.4

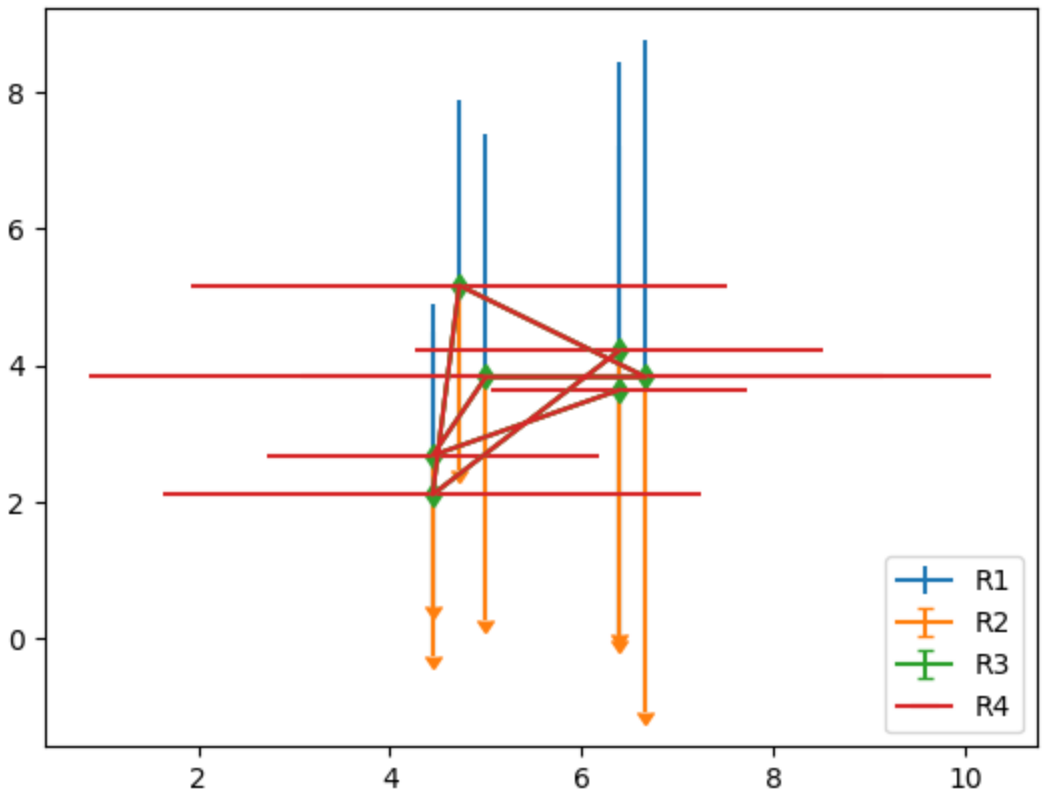

Fig 3. The normalization values.

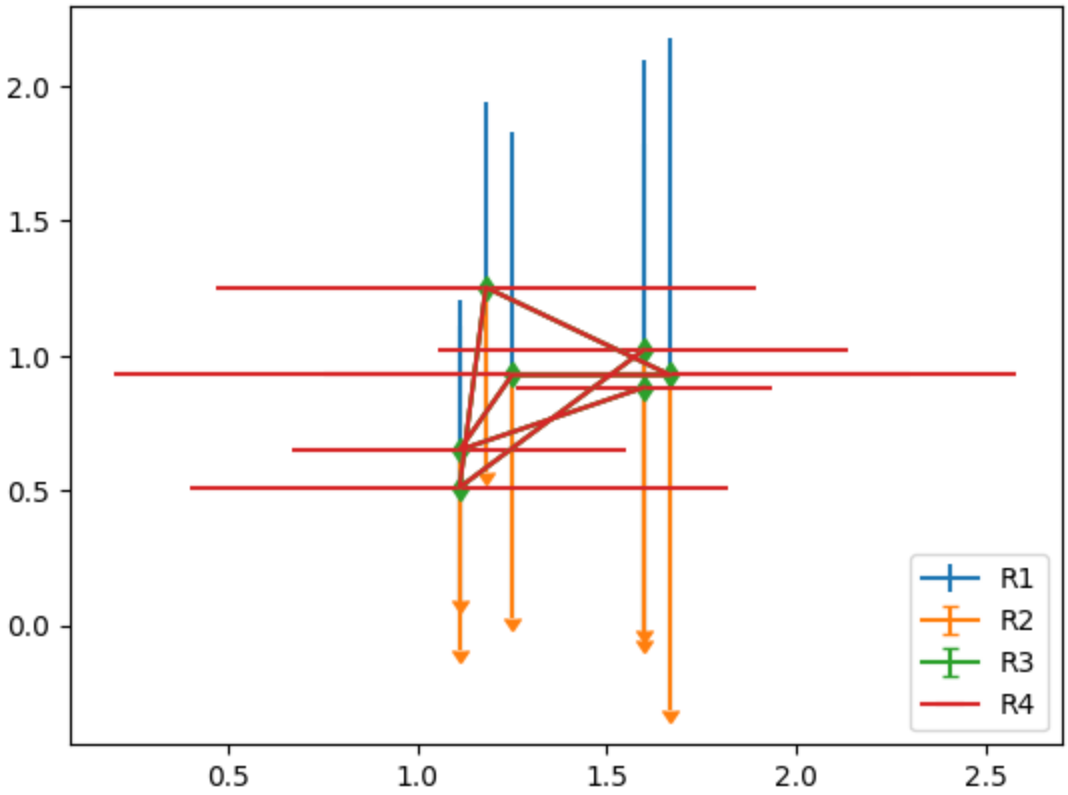

Fig 4. expert-weighted criteria decision matrix.

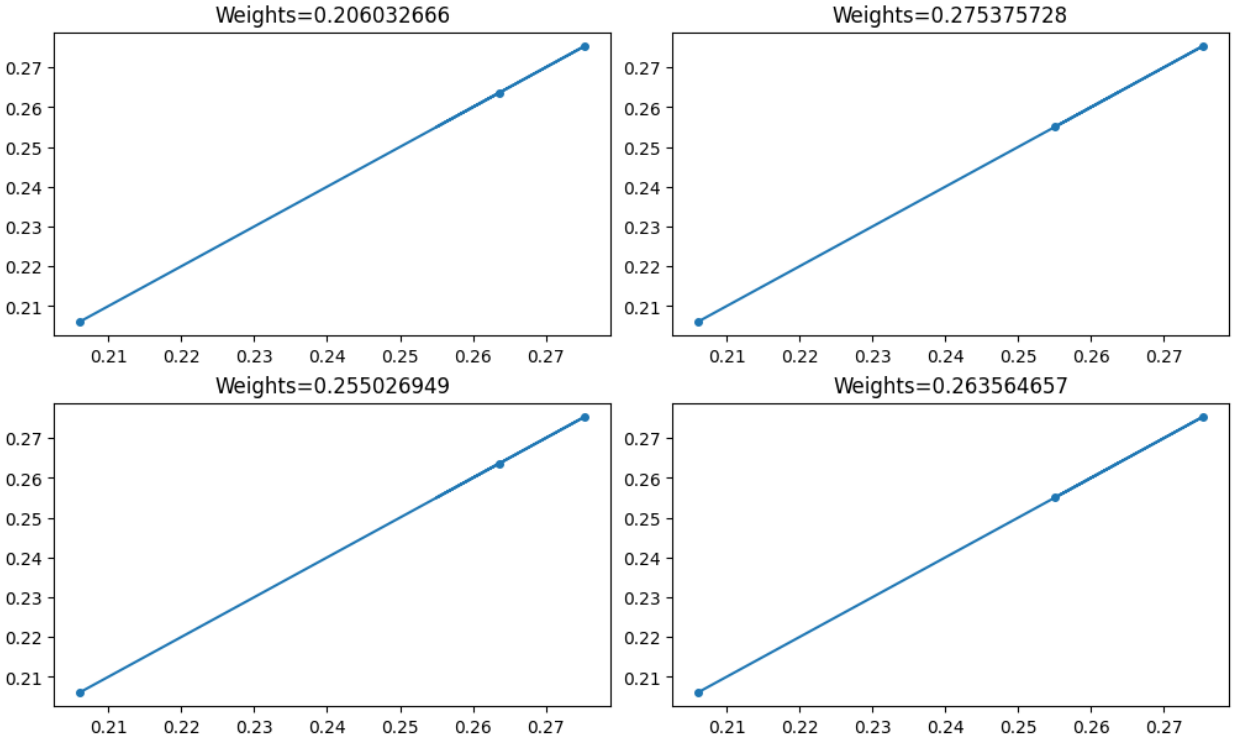

Fig 5. The criteria weights.

COPRAS Method Results

We normalize the decision matrix using Eq. (7) as shown in Fig 6.

We compute the weighted decision matrix using Eq. (8) as shown in Fig 7.

Then we compute the max and min indexes for positive and cost criteria using Eqs. (9 and 10).

Then we compute the relative value using Eq. (11) as shown in Fig 8. Then we rank the alternatives as shown in Fig 9.

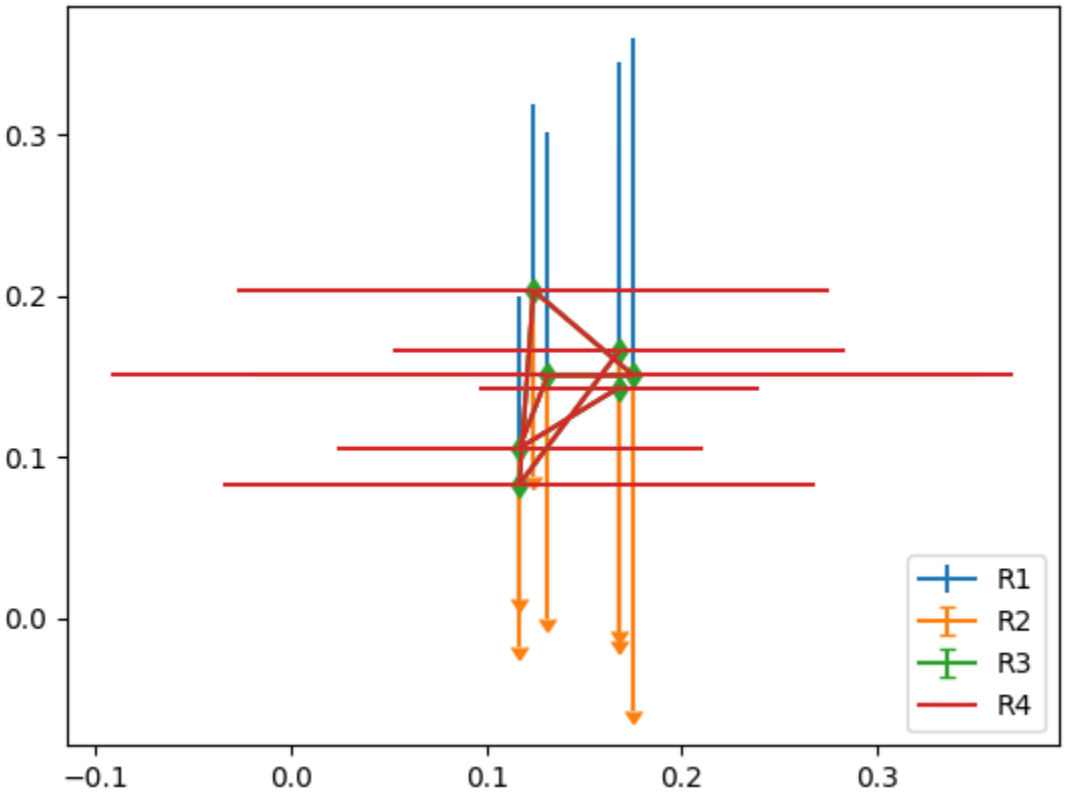

Fig 6. The normalization values by the COPRAS method.

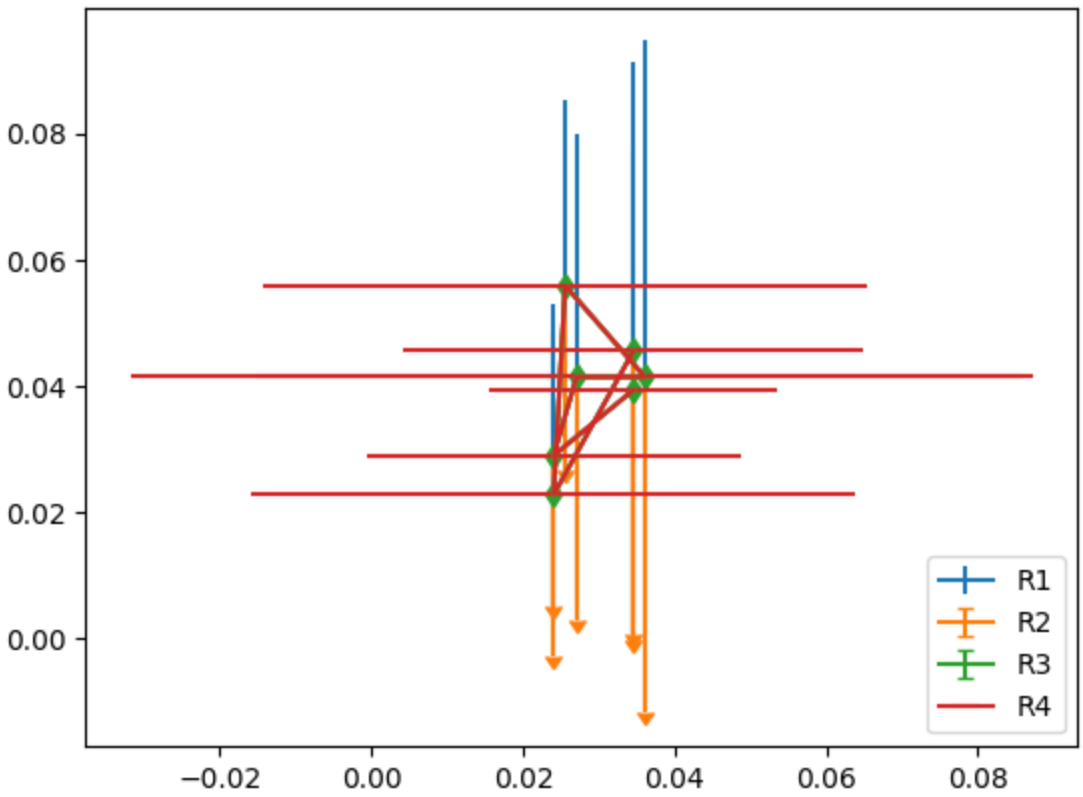

Fig 7. The weighted decision matrix.

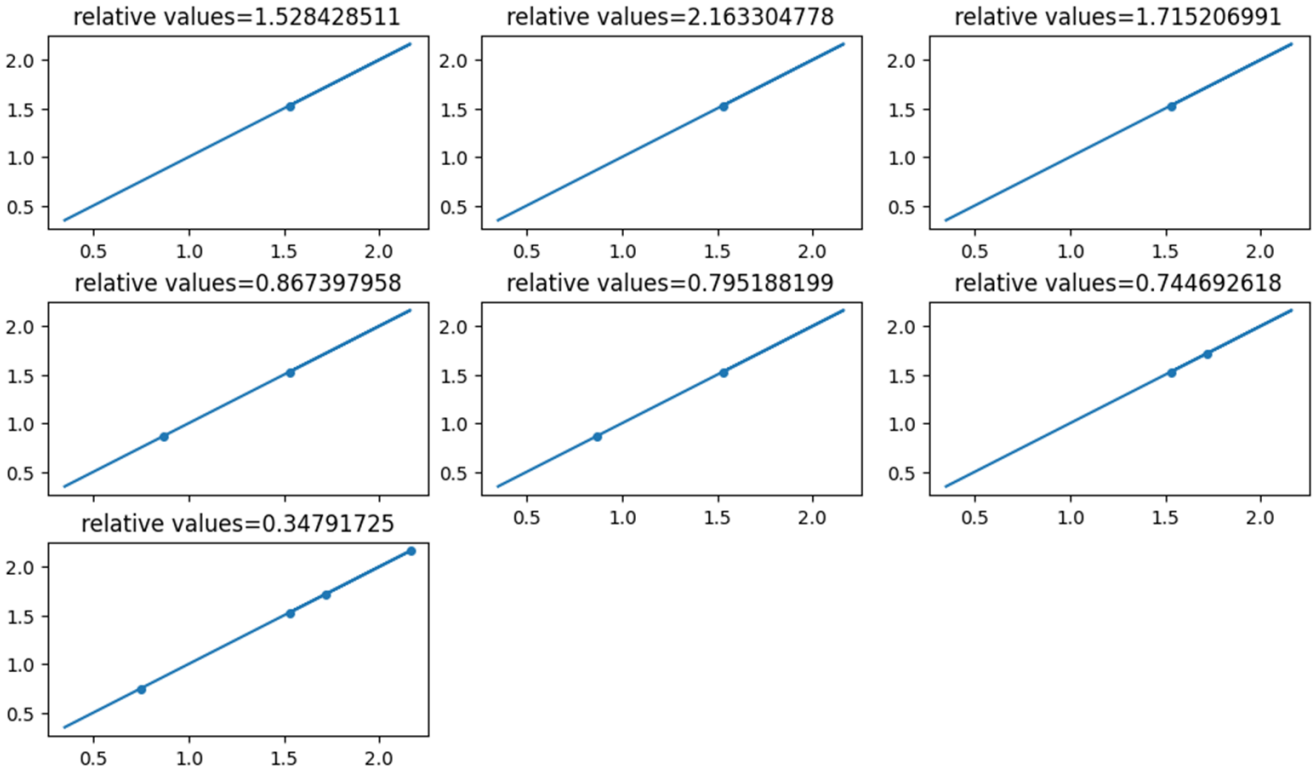

Fig 8. The relative values.

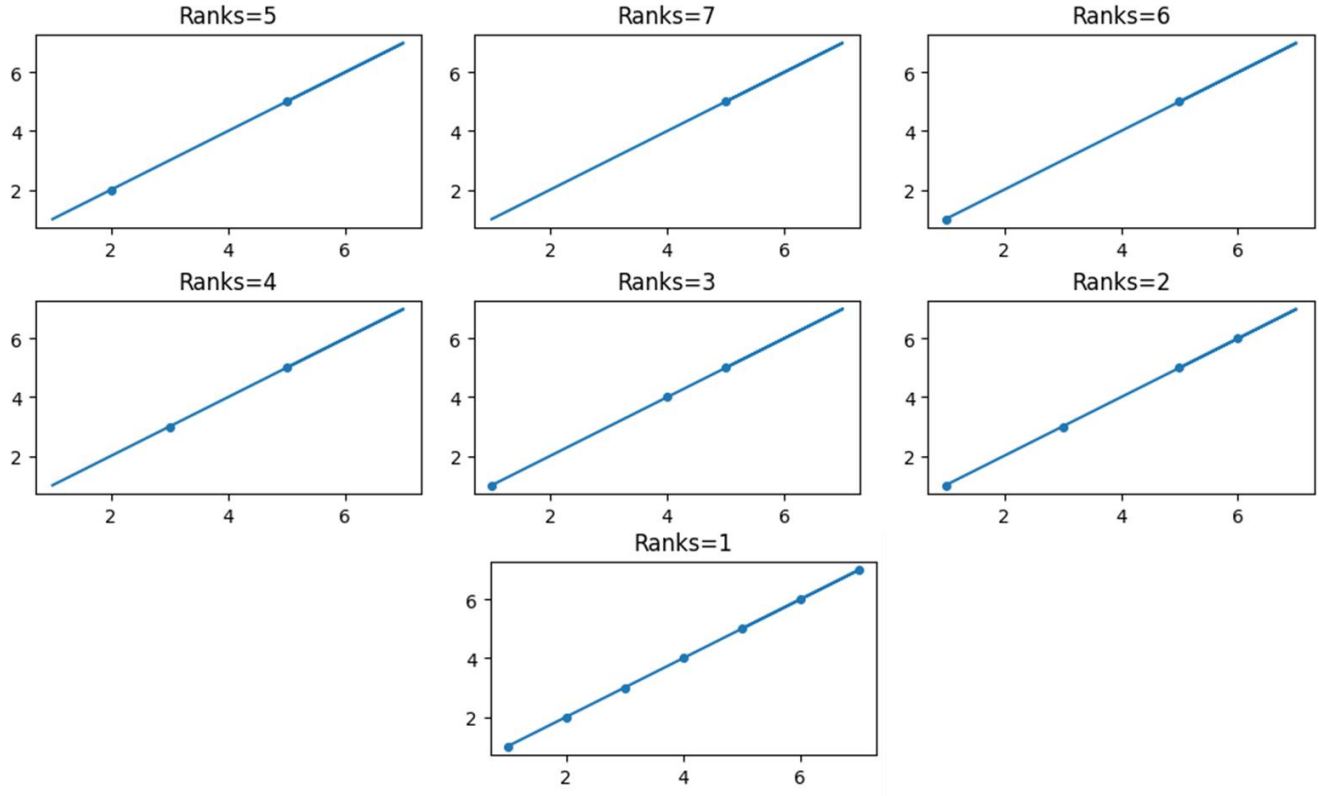

Fig 9. The rank of alternatives.

4. Comparative Analysis

This section shows the comparative analysis between the proposed approach and other MCDM methods to show the validation and effectiveness of the proposed approach. We compared the proposed approach with four different methods. The results show alternative 2 is the best and alternative 7 is the worst. The results show the proposed approach is effective compared to other MCDM methods. Fig 10 shows the comparative ranks.

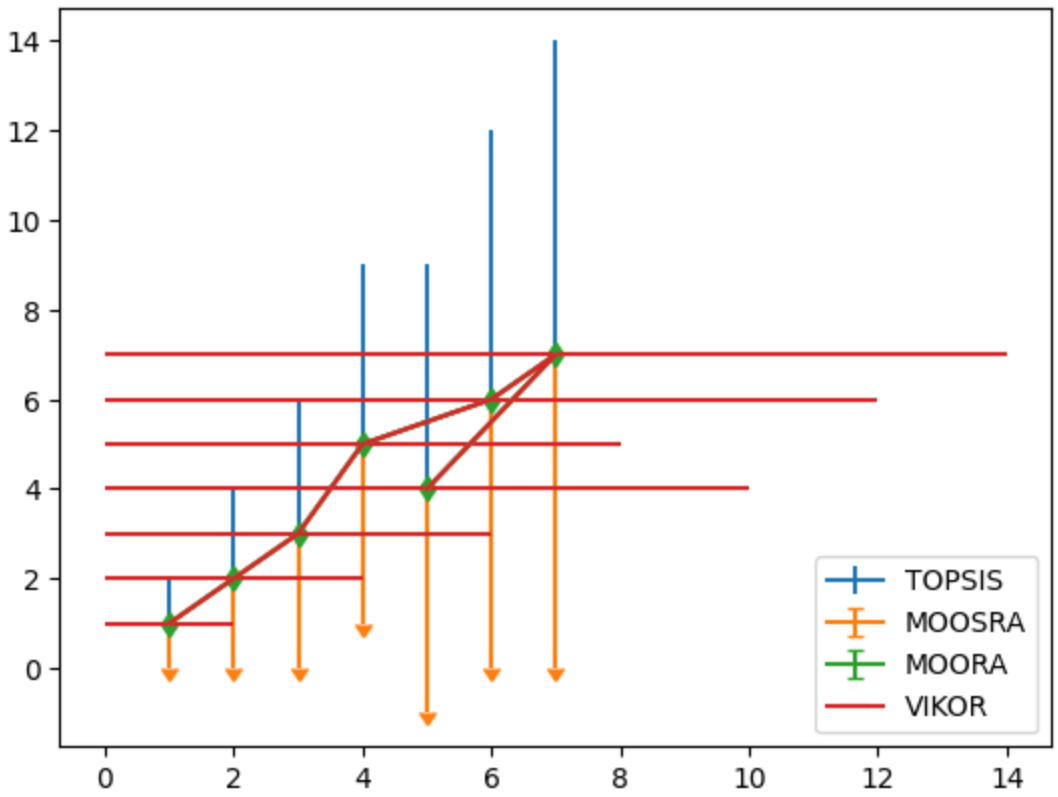

Fig 10. The comparative ranks.

5. Conclusions and Future Study

This study proposed a MCDM approach for Performance Evaluation of Human Resource Management in Power Enterprises. We proposed two MCDM methods such as CIMAS method to compute the criteria weights and the COPRAS method to rank the alternatives. These methods are used with the TreeSoft Set to dela with different criteria and sub criteria. Four criteria and different sub criteria are printed as tree with three levels, then we applied the MCDM methods. The results show alternative 2 is the best and alternative 7 is the worst. TreeSoft set can be used in the future study with other uncertainty models such as fuzzy sets, neutrosophic sets. Different MCDM methods can be used to compute the criteria weights and rank the alternatives.

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$\mathcal{SV}$ −Łukasiewicz Neutrosophic Fuzzy Matrix: A Robust Approach to Identifying an Ultimate Laptop Model

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Abstract: Neutrosophic $\mathcal{F}$uzzy Sets deal with complex, incomplete and imprecise data. For the robust approach in such data, we have constructed an $\mathcal{SV}$ −Łukasiewicz Neutrosophic $\mathcal{F}$uzzy matrix. To approach this study with real-life problem, we took a decision-making problem on choosing an ultimate Laptop Model on behalf of the needs of an individual. We have established the operations and types of $\mathcal{SV}$ −Łukasiewicz Neutrosophic $\mathcal{F}$uzzy matrix. These are benefited in the procedure of achieving the best outcome.

Keywords: $\mathcal{SV}$ − Neutrosophic $\mathcal{F}$uzzy Set, $\mathcal{SV}$ − Łukasiewicz Neutrosophic $\mathcal{F}$uzzy Set, Neutrosophic $\mathcal{F}$uzzy Matrix, $\mathcal{SV}$ −Neutrosophic $\mathcal{F}$uzzy Matrix, $\mathcal{SV}$ − Łukasiewicz Neutrosophic $\mathcal{F}$uzzy Matrix.

1. Introduction

$\mathcal{F}$uzzy is a powerful mathematical technique that forms the basis of set theory and logic. It can handle imprecise, inconsistent and ambiguous data by providing membership grades in the interval [0,1] on the basis of truth or belongingness. In addition to this grade, Atanassov included the non-membership grade in the set theory, for considering the falsity or non-belonginess. It was named as Intuitionistic Fuzzy set. These set theories lack in handling indetermined data. [2] Smarandache handled this lack by introducing Neutrosophic set, which consists of truth, falsity and indeterminacy grades in non-standard intervals. Then, a view of Neutrosophic $\mathcal{F}$uzzy set was first proposed by S. Das et al [7]. This set theory blends with the characteristics of Neutrosophic set and $\mathcal{F}$uzzy set. The elements in the set are in the form of four tuple that defines the membership value with varying

Accuracy, Inaccuracy, and Indecisiveness degrees. This provides more accuracy in handling real-life decision-making problem. Also, it doesn't suit for complex problem due to the representation non-standard intervals for the Neutrosophic components. This issue is overcome by the proposal of $\mathcal{SV}$ −Neutrosophic $\mathscr{F}$uzzy set [9]. As it works with the components of Neutrosophic sets in [0,1]. This helps to work in the complex field of pattern recognition, artificial intelligence and so on. Furthermore, using computational instances, the authors developed a range of set theoretic operations and properties. [10,11] Later, they extended this concept in matrix theory for the utilization of Neutrosophic $\mathscr{F}$uzzy Matrix. This contributes to the creation of Single-valued Neutrosophic $\mathscr{F}$uzzy Matrix or $\mathcal{SV}$ −Neutrosophic $\mathscr{F}$uzzy Matrix, which allows the entries of the matrix to possess the four tuples namely membership value and its degree of Accuracy, Indecisiveness and Inaccuracy [13]. As an extension of set theory to matrix theory, one can work on real life data in various ways with the use of their operations. Some authors incorporated this concept in decision making problem by conducting research in the perspective of algebraic structures and investigated the algebraic operations. Using which some authors constructed the score matrix for the application in decision-making problem by use of matrix operations [14]. This $\mathcal{SV}$ −Neutrosophic $\mathscr{F}$uzzy Matrix gives better and more accurate result in decision-making problem rather than other theories. To bring advancement in this concept, we combined this matrix theory with Łukasiewicz Fuzzy Set theory. It has both logical interpretations and a suitable basis for implementation. This was first proposed by a polish Mathematician, Jan Łukasiewicz, who was a logician and Philosopher. He gave rise to the worthwhile logic called Łukasiewicz logic. This logic provides a powerful framework with diverse application across various disciplines. As a part of it, many professionals incorporate the above logic to fuzzy set theory and presented the idea of Łukasiewicz $\mathscr{F}$uzzy Set in various algebra [15-18].

In this study, we have extended the Łukasiewicz set theory to Matrix theory and fused the concept of Łukasiewicz theory to $\mathcal{SV}$ −Neutrosophic $\mathscr{F}$uzzy Matrix for bringing the discussion on operations of the matrices to earn the betterment of result in decision making problem. We have addressed a specific problem on choosing a best laptop model under the circumstance of user's requirements and budgetary constraints.

2. Preliminaries

2.1. $\mathscr{F}$uzzy Set ($\mathfrak{FS}$)

A $\mathfrak{FS}$ $\dot{\mathbb{A}}$ over $\acute{\mathbb{U}}$ is referred as $\dot{\mathbb{A}} = \{(\ddot{\mathbf{a}}, \boldsymbol{\mu}_{\dot{\mathbb{A}}}(\ddot{\mathbf{a}}))/\ddot{\mathbf{a}} \in \acute{\mathbb{U}}\}$, and $\boldsymbol{\mu}_{\dot{\mathbb{A}}}: \acute{\mathbb{U}} \to [\mathbf{0}, \mathbf{1}]$ be membership function of $\dot{\mathbb{A}}$.

2.2. $\mathcal{SV}$ −Neutrosophic Set (𝔖𝔑𝔖)

A 𝔖𝔑𝔖 $\dot{\mathbb{B}}$ over $\acute{\mathrm{U}}$ is referred as

$$\dot{\mathbb{B}} = \left\{\left(\ddot{\mathbf{a}}, \acute{\mathcal{T}}_{\dot{\mathbb{B}}}(\ddot{\mathbf{a}}), \acute{\mathcal{I}}_{\dot{\mathbb{B}}}(\ddot{\mathbf{a}}), \acute{\mathcal{F}}_{\dot{\mathbb{B}}}(\ddot{\mathbf{a}})\right) | \ddot{\mathbf{a}} \in \acute{\mathrm{U}}\right\},$$

where $\acute{\mathcal{T}}_{\dot{\mathbb{B}}}, \acute{\mathcal{I}}_{\dot{\mathbb{B}}}$ and $\acute{\mathcal{F}}_{\dot{\mathbb{B}}}$ are functions of Accuracy, Indecisiveness, and Inaccuracy degrees respectively. For each point $\ddot{\mathbf{a}}$ in $\acute{\mathrm{U}}$, $\acute{\mathcal{T}}_{\dot{\mathbb{B}}}, \acute{\mathcal{I}}_{\dot{\mathbb{B}}}, \acute{\mathcal{F}}_{\dot{\mathbb{B}}} \in [\mathbf{0}, \mathbf{1}]$.

2.3. $\mathcal{SV}$ −Neutrosophic $\mathcal{F}$uzzy Set (𝔑𝔉𝔖)

A 𝔑𝔉𝔖 $\dot{\mathbb{C}}$ on $\acute{\mathrm{U}}$ is referred as

$$\dot{\mathbb{C}} = \left\{\left(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}}_{\dot{\mathbb{C}}}(\ddot{\mathbf{a}}), \acute{\mathcal{T}}_{\dot{\mathbb{C}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{I}}_{\dot{\mathbb{C}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{F}}_{\dot{\mathbb{C}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}})\right) | \ddot{\mathbf{a}} \in \acute{\mathrm{U}}\right\}$$

where its membership value $\acute{\boldsymbol{\mu}}_{\dot{\mathbb{C}}}$ is described by its Accuracy $\acute{\mathcal{T}}_{\dot{\mathbb{C}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}})$, Indecisiveness $\acute{\mathcal{I}}_{\dot{\mathbb{C}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}})$ and Inaccuracy $\acute{\mathcal{F}}_{\dot{\mathbb{C}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}})$functions. Moreover $\acute{\mathcal{T}}_{\dot{\mathbb{C}}}, \acute{\mathcal{I}}_{\dot{\mathbb{C}}}$ and $\acute{\mathcal{F}}_{\dot{\mathbb{C}}}$ are either standard or non-standard form of $]\mathbf{0}^-, \mathbf{1}^+[$. That is, $\acute{\mathcal{T}}_{\dot{\mathbb{C}}}: \acute{\mathrm{U}} \to]\mathbf{0}^-, \mathbf{1}^+[$, $\acute{\mathcal{I}}_{\dot{\mathbb{C}}}: \acute{\mathrm{U}} \to]\mathbf{0}^-, \mathbf{1}^+[$ and $\acute{\mathcal{F}}_{\dot{\mathbb{C}}}: \acute{\mathrm{U}} \to]\mathbf{0}^-, \mathbf{1}^+[$. Also $\mathbf{0}^- \leq \boldsymbol{Sup}\left(\acute{\mathcal{T}}_{\dot{\mathbb{C}}}\right) + \boldsymbol{Sup}\left(\acute{\mathcal{I}}_{\dot{\mathbb{C}}}\right) + \boldsymbol{Sup}\left(\acute{\mathcal{F}}_{\dot{\mathbb{C}}}\right) \leq \mathbf{3}^+$.

2.4. $\mathcal{SV}$ −Neutrosophic $\mathcal{F}$uzzy Set (𝔖𝔑𝔉𝔖)

The 𝔖𝔑𝔉𝔖 $\dot{\mathbb{D}}$ on $\acute{\mathrm{U}}$ is defined by

$$\dot{\mathbb{D}} = \left\{\left(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}}_{\dot{\mathbb{D}}}(\ddot{\mathbf{a}}), \acute{\mathcal{T}}_{\dot{\mathbb{D}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{I}}_{\dot{\mathbb{D}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{F}}_{\dot{\mathbb{D}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}})\right) | \ddot{\mathbf{a}} \in \acute{\mathrm{U}}\right\},$$

where $\acute{\mathcal{T}}_{\dot{\mathbb{C}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}})$, $\acute{\mathcal{I}}_{\dot{\mathbb{C}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{F}}_{\dot{\mathbb{C}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}}) \in [\mathbf{0}, \mathbf{1}]$ and $\mathbf{0} \leq \acute{\mathcal{T}}_{\dot{\mathbb{C}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}}) + \acute{\mathcal{I}}_{\dot{\mathbb{C}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}}) + \acute{\mathcal{F}}_{\dot{\mathbb{C}}}(\ddot{\mathbf{a}}, \acute{\boldsymbol{\mu}}) \leq \mathbf{3}$.

2.5. $\mathcal{SV}$ −Neutrosophic Matrix (𝔖𝔑𝔐)

A 𝔖𝔑𝔐 is defined as

$$\dot{\mathbb{M}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\hat{p}\times\hat{q}} = \left[\langle \ddot{\mathbf{a}}_{ij}^{\acute{\mathcal{T}}}, \ddot{\mathbf{a}}_{ij}^{\acute{\mathcal{I}}}, \ddot{\mathbf{a}}_{ij}^{\acute{\mathcal{F}}} \rangle\right]_{\hat{p}\times\hat{q}}$$

where $\ddot{\mathbf{a}}_{ij}^{\acute{\mathcal{T}}}, \ddot{\mathbf{a}}_{ij}^{\acute{\mathcal{I}}}, \ddot{\mathbf{a}}_{ij}^{\acute{\mathcal{F}}}$ are referred to as Accuracy, Indecisiveness and Inaccuracy degrees of the $\boldsymbol{ij^{th}}$ element in $\dot{\mathbb{M}}$. These degrees satisfy $\mathbf{0} \leq \ddot{\mathbf{a}}_{ij}^{\acute{\mathcal{T}}} + \ddot{\mathbf{a}}_{ij}^{\acute{\mathcal{I}}} + \ddot{\mathbf{a}}_{ij}^{\acute{\mathcal{F}}} \leq \mathbf{3}$ $\forall\, \boldsymbol{i}, \boldsymbol{j}$.

2.6. Neutrosophic $\mathcal{F}$uzzy Matrix (𝔑𝔉𝔐)

The 𝔑𝔉𝔐 is defined as

$$\dot{\mathbb{H}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\hat{p}\times\hat{q}} = \left[\langle \ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}_{\dot{\mathbb{H}}}\left(\ddot{\mathbf{a}}_{ij}\right), \acute{\mathcal{T}}_{\dot{\mathbb{H}}}\left(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}\right), \acute{\mathcal{I}}_{\dot{\mathbb{H}}}\left(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}\right), \acute{\mathcal{F}}_{\dot{\mathbb{H}}}\left(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}\right)\rangle\right]_{\hat{p}\times\hat{q}},$$

where its membership value has been represented by an Accuracy, Indecisiveness and Inaccuracy degrees which are subsets of either standard or non-standard form of $]\mathbf{0}^{-}, \mathbf{1}^{+}[$ and $\mathbf{0}^{-} \leq \boldsymbol{Sup}(\dot{\mathcal{T}}_{\dot{\mathbb{H}}}) + \boldsymbol{Sup}(\dot{\mathcal{J}}_{\dot{\mathbb{H}}}) + \boldsymbol{Sup}(\dot{\mathcal{F}}_{\dot{\mathbb{H}}}) \leq \mathbf{3}^{+}$.

2.7. $\mathcal{SV}$ −Neutrosophic $\mathscr{F}$uzzy Matrix ($\mathfrak{SNFM}$)

The $\mathfrak{SNFM}$ is defined as

$$\dot{\mathbb{G}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\hat{p}\times\hat{q}} = \left[\dot{\boldsymbol{\mu}}_{\dot{\mathbb{G}}}(\ddot{\mathbf{a}}_{ij}), \dot{\mathcal{T}}_{\dot{\mathbb{G}}}(\ddot{\mathbf{a}}_{ij}, \dot{\boldsymbol{\mu}}), \dot{\mathcal{J}}_{\dot{\mathbb{G}}}(\ddot{\mathbf{a}}_{ij}, \dot{\boldsymbol{\mu}}), \dot{\mathcal{F}}_{\dot{\mathbb{G}}}(\ddot{\mathbf{a}}_{ij}, \dot{\boldsymbol{\mu}})\right]_{\hat{p}\times\hat{q}},$$

where its membership value has been represented by an Accuracy, Indecisiveness and Inaccuracy membership and $\dot{\mathcal{T}}_{\dot{\mathbb{G}}}(\ddot{\mathbf{a}}_{ij}, \dot{\boldsymbol{\mu}}),\ \dot{\mathcal{J}}_{\dot{\mathbb{G}}}(\ddot{\mathbf{a}}_{ij}, \dot{\boldsymbol{\mu}}), \dot{\mathcal{F}}_{\dot{\mathbb{G}}}(\ddot{\mathbf{a}}_{ij}, \dot{\boldsymbol{\mu}}) \in [\mathbf{0}, \mathbf{1}]$ which satisfy the condition that $\mathbf{0} \leq \dot{\mathcal{T}}_{\dot{\mathbb{G}}}(\ddot{\mathbf{a}}_{ij}, \dot{\boldsymbol{\mu}}) + \dot{\mathcal{J}}_{\dot{\mathbb{G}}}(\ddot{\mathbf{a}}_{ij}, \dot{\boldsymbol{\mu}}) + \dot{\mathcal{F}}_{\dot{\mathbb{G}}}(\ddot{\mathbf{a}}_{ij}, \dot{\boldsymbol{\mu}}) \leq \mathbf{3}$.

3. $\mathcal{SV}$ −Łukasiewicz Neutrosophic $\mathscr{F}$uzzy Matrix

3.1. Definitions

3.1.1. $\ddot{\boldsymbol{\theta}}-$ Łukasiewicz Neutrosophic $\mathscr{F}$uzzy Set ($\mathfrak{LNFS}$)

Let $\dot{\mathbb{A}}$ be a $\mathfrak{FS}$ in $\dot{\mathbb{U}}$ and let $\ddot{\boldsymbol{\theta}} = [\mathbf{0}, \mathbf{1}]$. The $\mathfrak{LNFS}$ $Ł_{\dot{\mathbb{A}}}^{\ddot{\boldsymbol{\theta}}}$ is defined as

$$Ł_{\dot{\mathbb{A}}}^{\ddot{\boldsymbol{\theta}}}: \dot{\mathbb{U}} \to [\mathbf{0}, \mathbf{1}], \ddot{\mathbf{a}} \to \boldsymbol{max}\{\mathbf{0}, \dot{\mathbb{A}}(\ddot{\mathbf{a}}) + \ddot{\boldsymbol{\theta}} - \mathbf{1}\}.$$

3.1.2. $\mathcal{SV}-\ddot{\boldsymbol{\theta}}-$ Łukasiewicz Neutrosophic $\mathscr{F}$uzzy Set ($\mathfrak{SLNFS}$)

Let $\dot{\mathbb{D}} = (\dot{\boldsymbol{\mu}}_{\dot{\mathbb{D}}}(\ddot{\mathbf{a}}), \dot{\mathcal{T}}_{\dot{\mathbb{D}}}(\ddot{\mathbf{a}}, \dot{\boldsymbol{\mu}}), \dot{\mathcal{J}}_{\dot{\mathbb{D}}}(\ddot{\mathbf{a}}, \dot{\boldsymbol{\mu}}), \dot{\mathcal{F}}_{\dot{\mathbb{D}}}(\ddot{\mathbf{a}}, \dot{\boldsymbol{\mu}}))$ be a $\mathfrak{SNFS}$ in $\dot{\mathbb{U}}$ and let $\ddot{\boldsymbol{\theta}} = [\mathbf{0}, \mathbf{1}]$. A $\mathfrak{SLNFS}$ $Ł_{\dot{\mathbb{D}}}^{\ddot{\boldsymbol{\theta}}}$ is defined as

$$Ł_{\dot{\mathbb{D}}}^{\ddot{\boldsymbol{\theta}}} = \left\{(\dot{\boldsymbol{\mu}}_{Ł_{\dot{\mathbb{D}}}^{\ddot{\boldsymbol{\theta}}}}(\ddot{\mathbf{a}}), \dot{\mathcal{T}}_{Ł_{\dot{\mathbb{D}}}^{\ddot{\boldsymbol{\theta}}}}(\ddot{\mathbf{a}}, \dot{\boldsymbol{\mu}}), \dot{\mathcal{J}}_{Ł_{\dot{\mathbb{D}}}^{\ddot{\boldsymbol{\theta}}}}(\ddot{\mathbf{a}}, \dot{\boldsymbol{\mu}}), \dot{\mathcal{F}}_{Ł_{\dot{\mathbb{D}}}^{\ddot{\boldsymbol{\theta}}}}(\ddot{\mathbf{a}}, \dot{\boldsymbol{\mu}}))\middle|\ddot{\mathbf{a}} \in \dot{\mathbb{U}}\right\},$$

where $\dot{\boldsymbol{\mu}}_{Ł_{\dot{\mathbb{D}}}^{\ddot{\boldsymbol{\theta}}}}: \dot{\mathbb{U}} \to [\mathbf{0}, \mathbf{1}], \ddot{\mathbf{a}} \to \boldsymbol{max}\{\mathbf{0}, \dot{\boldsymbol{\mu}}_{\dot{\mathbb{D}}}(\ddot{\mathbf{a}}) + \ddot{\boldsymbol{\theta}} - \mathbf{1}\}$, $\dot{\mathcal{T}}_{Ł_{\dot{\mathbb{D}}}^{\ddot{\boldsymbol{\theta}}}}: \dot{\mathbb{U}} \to [\mathbf{0}, \mathbf{1}], \ddot{\mathbf{a}} \to \boldsymbol{max}\{\mathbf{0}, \dot{\mathcal{T}}_{\dot{\mathbb{D}}}(\ddot{\mathbf{a}}) + \ddot{\boldsymbol{\theta}} - \mathbf{1}\}$, $\dot{\mathcal{J}}_{Ł_{\dot{\mathbb{D}}}^{\ddot{\boldsymbol{\theta}}}}: \dot{\mathbb{U}} \to [\mathbf{0}, \mathbf{1}], \ddot{\mathbf{a}} \to \boldsymbol{max}\{\mathbf{0}, \dot{\mathcal{J}}_{\dot{\mathbb{D}}}(\ddot{\mathbf{a}}) + \ddot{\boldsymbol{\theta}} - \mathbf{1}\}$ and $\dot{\mathcal{F}}_{Ł_{\dot{\mathbb{D}}}^{\ddot{\boldsymbol{\theta}}}}: \dot{\mathbb{U}} \to [\mathbf{0}, \mathbf{1}], \ddot{\mathbf{a}} \to \boldsymbol{max}\{\mathbf{0}, \dot{\mathcal{F}}_{\dot{\mathbb{D}}}(\ddot{\mathbf{a}}) + \ddot{\boldsymbol{\theta}} - \mathbf{1}\}$ are membership value and its Accuracy, Indecisiveness and Inaccuracy memberships, which satisfies the condition that $\mathbf{0} \leq \dot{\mathcal{T}}_{Ł_{\dot{\mathbb{D}}}^{\ddot{\boldsymbol{\theta}}}}(\ddot{\mathbf{a}}, \dot{\boldsymbol{\mu}}) + \dot{\mathcal{J}}_{Ł_{\dot{\mathbb{D}}}^{\ddot{\boldsymbol{\theta}}}}(\ddot{\mathbf{a}}, \dot{\boldsymbol{\mu}}) + \dot{\mathcal{F}}_{Ł_{\dot{\mathbb{D}}}^{\ddot{\boldsymbol{\theta}}}}(\ddot{\mathbf{a}}, \dot{\boldsymbol{\mu}}) \leq \mathbf{3}$.

3.1.3. $\mathcal{SV}-\ddot{\boldsymbol{\theta}}-$ Łukasiewicz Neutrosophic $\mathscr{F}$uzzy Matrix ($\mathfrak{SLNFM}$)

The $\mathfrak{SLNFM}$ is defined as

$$Ł_{\dot{\mathbb{G}}}^{\ddot{\boldsymbol{\theta}}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\hat{p}\times\hat{q}} = \left[(\dot{\boldsymbol{\mu}}_{Ł_{\dot{\mathbb{G}}}^{\ddot{\boldsymbol{\theta}}}}(\ddot{\mathbf{a}}_{ij}), \dot{\mathcal{T}}_{Ł_{\dot{\mathbb{G}}}^{\ddot{\boldsymbol{\theta}}}}(\ddot{\mathbf{a}}_{ij}, \dot{\boldsymbol{\mu}}), \dot{\mathcal{J}}_{Ł_{\dot{\mathbb{G}}}^{\ddot{\boldsymbol{\theta}}}}(\ddot{\mathbf{a}}_{ij}, \dot{\boldsymbol{\mu}}), \dot{\mathcal{F}}_{Ł_{\dot{\mathbb{G}}}^{\ddot{\boldsymbol{\theta}}}}(\ddot{\mathbf{a}}_{ij}, \dot{\boldsymbol{\mu}})\right]_{\hat{p}\times\hat{q}},$$

where its membership value has been expressed by an Accuracy, Indecisiveness and Inaccuracy degrees as $\acute{\boldsymbol{\mu}}_{Ł_{\mathbb{G}}^{\ddot{\theta}}}: \acute{\mathbb{U}} \to [\mathbf{0},\mathbf{1}], \ddot{\mathbf{a}}_{ij} \to \boldsymbol{max}\{\mathbf{0}, \acute{\boldsymbol{\mu}}_{\mathbb{G}}(\ddot{\mathbf{a}}_{ij}) + \ddot{\boldsymbol{\theta}} - \mathbf{1}\}$, $\acute{\boldsymbol{\mathcal{T}}}_{Ł_{\mathbb{G}}^{\ddot{\theta}}}: \acute{\mathbb{U}} \to [\mathbf{0},\mathbf{1}], \ddot{\mathbf{a}}_{ij} \to \boldsymbol{max}\{\mathbf{0}, \acute{\boldsymbol{\mathcal{T}}}_{\mathbb{G}}(\ddot{\mathbf{a}}_{ij}) + \ddot{\boldsymbol{\theta}} - \mathbf{1}\}$, $\acute{\boldsymbol{\mathcal{I}}}_{Ł_{\mathbb{G}}^{\ddot{\theta}}}: \acute{\mathbb{U}} \to [\mathbf{0},\mathbf{1}], \ddot{\mathbf{a}}_{ij} \to \boldsymbol{max}\{\mathbf{0}, \acute{\boldsymbol{\mathcal{I}}}_{\mathbb{G}}(\ddot{\mathbf{a}}_{ij}) + \ddot{\boldsymbol{\theta}} - \mathbf{1}\}$ and $\acute{\boldsymbol{\mathcal{F}}}_{Ł_{\mathbb{G}}^{\ddot{\theta}}}: \acute{\mathbb{U}} \to [\mathbf{0},\mathbf{1}], \ddot{\mathbf{a}}_{ij} \to \boldsymbol{max}\{\mathbf{0}, \acute{\boldsymbol{\mathcal{F}}}_{\mathbb{G}}(\ddot{\mathbf{a}}_{ij}) + \ddot{\boldsymbol{\theta}} - \mathbf{1}\}$ respectively, which satisfy the condition that $\mathbf{0} \leq \acute{\boldsymbol{\mathcal{T}}}_{Ł_{\mathbb{G}}^{\ddot{\theta}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) + \acute{\boldsymbol{\mathcal{I}}}_{Ł_{\mathbb{G}}^{\ddot{\theta}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) + \acute{\boldsymbol{\mathcal{F}}}_{Ł_{\mathbb{G}}^{\ddot{\theta}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) \leq \mathbf{3}$.

3.2. Operations on $\mathcal{SV}$ −Łukasiewicz Neutrosophic $\mathscr{F}$uzzy Matrix

3.2.1. Matrix Addition

Let $\ddot{\mathbb{P}} = [\mathbb{p}_{ij}]_{\ddot{m}\times\ddot{n}}$ and $\ddot{\mathbb{Q}} = [\mathbb{q}_{ij}]_{\ddot{m}\times\ddot{n}}$ be two $\mathbb{SLNFM}$s. The matrix addition is in the form

$$\ddot{\mathbb{P}} + \ddot{\mathbb{Q}} = \left[\langle \acute{\boldsymbol{\mu}}_{\mathbb{P}+\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mathcal{T}}}_{\mathbb{P}+\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}+\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{F}}}_{\mathbb{P}+\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\rangle\right]_{\ddot{m}\times\ddot{n}}$$

where $\acute{\boldsymbol{\mu}}_{\mathbb{P}+\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}) = \boldsymbol{max}\left(\acute{\boldsymbol{\mu}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mu}}_{\mathbb{Q}}(\ddot{\mathbf{a}}_{ij})\right)$, $\acute{\boldsymbol{\mathcal{T}}}_{\mathbb{P}+\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) = \boldsymbol{max}\left(\acute{\boldsymbol{\mathcal{T}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{T}}}_{\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right)$

$\acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}+\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) = \boldsymbol{max}\left(\acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{I}}}_{\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right)$ and $\acute{\boldsymbol{\mathcal{F}}}_{\mathbb{P}+\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) = \boldsymbol{min}\left(\acute{\boldsymbol{\mathcal{F}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{F}}}_{\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right)$.

3.2.2. Matrix Subtraction

Let $\ddot{\mathbb{P}} = [\mathbb{p}_{ij}]_{\ddot{m}\times\ddot{n}}$ and $\ddot{\mathbb{Q}} = [\mathbb{q}_{ij}]_{\ddot{m}\times\ddot{n}}$ be two $\mathbb{SLNFM}$s. The matrix subtraction is in the form

$$\ddot{\mathbb{P}} - \ddot{\mathbb{Q}} = \left[\langle \acute{\boldsymbol{\mu}}_{\mathbb{P}-\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mathcal{T}}}_{\mathbb{P}-\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}-\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{F}}}_{\mathbb{P}-\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\rangle\right]_{\ddot{m}\times\ddot{n}}$$

where $\acute{\boldsymbol{\mu}}_{\mathbb{P}-\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}) = \begin{cases} \acute{\boldsymbol{\mu}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}), & \boldsymbol{if}\ \acute{\boldsymbol{\mu}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}) \geq \acute{\boldsymbol{\mu}}_{\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}) \\ \mathbf{0}, & \boldsymbol{otherwise} \end{cases}$,

$\acute{\boldsymbol{\mathcal{T}}}_{\mathbb{P}-\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) = \begin{cases} \acute{\boldsymbol{\mathcal{T}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), & \boldsymbol{if}\ \acute{\boldsymbol{\mathcal{T}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) \geq \acute{\boldsymbol{\mathcal{T}}}_{\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) \\ \mathbf{0}, & \boldsymbol{otherwise} \end{cases}$,

$\acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}-\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) = \begin{cases} \acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), & \boldsymbol{if}\ \acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) \geq \acute{\boldsymbol{\mathcal{I}}}_{\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) \\ \mathbf{0}, & \boldsymbol{otherwise} \end{cases}$ and

$\acute{\boldsymbol{\mathcal{F}}}_{\mathbb{P}-\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) = \begin{cases} \acute{\boldsymbol{\mathcal{F}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), & \boldsymbol{if}\ \acute{\boldsymbol{\mathcal{F}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) \leq \acute{\boldsymbol{\mathcal{F}}}_{\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) \\ \mathbf{0}, & \boldsymbol{otherwise} \end{cases}$.

3.2.3. Component wise Matrix Multiplication

Let $\ddot{\mathbb{P}} = [\mathbb{p}_{ij}]_{\ddot{m}\times\ddot{n}}$ and $\ddot{\mathbb{Q}} = [\mathbb{q}_{ij}]_{\ddot{m}\times\ddot{n}}$ be two $\mathbb{SLNFM}$s. The component wise matrix multiplication will be

$$\ddot{\mathbb{P}} \circ \ddot{\mathbb{Q}} = \left[\langle \acute{\boldsymbol{\mu}}_{\mathbb{P}\circ\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mathcal{T}}}_{\mathbb{P}\circ\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}\circ\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{F}}}_{\mathbb{P}\circ\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\rangle\right]_{\ddot{m}\times\ddot{n}'}$$

where $\acute{\boldsymbol{\mu}}_{\mathbb{P}\circ\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}) = \boldsymbol{min}\left(\acute{\boldsymbol{\mu}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mu}}_{\mathbb{Q}}(\ddot{\mathbf{a}}_{ij})\right)$, $\acute{\boldsymbol{\mathcal{T}}}_{\mathbb{P}\circ\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) = \boldsymbol{min}\left(\acute{\boldsymbol{\mathcal{T}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{T}}}_{\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right)$

$\acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}\circ\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) = \boldsymbol{min}\left(\acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{I}}}_{\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right)$ and $\acute{\boldsymbol{\mathcal{F}}}_{\mathbb{P}\circ\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) = \boldsymbol{max}\left(\acute{\boldsymbol{\mathcal{F}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{F}}}_{\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right)$.

3.2.4. Scalar Product

The scalar product of $\mathbb{SLNFM}$ $\breve{\mathbb{P}} = \left[\langle \acute{\boldsymbol{\mu}}_{\breve{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mathcal{T}}}_{\breve{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{I}}}_{\breve{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{F}}}_{\breve{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\rangle\right]_{\ddot{\boldsymbol{m}}\times\ddot{\boldsymbol{n}}}$ be defined to be

$$\acute{\boldsymbol{k}}\breve{\mathbb{P}} = \left[\langle \boldsymbol{min}\left(\acute{\boldsymbol{k}}, \acute{\boldsymbol{\mu}}_{\breve{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij})\right), \boldsymbol{min}\left(\acute{\boldsymbol{k}}, \acute{\boldsymbol{\mathcal{T}}}_{\breve{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right), \boldsymbol{min}\left(\acute{\boldsymbol{k}}, \acute{\boldsymbol{\mathcal{I}}}_{\breve{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right), \boldsymbol{max}\left(\mathbf{1} - \acute{\boldsymbol{k}}, \acute{\boldsymbol{\mathcal{F}}}_{\breve{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right)\rangle\right]_{\ddot{\boldsymbol{m}}\times\ddot{\boldsymbol{n}}}.$$

3.2.5. Matrix Product

The matrix product of two $\mathbb{SLNFM}$s $\breve{\mathbb{P}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\ddot{\boldsymbol{m}}\times\ddot{\boldsymbol{n}}}$ and $\breve{\mathbb{Q}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\ddot{\boldsymbol{n}}\times\ddot{\boldsymbol{r}}}$ are defined to be

$$\breve{\mathbb{P}} * \breve{\mathbb{Q}} = \left[\langle \acute{\boldsymbol{\mu}}_{\breve{\mathbb{P}}*\breve{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mathcal{T}}}_{\breve{\mathbb{P}}*\breve{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{I}}}_{\breve{\mathbb{P}}*\breve{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{F}}}_{\breve{\mathbb{P}}*\breve{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\rangle\right]_{\ddot{\boldsymbol{m}}\times\ddot{\boldsymbol{r}}'}$$

where $\acute{\boldsymbol{\mu}}_{\mathbb{P}*\mathbb{Q}}(\ddot{\mathbf{a}}_{ij}) = \boldsymbol{max}_{\boldsymbol{k}}\left\{\boldsymbol{min}\{\acute{\boldsymbol{\mu}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ik}), \acute{\boldsymbol{\mu}}_{\mathbb{Q}}(\ddot{\mathbf{a}}_{kj})\}\right\}$

$\acute{\boldsymbol{\mathcal{T}}}_{\mathbb{P}*\breve{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) = \boldsymbol{max}_{\boldsymbol{k}}\left\{\boldsymbol{min}\{\acute{\boldsymbol{\mathcal{T}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ik}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{T}}}_{\breve{\mathbb{Q}}}(\ddot{\mathbf{a}}_{kj}, \acute{\boldsymbol{\mu}})\}\right\}$,

$\acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}*\breve{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) = \boldsymbol{min}_{\boldsymbol{k}}\left\{\boldsymbol{max}\{\acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ik}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{I}}}_{\breve{\mathbb{Q}}}(\ddot{\mathbf{a}}_{kj}, \acute{\boldsymbol{\mu}})\}\right\}$ and

$\acute{\boldsymbol{\mathcal{F}}}_{\mathbb{P}*\breve{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) = \boldsymbol{min}_{\boldsymbol{k}}\left\{\boldsymbol{max}\{\acute{\boldsymbol{\mathcal{F}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ik}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{F}}}_{\breve{\mathbb{Q}}}(\ddot{\mathbf{a}}_{kj}, \acute{\boldsymbol{\mu}})\}\right\}$, $\boldsymbol{k} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}$, $\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{m}$ and $\boldsymbol{j} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{r}$.

3.2.6. Score Matrix ($\mathbb{SM}$)

Let $\ddot{\boldsymbol{\kappa}}_{\mathbf{1}}$ and $\ddot{\boldsymbol{\kappa}}_{\mathbf{2}}$ be two $\mathbb{SLNFM}$s. Its $\mathbb{SM}$ is defined as

$$\boldsymbol{\mathcal{S}}_{\boldsymbol{c}}(\ddot{\boldsymbol{\kappa}}_{\mathbf{1}}, \ddot{\boldsymbol{\kappa}}_{\mathbf{2}}) = \left[\breve{\mathcal{U}} - \breve{\mathcal{V}}\right]$$

where $\breve{\mathcal{U}} = \left[\boldsymbol{u}_{ij}\right]_{\ddot{\boldsymbol{m}}\times\ddot{\boldsymbol{n}}'}$ $\boldsymbol{u}_{ij} = \acute{\boldsymbol{\mu}}_{\ddot{\kappa}_1}(\ddot{\mathbf{a}}_{ij}) + \acute{\boldsymbol{\mathcal{T}}}_{\ddot{\kappa}_1}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) + \acute{\boldsymbol{\mathcal{I}}}_{\ddot{\kappa}_1}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) - \acute{\boldsymbol{\mathcal{F}}}_{\ddot{\kappa}_1}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})$ and

$\breve{\mathcal{V}} = \left[\boldsymbol{v}_{ij}\right]_{\ddot{\boldsymbol{m}}\times\ddot{\boldsymbol{n}}'}$ $\boldsymbol{v}_{ij} = \acute{\boldsymbol{\mu}}_{\ddot{\kappa}_2}(\ddot{\mathbf{a}}_{ij}) + \acute{\boldsymbol{\mathcal{T}}}_{\ddot{\kappa}_2}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) + \acute{\boldsymbol{\mathcal{I}}}_{\ddot{\kappa}_2}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) - \acute{\boldsymbol{\mathcal{F}}}_{\ddot{\kappa}_2}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})$.

3.2.7. Transpose of the Matrix

Let $\breve{\mathbb{P}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\ddot{\boldsymbol{m}}\times\ddot{\boldsymbol{n}}}$ be $\mathbb{SLNFM}$ and its transpose is referred to be

$$\mathbb{P}^{T} = \left[\langle \boldsymbol{\mu}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ji}), \boldsymbol{\mathcal{T}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ji}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ji}, \acute{\boldsymbol{\mu}}), \boldsymbol{\mathcal{F}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ji}, \acute{\boldsymbol{\mu}})\rangle\right]_{\ddot{m}\times\ddot{n}}.$$

3.2.8. Complement of the Matrix

Let $\ddot{\mathbb{P}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\ddot{m}\times\ddot{n}}$ be $\mathbb{SLNFM}$ and its complement is referred to be

$$\bar{\ddot{\mathbb{P}}} = \left[\langle \acute{\boldsymbol{\mathcal{F}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \mathbf{1} - \acute{\boldsymbol{\mathcal{I}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mathcal{T}}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\boldsymbol{\mu}}_{\mathbb{P}}(\ddot{\mathbf{a}}_{ij})\rangle\right]_{\ddot{m}\times\ddot{n}}.$$

3.2.9. Trace of the Matrix

Let $\ddot{\mathbb{P}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\ddot{m}\times\ddot{m}}$ be $\mathbb{SLNFM}$ and its trace is defined as

$$\boldsymbol{tr}(\ddot{\mathbb{P}}) = \sum_{i=1}^{n} \ddot{\mathbf{a}}_{ii}.$$

3.3. Types of $\mathcal{SV}$ −Łukasiewicz Neutrosophic $\mathcal{F}$uzzy Matrix

3.3.1. $\mathcal{SV}$ −Zero Łukasiewicz Neutrosophic $\mathcal{F}$uzzy Matrix ($\mathbb{SZLNFM}$)

Let $\ddot{\mathbb{P}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\ddot{m}\times\ddot{n}}$ be a $\mathbb{SLNFM}$. The matrix is known to be $\mathbb{SZLNFM}$, if all the entries of $\ddot{\mathbb{P}}$ are $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1})$ and is denoted by 0.

3.3.2. $\mathcal{SV}$ −Universal Łukasiewicz Neutrosophic $\mathcal{F}$uzzy Matrix ($\mathbb{SULNFM}$)

Let $\ddot{\mathbb{P}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\ddot{m}\times\ddot{n}}$ be a $\mathbb{SLNFM}$. The matrix is known to be $\mathbb{SULNFM}$, if all the entries of $\ddot{\mathbb{P}}$ are $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{0})$ and is denoted by I.

3.3.3. $\mathcal{SV}$ −Symmetric Łukasiewicz Neutrosophic $\mathcal{F}$uzzy Matrix ($\mathbb{SSLNFM}$)

Let $\ddot{\mathbb{P}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\ddot{m}\times\ddot{m}}$ be a $\mathbb{SLNFM}$. The matrix $\ddot{\mathbb{P}}$ is known to be $\mathbb{SSLNFM}$ if $\ddot{\mathbb{P}}_{ij} = \ddot{\mathbb{P}}_{ji}$, $\boldsymbol{i}, \boldsymbol{j} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}$.

3.3.4. $\mathcal{SV}$ −Upper triangular Łukasiewicz Neutrosophic $\mathcal{F}$uzzy Matrix ($\mathbb{SUTLNFM}$)

Suppose $\ddot{\mathbb{P}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\ddot{m}\times\ddot{m}}$ be a $\mathbb{SLNFM}$. Then the matrix $\ddot{\mathbb{P}}$ is known to be $\mathbb{SUTLNFM}$ if $\ddot{\mathbb{P}}_{ij} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1})$, $\forall\, \boldsymbol{i} > \boldsymbol{j}$ and $\boldsymbol{i}, \boldsymbol{j} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}$.

3.3.5. $\mathcal{SV}$ −Lower Triangular Łukasiewicz Neutrosophic $\mathcal{F}$uzzy Matrix ($\mathbb{SLTLNFM}$)

Suppose $\ddot{\mathbb{P}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\ddot{m}\times\ddot{m}}$ be a $\mathbb{SLNFM}$. Then the matrix $\ddot{\mathbb{P}}$ is known to be $\mathbb{SLTLNFM}$ if $\ddot{\mathbb{P}}_{ij} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1})$, $\forall\, \boldsymbol{i} < \boldsymbol{j}$ and $\boldsymbol{i}, \boldsymbol{j} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}$.

3.3.6. Theorem

Let $\ddot{\mathbb{P}}$ be $\mathbb{SLNFS}$. Show that $\ddot{\mathbb{P}}$ is a vector space under $\mathbb{SLNFM}$ addition and scalar multiplication.

Proof

Let us examine all the axioms of Vector space.

$\boldsymbol{A_1}$: Let $\ddot{\mathbb{P}} = \left[\langle \acute{\mu}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}), \acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu})\rangle\right]_{\ddot{m}\times\ddot{m}}$ and

$\ddot{\mathbb{Q}} = \left[\langle \acute{\mu}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}), \acute{\mathcal{T}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu})\rangle\right]_{\ddot{m}\times\ddot{m}}$ be two $\mathbb{SLNFM}$. In accordance with matrix addition, it is obvious that $\ddot{\mathbb{P}} + \ddot{\mathbb{Q}}$ also belongs to the $\mathbb{SLNFM}$ of order $\ddot{m} \times \ddot{m}$.

$\boldsymbol{A_2}$: Let $\ddot{\mathbb{P}} = \left[\langle \acute{\mu}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}), \acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu})\rangle\right]_{\ddot{m}\times\ddot{m}}$

$\ddot{\mathbb{Q}} = \left[\langle \acute{\mu}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}), \acute{\mathcal{T}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu})\rangle\right]_{\ddot{m}\times\ddot{m}}$ and

$\ddot{\mathbb{R}} = \left[\langle \acute{\mu}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}), \acute{\mathcal{T}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu})\rangle\right]_{\ddot{m}\times\ddot{m}}$ be three $\mathbb{SLNFM}$.

$$\ddot{\mathbb{P}} + \left(\ddot{\mathbb{Q}} + \ddot{\mathbb{R}}\right) = \left[\langle \acute{\mu}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}), \acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu})\rangle\right] + \left[\left\langle \begin{matrix} \max\left(\acute{\mu}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}), \acute{\mu}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij})\right), \max\left(\acute{\mathcal{T}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{T}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu})\right), \\ \max\left(\acute{\mathcal{I}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu})\right), \min\left(\acute{\mathcal{F}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu})\right) \end{matrix} \right\rangle\right]$$

$$= \left[\left\langle \begin{matrix} \max\left(\acute{\mu}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}), \acute{\mu}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}), \acute{\mu}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij})\right), \max\left(\acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{T}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{T}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu})\right), \\ \max\left(\acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu})\right), \min\left(\acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu})\right) \end{matrix} \right\rangle\right]$$

$$\left(\ddot{\mathbb{P}} + \ddot{\mathbb{Q}}\right) + \ddot{\mathbb{R}} = \left[\left\langle \begin{matrix} \max\left(\acute{\mu}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}), \acute{\mu}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij})\right), \max\left(\acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{T}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu})\right), \\ \max\left(\acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu})\right), \min\left(\acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu})\right) \end{matrix} \right\rangle\right] + \left[\langle \acute{\mu}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}), \acute{\mathcal{T}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu})\rangle\right]$$

$$= \left[\left\langle \begin{matrix} \max\left(\acute{\mu}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}), \acute{\mu}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}), \acute{\mu}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij})\right), \max\left(\acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{T}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{T}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu})\right), \\ \max\left(\acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu})\right), \min\left(\acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{R}}}(\ddot{a}_{ij}, \acute{\mu})\right) \end{matrix} \right\rangle\right]$$

Hence $\ddot{\mathbb{P}} + \left(\ddot{\mathbb{Q}} + \ddot{\mathbb{R}}\right) = \left(\ddot{\mathbb{P}} + \ddot{\mathbb{Q}}\right) + \ddot{\mathbb{R}}$.

In a similar manner, we can show, $\ddot{\mathbb{P}} \circ \left(\ddot{\mathbb{Q}} \circ \ddot{\mathbb{R}}\right) = \left(\ddot{\mathbb{P}} \circ \ddot{\mathbb{Q}}\right) \circ \ddot{\mathbb{R}}$.

$$\boldsymbol{A_3}\text{: } \ddot{\mathbb{P}} + \ddot{\mathbb{Q}} = \left[\left\langle \begin{matrix} \max\left(\acute{\mu}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}), \acute{\mu}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij})\right), \max\left(\acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{T}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu})\right), \\ \max\left(\acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{I}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu})\right), \min\left(\acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{a}_{ij}, \acute{\mu}), \acute{\mathcal{F}}_{\ddot{\mathbb{Q}}}(\ddot{a}_{ij}, \acute{\mu})\right) \end{matrix} \right\rangle\right]$$

$$= \begin{bmatrix} \langle & \boldsymbol{max}\left(\acute{\boldsymbol{\mu}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mu}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij})\right), \boldsymbol{max}\left(\acute{\mathcal{T}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right), \\ & \boldsymbol{max}\left(\acute{\mathcal{I}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right), \boldsymbol{min}\left(\acute{\mathcal{F}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right) & \rangle \end{bmatrix} = \ddot{\mathbb{Q}} + \ddot{\mathbb{P}}.$$

$\boldsymbol{A_4}$: If $\ddot{\mathbb{P}} + \mathbf{0} = \ddot{\mathbb{P}}$, for any $\mathbb{SLNFM}$ $\ddot{\mathbb{P}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\ddot{m}\times\ddot{m}'}$ then $\mathbb{SZLNFM}$, 0 is the additive identity.

If $\ddot{\mathbb{P}} \circ \mathbf{I} = \ddot{\mathbb{P}}$, for any $\mathbb{SLNFM}$ $\ddot{\mathbb{P}} = \left[\ddot{\mathbf{a}}_{ij}\right]_{\ddot{m}\times\ddot{m}'}$ then $\mathbb{SULNFM}$, I is the identity for multiplication.

$\boldsymbol{M_1}$: Let us say $\ddot{\mathbb{P}}$ be a $\mathbb{SLNFM}$ and $\acute{k}$ be any scalar. In accordance with the scalar multiplication for $\mathbb{SLNFM}$, it is obvious that $\acute{k}\ddot{\mathbb{P}}$ is also a $\mathbb{SLNFM}$.

$\boldsymbol{M_2}$: Now to prove distributive law. Suppose that $\ddot{\mathbb{P}} \subseteq \ddot{\mathbb{Q}}, \ddot{\mathbb{R}}$. Then $\acute{\boldsymbol{\mu}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}) \leq \acute{\boldsymbol{\mu}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mu}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij})$; $\acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) \leq \acute{\mathcal{T}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{T}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})$; $\acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) \leq \acute{\mathcal{I}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{I}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})$; $\acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) \geq \acute{\mathcal{F}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{F}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})$. Thus,

$$\ddot{\mathbb{P}} \circ (\ddot{\mathbb{Q}} + \ddot{\mathbb{R}}) = \left[\langle \acute{\boldsymbol{\mu}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}), \acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}) \rangle\right] \circ \begin{bmatrix} \langle & \boldsymbol{max}\left(\acute{\boldsymbol{\mu}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mu}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij})\right), \boldsymbol{max}\left(\acute{\mathcal{T}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{T}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right), \\ & \boldsymbol{max}\left(\acute{\mathcal{I}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{I}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right), \boldsymbol{min}\left(\acute{\mathcal{F}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{F}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right) & \rangle \end{bmatrix}$$

$$= \begin{bmatrix} & \boldsymbol{min}\left(\acute{\boldsymbol{\mu}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}), \boldsymbol{max}\left(\acute{\boldsymbol{\mu}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mu}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij})\right)\right), \\ \langle & \boldsymbol{min}\left(\acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \boldsymbol{max}\left(\acute{\mathcal{T}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{T}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right)\right), \\ & \boldsymbol{min}\left(\acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \boldsymbol{max}\left(\acute{\mathcal{I}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{I}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right)\right), & \rangle \\ & \boldsymbol{max}\left(\acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \boldsymbol{min}\left(\acute{\mathcal{F}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{F}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right)\right) \end{bmatrix} = \ddot{\mathbb{P}}$$

$$(\ddot{\mathbb{P}} \circ \ddot{\mathbb{Q}}) + (\ddot{\mathbb{P}} \circ \ddot{\mathbb{R}}) = \begin{bmatrix} \langle & \boldsymbol{min}\left(\acute{\boldsymbol{\mu}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mu}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij})\right), \boldsymbol{min}\left(\acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{T}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right), \\ & \boldsymbol{min}\left(\acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{I}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right), \boldsymbol{max}\left(\acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{F}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right) & \rangle \end{bmatrix}$$

$$+ \begin{bmatrix} \langle & \boldsymbol{min}\left(\acute{\boldsymbol{\mu}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mu}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij})\right), \boldsymbol{min}\left(\acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{T}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right), \\ & \boldsymbol{min}\left(\acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{I}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right), \boldsymbol{max}\left(\acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{F}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right) & \rangle \end{bmatrix}$$

$$= \begin{bmatrix} & \boldsymbol{max}\left(\boldsymbol{min}\left(\acute{\boldsymbol{\mu}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mu}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij})\right), \boldsymbol{min}\left(\acute{\boldsymbol{\mu}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}), \acute{\boldsymbol{\mu}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij})\right)\right), \\ \langle & \boldsymbol{max}\left(\boldsymbol{min}\left(\acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{T}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right), \boldsymbol{min}\left(\acute{\mathcal{T}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{T}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right)\right), \\ & \boldsymbol{max}\left(\boldsymbol{min}\left(\acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{I}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right), \boldsymbol{min}\left(\acute{\mathcal{I}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{I}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right)\right), & \rangle \\ & \boldsymbol{min}\left(\boldsymbol{max}\left(\acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{F}}_{\ddot{\mathbb{Q}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right), \boldsymbol{max}\left(\acute{\mathcal{F}}_{\ddot{\mathbb{P}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}}), \acute{\mathcal{F}}_{\ddot{\mathbb{R}}}(\ddot{\mathbf{a}}_{ij}, \acute{\boldsymbol{\mu}})\right)\right) \end{bmatrix} = \ddot{\mathbb{P}}.$$

Similarly, we can prove if $\ddot{\mathbb{P}} \supseteq \ddot{\mathbb{Q}}, \ddot{\mathbb{R}}$. Thus, the distributive law holds.

$\boldsymbol{M_3}$: For any two scalars $\acute{\boldsymbol{k}}_1$ and $\acute{\boldsymbol{k}}_2$, we have

$$(\acute{\boldsymbol{k}}_1 + \acute{\boldsymbol{k}}_2)\breve{\mathbb{P}} = (\acute{\boldsymbol{k}}_1 + \acute{\boldsymbol{k}}_2)\mathbf{I} \circ \breve{\mathbb{P}} = (\acute{\boldsymbol{k}}_1\mathbf{I} + \acute{\boldsymbol{k}}_2\mathbf{I}) \circ \breve{\mathbb{P}} = \acute{\boldsymbol{k}}_1\mathbf{I} \circ \breve{\mathbb{P}} + \acute{\boldsymbol{k}}_2\mathbf{I} \circ \breve{\mathbb{P}} = \acute{\boldsymbol{k}}_1 \circ \breve{\mathbb{P}} + \acute{\boldsymbol{k}}_2 \circ \breve{\mathbb{P}}.$$

Therefore, $\mathbb{SLNFM}$ forms a vector space.

4. Case study on $\mathcal{SV}$ −Łukasiewicz Neutrosophic Fuzzy Matrix

The $\mathbb{SLNFM}$ is widely applied in areas requiring decision-making under uncertainty, as it effectively handles imprecise, inconsistent, and incomplete information. We applied this idea on a decision-making problem to demonstrate how to apply it in real-life situations. In the current market, there are numerous varieties of models in electronic devices. The process of buying an electronic device deal with lot of issues due to the lack of knowledge about the device's features or confusion in selecting the appropriate model based on their specific needs. They are typically purchased infrequently. So, the decision-making process is crucial to align with both the user's requirements and their budgetary constraints. As a note of it, we have addressed the problem specifically on the selection of best Laptop model from the available data using the theory of $\mathcal{SV}$ −Łukasiewicz Neutrosophic $\mathcal{F}$uzzy Matrix. For the case analysis, we have collected the data from five persons $\ddot{\boldsymbol{\mathcal{P}}} = \{\ddot{\boldsymbol{\mathcal{P}}}_1, \ddot{\boldsymbol{\mathcal{P}}}_2, \ddot{\boldsymbol{\mathcal{P}}}_3, \ddot{\boldsymbol{\mathcal{P}}}_4, \ddot{\boldsymbol{\mathcal{P}}}_5\}$ and noted their preferences in the features such as $\ddot{\boldsymbol{\mathcal{F}}} = \{\textbf{Processor}, \textbf{Price}, \textbf{RAM}, \textbf{Display}, \textbf{Rating}\}$ of the Laptops. Also, we have used the data of five different laptop models $\ddot{\boldsymbol{\mathcal{L}}} = \{\ddot{\boldsymbol{\mathcal{L}}}_1, \ddot{\boldsymbol{\mathcal{L}}}_2, \ddot{\boldsymbol{\mathcal{L}}}_3, \ddot{\boldsymbol{\mathcal{L}}}_4, \ddot{\boldsymbol{\mathcal{L}}}_5\}$ and their features. Now we have to deduce which laptop model is suitable for the persons, we have chosen. Table 6.1 is the data of persons and their preferences in laptop features and Table 6.2 is the data about the laptop models and its features.

Person	**Feature Preferences**				
	Processor	**Price (in ₹)**	**DDR4 RAM**	**Display (in inches)**	**Rating**
Arun $(\ddot{\boldsymbol{\mathcal{P}}}_1)$	Intel Core i5 Processor	60000	8 GB	15.5	4.4
Fredrick $(\ddot{\boldsymbol{\mathcal{P}}}_2)$	Intel Core i3 Processor	55000	4 GB	14	4.1
John $(\ddot{\boldsymbol{\mathcal{P}}}_3)$	Apple M1 Processor	75000	16 GB	13.5	4.6
Max $(\ddot{\boldsymbol{\mathcal{P}}}_4)$	Intel Core i5 Processor	65000	16 GB	14.5	4.5
Thomas $(\ddot{\boldsymbol{\mathcal{P}}}_5)$	Intel Core i3 Processor	50000	8 GB	15	3.8

Table 1. Persons and their preferences in the feature of the laptops

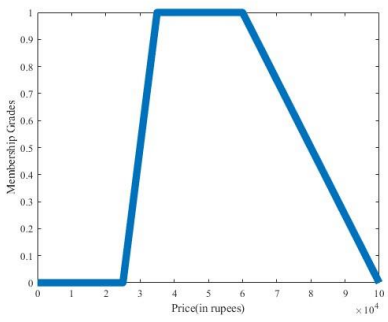
Laptop Model	**Features**				
	Processor	**Price (in ₹)**	**DDR4 RAM**	**Display (in inches)**	**Rating**
Lenovo V15 G2 Core i3 11th Gen $(\ddot{\boldsymbol{\mathcal{L}}}_1)$	Intel Core i3 Processor	37500	8 GB	15.6	4.4

ASUS TUF Gaming F15 Core i5 10th Gen $(\ddot{\mathcal{L}}_2)$	Intel Core i5 Processor	49990	8 GB	15.6	4.4
ASUS Vivo Book 15 (2022) Core i3 10th Gen $(\ddot{\mathcal{L}}_3)$	Intel Core i3 Processor	33990	8 GB	15.6	4.3
Lenovo Athlon Dual Core $(\ddot{\mathcal{L}}_4)$	AMD Athlon Dual Core Processor	18999	4 GB	14	3.8
APPLE 2020 Mac book Air M1 $(\ddot{\mathcal{L}}_5)$	Apple M1 Processor	86990	8 GB	13.3	4.7

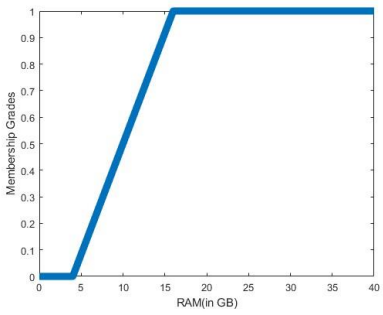
Table 2. Laptop Models and their features

We have defined the membership value for the above data sets as given below.

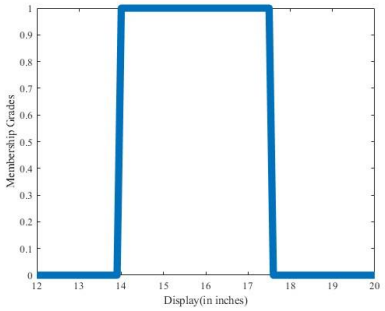
- Processor= $\left\{\frac{0.9}{Intel\ Core\ i5\ Processor}+\frac{0.7}{Intel\ Core\ i3\ Processor}+\frac{0.4}{AMD\ Athlon\ Dual\ Core\ Processor}+\frac{0.7}{Apple\ M1\ Processor}+\frac{0.8}{AMD\ Ryzen\ 5\ Hexa\ Core\ Processor}\right\}$
- Price= $\begin{cases} 0 & if\ p \le 25000\ and\ p \ge 100000 \\ \frac{p-25000}{10000} & if\ 25000 \le p \le 35000 \\ 1 & if\ 35000 \le p \le 60000 \\ \frac{100000-p}{40000} & if\ 60000 \le p \le 100000 \end{cases}$
- RAM = $\begin{cases} 0 & if\ p \le 4 \\ \frac{p-4}{12} & if\ 4 \le p \le 16 \\ 1 & if\ p \ge 16 \end{cases}$
- Display (in inches)= $\begin{cases} 0\ if\ p \le 14\ and\ p \ge 17.5 \\ 1\ if\ 14 < p < 17.5 \end{cases}$
- Rating= $\begin{cases} 0\ if\ p \le 3 \\ \frac{p-3}{1.5} if\ 3 \le p \le 4.5 \\ 1\ if\ 4.5 \le p \le 5 \end{cases}$



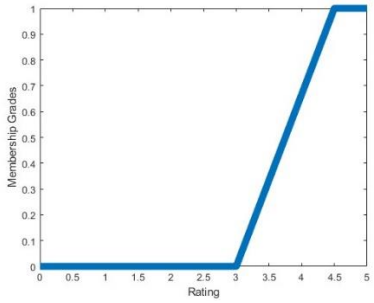
(a)



(b)



(c)



(d)

Figure 1. Graphical Representations of membership grades. (a) Membership Grade for Price, (b) Membership Grade for RAM, (c) Membership Grade for Display, (d) Membership Grade for Rating

4.1. Algorithm

1. Construct the $\mathfrak{SNFM}$ $\ddot{\mathbb{M}}$ using Table 1 and the membership values defined.

2. Construct the $\mathfrak{SNFM}$ $\ddot{\mathbb{N}}$ using Table 2 and the membership values defined.

3. Convert the $\mathfrak{SNFM}$s $\ddot{\mathbb{M}}$ and $\ddot{\mathbb{N}}$ to $\mathfrak{SLNFM}$s $Ł^{\ddot{\theta}}_{\ddot{\mathbb{M}}}$ and $Ł^{\ddot{\theta}}_{\ddot{\mathbb{N}}}$.

4. Determine the matrix complement of $Ł^{\ddot{\theta}}_{\ddot{\mathbb{N}}}$.

5. Determine the matrix $Ł^{\ddot{\theta}}_{\ddot{\mathbb{M}}} * Ł^{\ddot{\theta}}_{\ddot{\mathbb{N}}}$.

6. Determine the matrix $Ł^{\ddot{\theta}}_{\ddot{\mathbb{M}}} * Ł^{\ddot{\theta}^{c}}_{\ddot{\mathbb{N}}}$

7. Find $\breve{\mathcal{U}}$ and $\breve{\mathcal{V}}$ of $\mathfrak{SM}$.

8. Measure the $\mathfrak{SM}$ $\mathcal{S}_c\left(Ł^{\ddot{\theta}}_{\ddot{\mathbb{M}}} * Ł^{\ddot{\theta}}_{\ddot{\mathbb{N}}}, Ł^{\ddot{\theta}}_{\ddot{\mathbb{M}}} * Ł^{\ddot{\theta}^{c}}_{\ddot{\mathbb{N}}}\right)$.

Step-1

Construct a $\mathfrak{SNFM}$ $\ddot{\mathbb{M}}$ using Table 1 and defined Membership grades to represent the Person's preferences in the feature of the laptops.

$$\ddot{\mathbb{M}} = \begin{bmatrix} (0.9,0.6,0.2,0.1) & (1,0.7,0.1,0.2) & (0.3,0.6,0.1,0.1) & (1,0.8,0.1,0.1) & (0.93,0.6,0.2,0.1) \\ (0.7,0.7,0.1,0.2) & (1,0.5,0.2,0.1) & (0,0.3,0.8,0.2) & (0,0.6,0.3,0) & (0.73,0.5,0.4,0.2) \\ (0.7,0.9,0,0) & (0.63,0.4,0.2,0.3) & (1,0.9,0.1,0) & (0,0.8,0.1,0) & (1,0.9,0.1,0.1) \\ (0.9,0.5,0.1,0.2) & (0.88,0.6,0.1,0.1) & (1,0.8,0.2,0.2) & (1,0.7,0.1,0.1) & (1,0.8,0.1,0.2) \\ (0.7,0.6,0.2,0.1) & (1,0.3,0.5,0.1) & (0.3,0.7,0.4,0.3) & (1,0.4,0.3,0.1) & (0.53,0.5,0.5,0.2) \end{bmatrix}$$

Step-2

Construct a $\mathfrak{SNFM}$ $\ddot{\mathbb{N}}$ using Table 2 and defined Membership grades to represent the Laptop Model's features.

$$\ddot{\mathbb{N}} = \begin{bmatrix} (0.7,1,0,0) & (1,1,0,0) & (0.3,1,0,0) & (1,1,0,0) & (0.93,1,0,0) \\ (0.9,1,0,0) & (1,1,0,0) & (0.3,1,0,0) & (1,1,0,0) & (0.93,1,0,0) \\ (0.7,1,0,0) & (0.9,1,0,0) & (0.3,1,0,0) & (1,1,0,0) & (0.87,1,0,0) \\ (0.4,1,0,0) & (0,1,0,0) & (0,1,0,0) & (0,1,0,0) & (0.53,1,0,0) \\ (0.7,1,0,0) & (0.33,1,0,0) & (0.3,1,0,0) & (0,1,0,0) & (1,1,0,0) \end{bmatrix}$$

Step-3

Convert the $\mathfrak{SNFM}$s $\ddot{\mathbb{M}}$ and $\ddot{\mathbb{N}}$ to $\mathfrak{SLNFM}$s $Ł^{\ddot{\theta}}_{\ddot{\mathbb{M}}}$ and $Ł^{\ddot{\theta}}_{\ddot{\mathbb{N}}}$. Now this $Ł^{\ddot{\theta}}_{\ddot{\mathbb{M}}}$ and $Ł^{\ddot{\theta}}_{\ddot{\mathbb{N}}}$ represents the Person's preferences in the feature of the laptops and Laptop Model's features.

$$Ł^{\ddot{\theta}}_{\ddot{\mathbb{M}}} = \begin{bmatrix} (0.45,0.15,0,0) & (0.55,0.7,0,0) & (0,0.15,0,0) & (0.55,0.35,0,0) & (0.48,0.15,0,0) \\ (0.25,0.25,0,0) & (0.55,0.5,0,0) & (0,0,0.35,0) & (0,0.15,0,0) & (0.28,0.05,0,0) \\ (0.25,0.45,0,0) & (0.18,0,0,0) & (0.55,0.45,0,0) & (0,0.35,0,0) & (0.55,0.45,0,0) \\ (0.45,0.05,0,0) & (0.43,0.6,0,0) & (0.55,0.35,0,0) & (0.55,0.25,0,0) & (0.55,0.35,0,0) \\ (0.25,0.15,0,0) & (0.55,0,0.5,0) & (0,0.25,0,0) & (0.55,0,0,0) & (0.08,0.05,0.05,0) \end{bmatrix}$$

$$Ł^{\ddot{\theta}}_{\ddot{\mathbb{N}}} = \begin{bmatrix} (0.25,0.55,0,0) & (0.55,0.55,0,0) & (0,0.55,0,0) & (0.55,0.55,0,0) & (0.48,0.55,0,0) \\ (0.45,0.55,0,0) & (0.55,0.55,0,0) & (0,0.55,0,0) & (0.55,0.55,0,0) & (0.48,0.55,0,0) \\ (0.25,0.55,0,0) & (0.45,0.55,0,0) & (0,0.55,0,0) & (0.55,0.55,0,0) & (0.42,0.55,0,0) \\ (0,0.55,0,0) & (0,0.55,0,0) & (0,0.55,0,0) & (0,0.55,0,0) & (0.08,0.55,0,0) \\ (0.25,0.55,0,0) & (0,0.55,0,0) & (0,0.55,0,0) & (0,0.55,0,0) & (0.55,0.55,0,0) \end{bmatrix}$$

Step-4

Determine the defect of the laptop models by computing the complement of the matrix $Ł^{\ddot{\theta}}_{\ddot{\mathbb{N}}}$. It is represented as $Ł^{\ddot{\theta}^{c}}_{\ddot{\mathbb{N}}}$.

$$Ł^{\ddot{\theta}^{c}}_{\ddot{\mathbb{N}}} = \begin{bmatrix} (0,1,0.55,0.25) & (0,1,0.55,0.55) & (0,1,0.55,0) & (0,1,0.55,0.55) & (0,1,0.55,0.48) \\ (0,1,0.55,0.45) & (0,1,0.55,0.55) & (0,1,0.55,0) & (0,1,0.55,0.55) & (0,1,0.55,0.48) \\ (0,1,0.55,0.25) & (0,1,0.55,0.45) & (0,1,0.55,0) & (0,1,0.55,0.55) & (0,1,0.55,0.42) \\ (0,1,0.55,0) & (0,1,0.55,0) & (0,1,0.55,0) & (0,1,0.55,0) & (0,1,0.55,0.08) \\ (0,1,0.55,0.25) & (0,1,0.55,0) & (0,1,0.55,0) & (0,1,0.55,0) & (0,1,0.55,0.55) \end{bmatrix}$$

Step-5

Determine the relation between feature preferences of persons and laptop features, by taking matrix multiplication between them $\left(i.e., Ł_{\ddot{\mathbb{M}}}^{\ddot{\theta}} * Ł_{\ddot{\mathbb{N}}}^{\ddot{\theta}}\right)$.

$$Ł_{\ddot{\mathbb{M}}}^{\ddot{\theta}} * Ł_{\ddot{\mathbb{N}}}^{\ddot{\theta}} = \begin{bmatrix} (0.45,0.55,0,0) & (0.55,0.55,0,0) & (0,0.55,0,0) & (0.55,0.55,0,0) & (0.48,0.55,0,0) \\ (0.45,0.5,0,0) & (0.55,0.5,0,0) & (0,0.5,0,0) & (0.55,0.5,0,0) & (0.48,0.5,0,0) \\ (0.25,0.45,0,0) & (0.45,0.45,0,0) & (0,0.45,0,0) & (0.55,0.45,0,0) & (0.55,0.45,0,0) \\ (0.43,0.55,0,0) & (0.45,0.55,0,0) & (0,0.55,0,0) & (0.55,0.55,0,0) & (0.55,0.55,0,0) \\ (0.45,0.25,0,0) & (0.55,0.25,0,0) & (0,0.25,0,0) & (0.55,0.25,0,0) & (0.48,0.25,0,0) \end{bmatrix}$$

Step-6

Determine the relation between feature preferences of persons and defects of laptop, by taking matrix multiplication between them $\left(i.e., Ł_{\ddot{\mathbb{M}}}^{\ddot{\theta}} * Ł_{\ddot{\mathbb{N}}}^{\ddot{\theta}^{c}}\right)$.

$$Ł_{\ddot{\mathbb{M}}}^{\ddot{\theta}} * Ł_{\ddot{\mathbb{N}}}^{\ddot{\theta}^{c}} = \begin{bmatrix} (0,0.7,0.55,0) & (0,0.7,0.55,0) & (0,0.7,0.55,0) & (0,0.7,0.55,0) & (0,0.7,0.55,0.08) \\ (0,0.5,0.55,0) & (0,0.5,0.55,0) & (0,0.5,0.55,0) & (0,0.5,0.55,0) & (0,0.5,0.55,0.08) \\ (0,0.45,0.55,0) & (0,0.45,0.55,0) & (0,0.45,0.55,0) & (0,0.45,0.55,0) & (0,0.45,0.55,0.08) \\ (0,0.6,0.55,0) & (0,0.6,0.55,0) & (0,0.6,0.55,0) & (0,0.6,0.55,0) & (0,0.6,0.55,0.08) \\ (0,0.25,0.55,0) & (0,0.25,0.55,0) & (0,0.25,0.55,0) & (0,0.25,0.55,0) & (0,0.25,0.55,0.08) \end{bmatrix}$$

Step-7

Find $\breve{\mathcal{U}}$ and $\breve{\mathcal{V}}$ of $\mathbb{SVM}$.

$$\breve{\mathcal{U}} = \begin{bmatrix} 1 & 1.1 & 0.55 & 1.1 & 1.03 \\ 0.95 & 1.05 & 0.5 & 1.05 & 0.98 \\ 0.7 & 0.9 & 0.45 & 1 & 1 \\ 0.98 & 1 & 0.55 & 1.1 & 1.1 \\ 0.7 & 0.8 & 0.25 & 0.8 & 0.73 \end{bmatrix}$$

$$\breve{\mathcal{V}} = \begin{bmatrix} 1.25 & 1.25 & 1.25 & 1.25 & 1.17 \\ 1.05 & 1.05 & 1.05 & 1.05 & 0.97 \\ 1 & 1 & 1 & 1 & 0.92 \\ 1.15 & 1.15 & 1.15 & 1.15 & 1.07 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.72 \end{bmatrix}$$

Step-8

Measure the $\mathfrak{SM}$, $\mathcal{S}_c\left(Ł^{\ddot{\theta}}_{\ddot{\mathbb{M}}} * Ł^{\ddot{\theta}}_{\ddot{\mathbb{N}}}, Ł^{\ddot{\theta}}_{\ddot{\mathbb{M}}} * Ł^{\ddot{\theta}^{\,c}}_{\ddot{\mathbb{N}}}\right)$ to determine the result.

$$\mathcal{S}_c\left(Ł^{\ddot{\theta}}_{\ddot{\mathbb{M}}} * Ł^{\ddot{\theta}}_{\ddot{\mathbb{N}}}, Ł^{\ddot{\theta}}_{\ddot{\mathbb{M}}} * Ł^{\ddot{\theta}^{\,c}}_{\ddot{\mathbb{N}}}\right) = \begin{bmatrix} -0.25 & -0.15 & -0.7 & -0.15 & -0.14 \\ -0.1 & 0 & -0.55 & 0 & 0.01 \\ -0.3 & -0.1 & -0.55 & 0 & 0.08 \\ -0.17 & -0.15 & -0.6 & -0.05 & 0.03 \\ -0.1 & 0 & -0.55 & 0 & 0.01 \end{bmatrix}$$

From the above $\mathfrak{SM}$, the score of APPLE 2020 MacBook Air M1 is higher for all the five persons. So, we conclude that, APPLE 2020 MacBook Air M1 is the best laptop model for everyone from the given data. Utilizing $\mathfrak{SLNFM}$ enables us to identify the most accurate laptop model during the early stages of decision-making. This approach not only streamlines the selection process, but also minimizes the risk of investing in a laptop that does not align with the user's preferences, ultimately leading to cost savings. This study can also be applied for large and complex data too.

5. Conclusions

In order to address multicriteria situations with neutrophilic inputs, we have built the idea of $\mathcal{SV}-$ Łukasiewicz Neutrosophic $\mathcal{F}$uzzy Matrices by combining Łukasiewicz logic to $\mathcal{SV}-$Neutrosophic $\mathcal{F}$uzzy Matrices and discussed the operations and types associated with them. As a note of it, we address a decision-making problem on selecting the best laptop model based on the individual's feature preferences and provided a solution which is more applicable for every individual in the data. This study shows how one can carry out the benefits of $\mathcal{SV}-$Łukasiewicz Neutrosophic $\mathcal{F}$uzzy Matrices in hard computations. In future, $\mathcal{SV}-$Łukasiewicz Neutrosophic $\mathcal{F}$uzzy Matrix can be investigated in higher dimensional structures and one can incorporate this idea in Machine Learning for constructing AI Models to enhance the process of reinforcement or clustering algorithm.

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Conflicts of Interest: The authors declare no conflict of interest.

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IndetermSoft Set for Digital Marketing Effectiveness Evaluation Driven by Big Data Based on Consumer Behavior

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Abstract: This study proposes the multi-criteria decision-making (MCDM) methodology for Digital Marketing Effectiveness Evaluation Driven by Big Data Based on Consumer Behavior. This study uses two MCDM methods such as the Entropy method to compute the criteria weights and the MARCOS method to rank the alternatives. We used the IndetermSoft set to deal with indeterminacy with different values in the criteria. IndetermSoft is integrated with the MCDM approach. Three experts evaluated the criteria and alternatives. We use seven criteria and ten alternatives. We conducted a comparative analysis to show the effectiveness of the proposed approach. The results show the proposed approach is effective compared to other MCDM methods.

Keywords: IndetermSoft Set; Digital Marketing; Big Data; Consumer Behavior; MCDM Methodology.

1. Introduction

In the digital era, businesses leverage big data-driven digital marketing strategies to maximize consumer engagement, improve conversion rates, and enhance return on investment (ROI). The effectiveness of digital marketing is no longer solely dependent on traditional advertising but is now heavily influenced by consumer behavior insights derived from vast datasets. Companies collect and analyze real-time data on user interactions, purchasing patterns, and online preferences to personalize marketing strategies[1], [2]. This shift has made it essential to establish a comprehensive evaluation framework that assesses digital marketing effectiveness based on key performance indicators (KPIs) linked to consumer behavior, such as engagement levels, conversion rates, and personalization effectiveness. The evaluation of digital marketing

effectiveness requires a multi-criteria decision-making (MCDM) approach that integrates big data analytics with consumer behavior metrics. Various marketing channels, including social media advertising, search engine optimization (SEO), content marketing, and AI-powered personalization, yield different levels of engagement and profitability[2], [3]. By analyzing structured and unstructured data, companies can measure the impact of their marketing campaigns in real time. This data-driven approach not only enhances marketing efficiency but also optimizes resource allocation, allowing businesses to invest in strategies that yield the highest ROI. Moreover, advanced analytics techniques, such as predictive modeling and machine learning, enable organizations to anticipate consumer needs and improve marketing precision.

However, despite its advantages, evaluating digital marketing effectiveness presents challenges. The sheer volume of consumer data, data privacy concerns, and the dynamic nature of consumer behavior make it difficult to establish a one-size-fits-all evaluation model. Additionally, the rapid evolution of digital platforms requires continuous adaptation to new trends, algorithms, and consumer expectations[4], [5]. Businesses must ensure that their evaluation frameworks are scalable, adaptable, and capable of handling uncertainty, MCDM methods essential for balancing multiple factors such as cost-effectiveness, engagement, and brand influence. By employing big data-driven evaluation frameworks, companies can develop more targeted, efficient, and responsive marketing strategies. A well-structured evaluation approach enables businesses to identify weaknesses in their marketing efforts, optimize campaign performance, and build stronger relationships with consumers. The integration of big data and consumer behavior analytics into digital marketing effectiveness evaluation is not just a competitive advantage—it is a necessity in the modern digital landscape[6], [7]. As businesses continue to refine their approaches, the ability to accurately measure and adapt to changing consumer behaviors will determine long-term success in the highly competitive digital marketplace.

1.1 IndetermSoft

IndetermSoft is an advanced extension of soft set theory designed to handle indeterminate and uncertain information in decision-making problems. Traditional soft sets provide a mathematical framework for dealing with uncertainty, but they assume that the membership values of elements in the set are well-defined. However, in real-world scenarios, information can often be incomplete, inconsistent, or indeterminate, making it difficult to reach precise conclusions. IndetermSoft addresses this limitation by introducing indeterminate elements, allowing for a more flexible and realistic approach to uncertain data processing[8], [9]. In IndetermSoft theory, elements in a soft set can belong to a parameter partially, fully, or indeterminately. This means that instead of simply having a clear membership value, an element's association with a given attribute can be ambiguous or undefined due to missing data, conflicting information, or lack of certainty.[10], [11] This characteristic makes IndetermSoft particularly useful in medical diagnosis, environmental assessments, risk management, and artificial intelligence, where uncertain or conflicting data is common.

Example of IndetermSoft

Consider a medical diagnosis problem where a doctor needs to assess whether a patient has a disease based on symptoms. Traditional soft set theory assumes that a symptom is either present or absent, but in many cases, the presence of a symptom may be uncertain or indeterminate due to ambiguous test results or incomplete patient history.

The rest of this study is organized as follows: Section 2 shows the methods of this study to compute the criteria weights and rank the alternatives. Section 3 shows the results of the proposed approach. Section 4 shows the analysis of this study. Section 5 shows the conclusions of this study.

2. Methods

This section shows the steps of two MCDM methods to compute the criteria weights and rank the alternatives. These methods are used with IndetermSoft to deal with indeterminacy in the criteria and alternatives evaluation.

Entropy Approach

This method is used to compute the criteria weights.

Build the decision matrix.

Experts and decision-makers use a scale between 0.1 and 0.9 to evaluate the criteria and alternatives. Then we combined these values by the average method.

Normalize the decision matrix.

The decision matrix is normalized between the criteria and alternatives such as:

$$q_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \quad (1)$$

Where x_{ij} refers to the value in the decision matrix, $i = 1, \dots, m; j = 1, \dots, n$

Compute the entropy

$$p_j = -h \sum_{i=1}^{m} q_{ij} \ln q_{ij} \quad (2)$$

$$h = \frac{1}{\ln m} \quad (3)$$

Compute the criteria weights

$$w_j = \frac{1-p_j}{\sum_{j=1}^{n} 1-p_j} \quad (4)$$

MARCOS Approach

This method is used to rank the alternatives.

We start with the decision matrix to compute the ideal and anti-ideal solution.

$$AAI = \min_i x_{ij} \ positive \ and \ \max_i x_{ij} \ negative \tag{5}$$

$$AI = \max_i x_{ij} \ positive \ and \ \min_i x_{ij} \ negative \tag{6}$$

Normalize the decision matrix

$$s_{ij} = \frac{x_{ij}}{x_{ai}} \ positive \tag{7}$$

$$s_{ij} = \frac{x_{ai}}{x_{ij}} \ negative \tag{8}$$

Compute the weighted matrix

$$y_{ij} = w_j s_{ij} \tag{9}$$

Compute the utility degree of alternatives

$$D_i^- = \frac{k_i}{k_{aai}} \tag{10}$$

$$D_i^+ = \frac{k_i}{k_{ai}} \tag{11}$$

$$k_i = \sum_{i=}^{n} y_{ij} \tag{12}$$

Compute the utility function of alternatives

$$f(D) = \frac{D_i^- + D_i^+}{1 + \frac{1-f(D_i^-)}{f(D_i^-)} + \frac{1-f(D_i^+)}{f(D_i^+)}} \tag{13}$$

$$f(D_i^-) = \frac{D_i^+}{D_i^- + D_i^+} \tag{14}$$

$$f(D_i^+) = \frac{D_i^-}{D_i^- + D_i^+} \tag{15}$$

Rank the alternatives.

IndetermSoft Set

Smarandache presented it in 2022[12], [13], [14]. Let H be a non-empty subset of U, P(H) be the powerset of H, and U be a universe of discourse. Assume that a is an attribute and that A is a collection of its values.

Then $F: A \rightarrow P(H)$ is referred to as an IndetermSoft Set if: i) there is some indeterminacy in the set A; ii) or the set P(H) has some indeterminacy; iii) there exists at least one attribute-value $v \in A$ such that $F(v)$ = indeterminate (unclear, incomplete, conflicting, or not unique); iv) or any two or all three of the aforementioned circumstances

The IndetermSoft Set is a specific instance of the NeutroFunction, which was defined in 2014–2015 and is a function that is only partially well-defined (inner-defined), partially indeterminate,

and partially outer-defined due to its degree of indeterminacy. A completely well-defined extension of the classical function is the NeutroFunction.

Since certain sources are unable to give precise or comprehensive information on the sets A, H, or P(H), as well as on the function F, IndetermSoft Set, an extension of the classical (determinate) Soft Set, deals with indeterminate data. We discovered indeterminacy in the real world; we did not introduce any.

Since many sources provide knowledge that is not accurate like in the Soft Set but rather approximate, ambiguous, partial, and contradictory, we still must cope with these kinds of circumstances.

Example:

Let a set of attributes select the best location with a set of attributes. We have attributes of area and price, these attributes have values such as area = (large, small), price=(low, high). These values have indeterminacy.

We can select a location based on these values.

We select location 1 or location 2 if the area is large and the price is low.

We select location 3 or location 4 if the area is large and the price is high.

3. Results and Discussion

This section shows the results of the two MCDM methods through a case study to show the validation of the methods. Three experts have a scale between 0.1 to 0.9 to evaluate the criteria and alternatives. Seven criteria and ten alternatives are used in this study. The criteria of this study are: Customer Retention, Conversion Rate, Data-Driven Decision-Making, Customer Engagement, Return on Investment, Personalization Effectiveness, and Social Media Influence. Ten alternatives of this study such as Influencer Marketing, Social Media Advertising, Video Marketing, Search Engine Optimization, AI-powered chatbots and Personalization, Pay-Per-Click Advertising, Content Marketing, Email Marketing, Affiliate Marketing, and Mobile Marketing.

Entropy Approach Results

We build the decision matrix. Experts and decision-makers use a scale between 0.1 and 0.9 to evaluate the criteria and alternatives. Then we combined these values by the average method.

We normalize the decision matrix using Eq. (1) as shown in Fig 1.

We compute the entropy using Eq. (2) as shown in Fig 2.

We compute the criteria weights using Eq. (4) as shown in Fig 3.

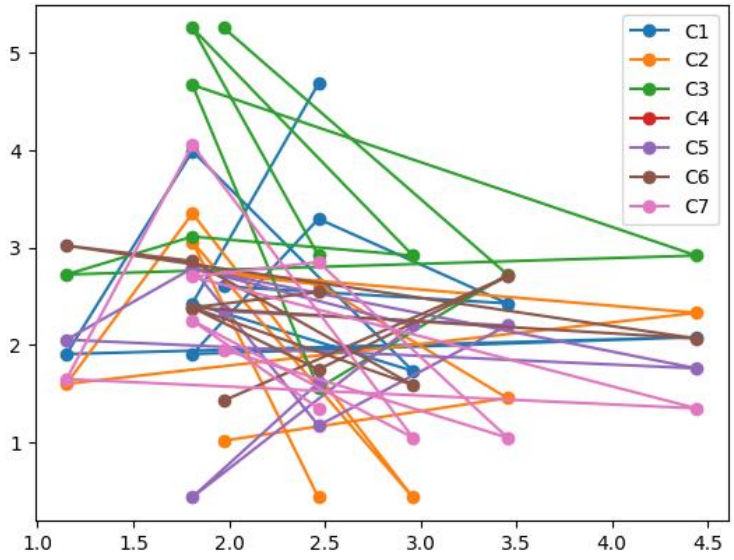

Fig 1. The normalization values by Entropy methods.

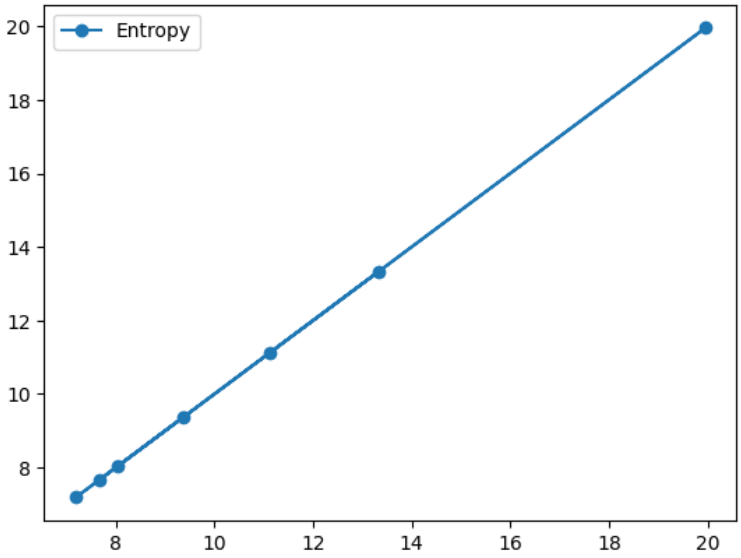

Fig 2. Entropy values.

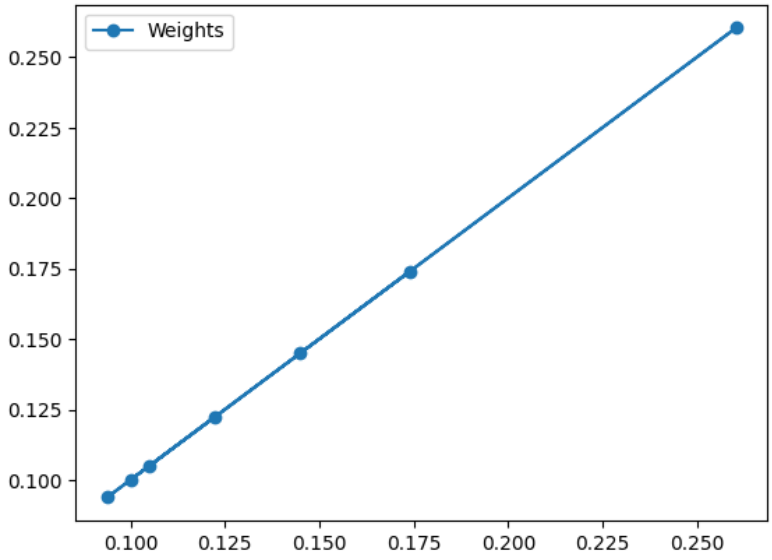

Fig 3. The criteria weights.

MARCOS Approach Results

We applied the IndetermSoft set to deal with indeterminacy in the criteria values. We have indeterminacy in the first criterion and its values are Strong, Moderate, and Weak. So, we ranked the alternatives based on these values. In the first, we let use the Strong value and then apply the MARCOS methods. The suggestions for using the values are:

$$\left\{\begin{array}{c}\text{Strong,}\\ \text{Moderate,}\\ \text{Weak,}\\ \text{Strong, Moderate}\\ \text{Strong, Weak}\\ \text{Moderate, Weak}\\ \text{Strong, Moderate, Weak}\end{array}\right\}$$

And other criteria have values of {High, Moderate, Medium, High, High, Strong}.

In the Strong value:

We normalize the decision matrix using Eq. (7) as shown in Fig 4.

We compute the weighted matrix using Eq. (9) as shown in Fig 5

Then we compute the utility degree of alternatives using Eqs. (10-12).

Then we compute the utility function of alternatives using Eq. (13).

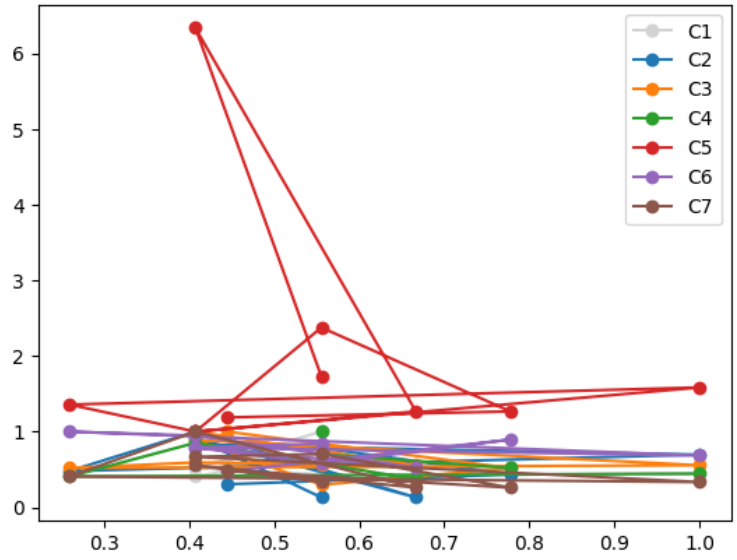

Fig 4. The normalization matrix.

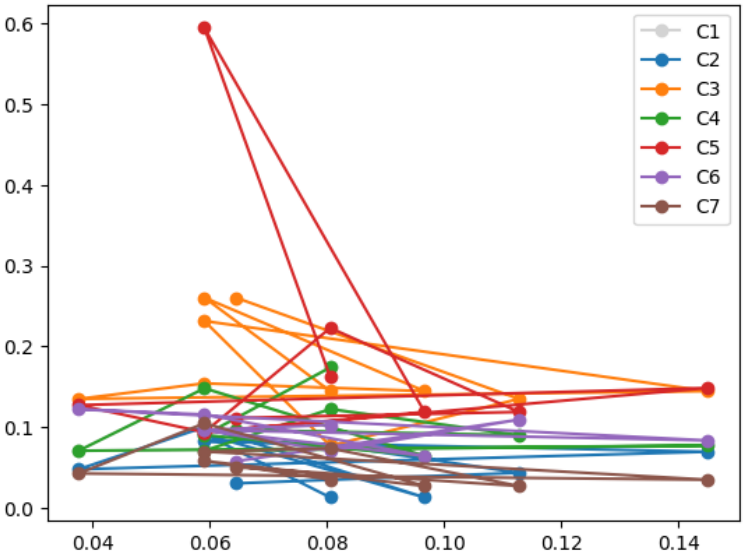

Fig 5. The weighted matrix.

In the Moderate value:

We normalize the decision matrix using Eq. (7) as shown in Fig 6.

We compute the weighted matrix using Eq. (9) as shown in Fig 7

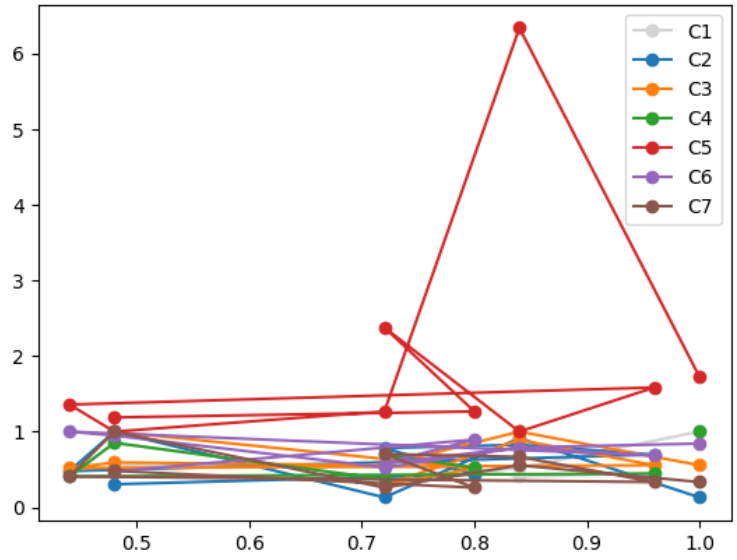

Fig 6. The normalization matrix.

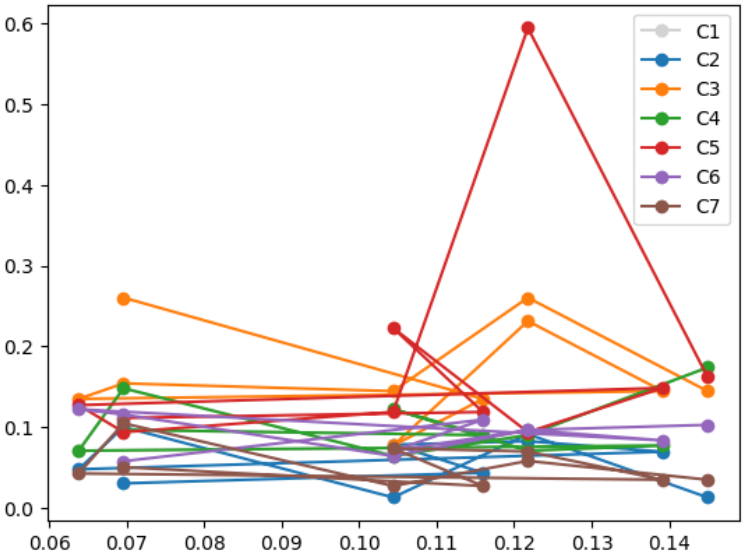

Fig 7. The weighted matrix.

In the Weak value:

We normalize the decision matrix using Eq. (7) as shown in Fig 8.

We compute the weighted matrix using Eq. (9) as shown in Fig 9.

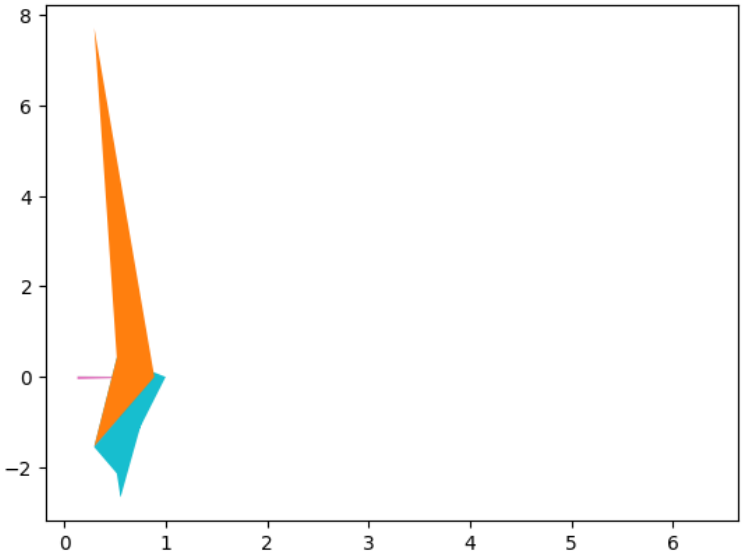

Fig 8. The normalization matrix.

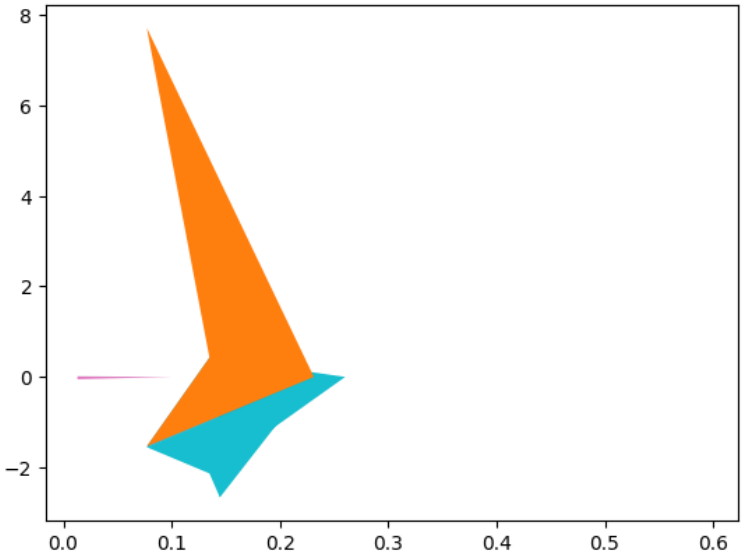

Fig 9. The weighted matrix.

In the Strong and Moderate value:

We normalize the decision matrix using Eq. (7) as shown in Fig 10.

We compute the weighted matrix using Eq. (9) as shown in Fig 11.

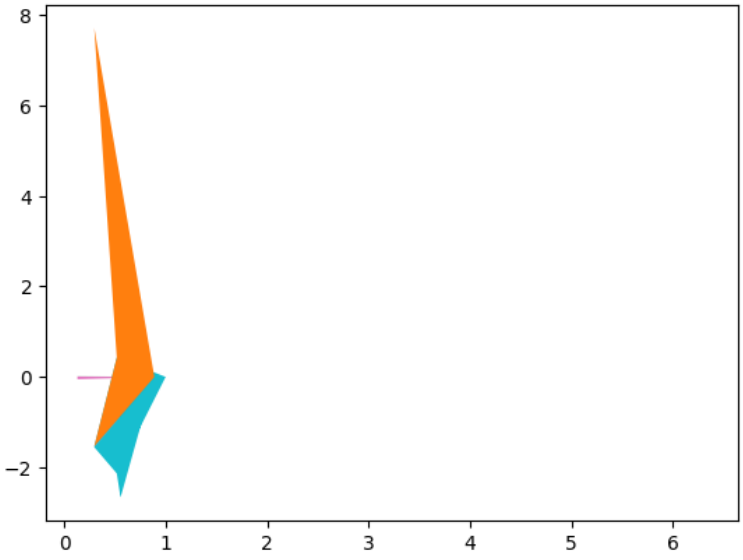

Fig 10. The normalization matrix.

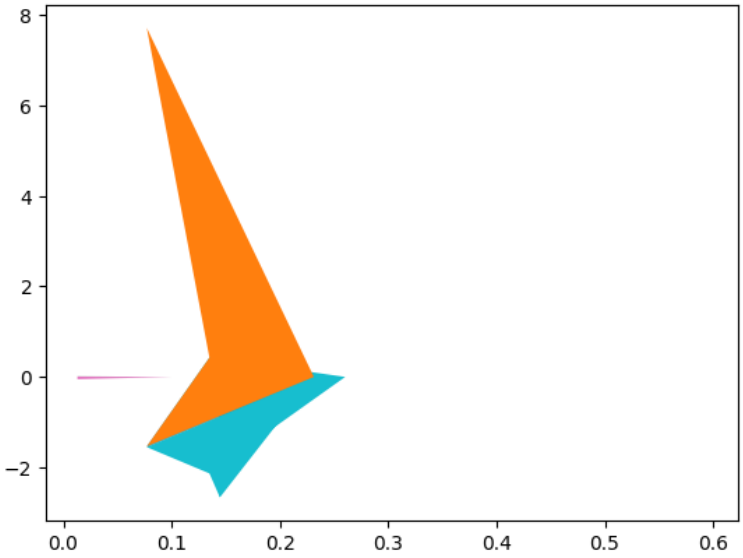

Fig 11. The weighted matrix.

In the Strong and Weak value:

We normalize the decision matrix using Eq. (7) as shown in Fig 12.

We compute the weighted matrix using Eq. (9) as shown in Fig 13.

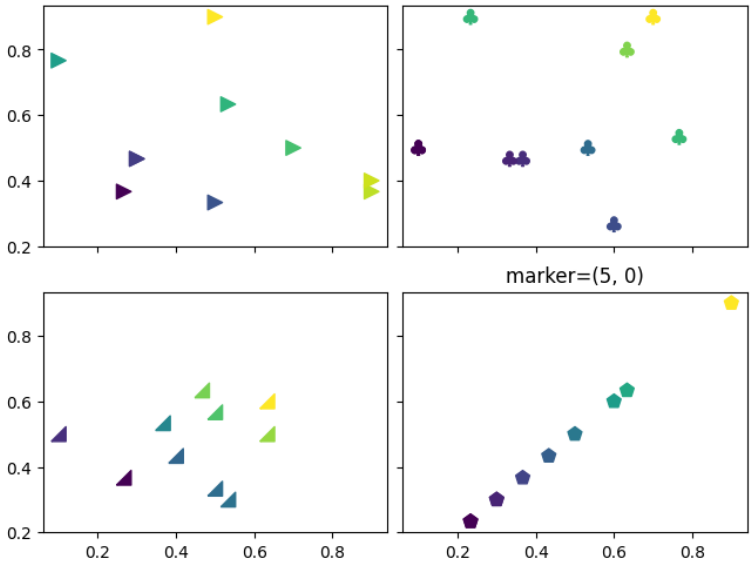

Fig 12. The normalization matrix.

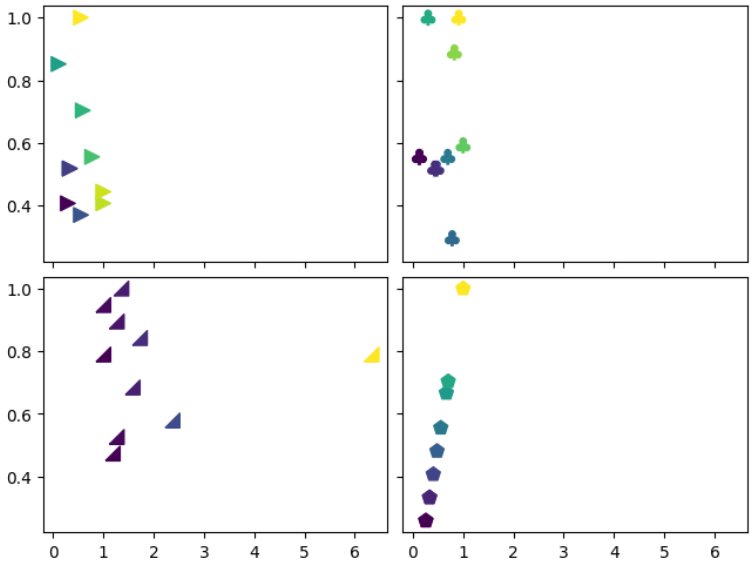

Fig 13. The weighted matrix.

In the Moderate and Weak value:

We normalize the decision matrix using Eq. (7) as shown in Fig 14.

We compute the weighted matrix using Eq. (9) as shown in Fig 15.

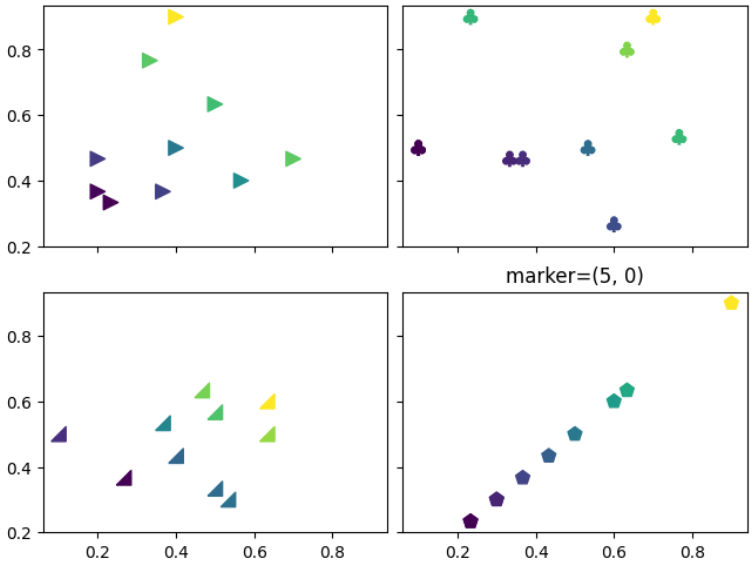

Fig 14. The normalization matrix.

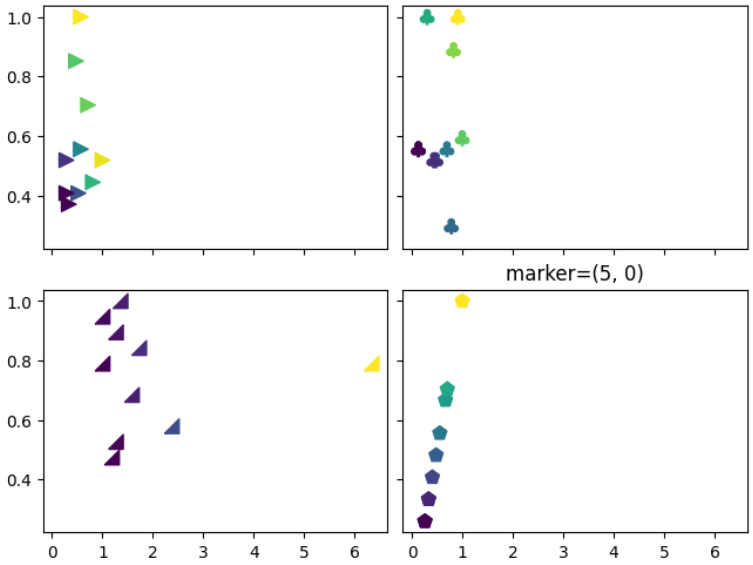

Fig 15. The weighted matrix.

In the Strong, Moderate, and Weak value:

We normalize the decision matrix using Eq. (7) as shown in Fig 16.

We compute the weighted matrix using Eq. (9) as shown in Fig 17.

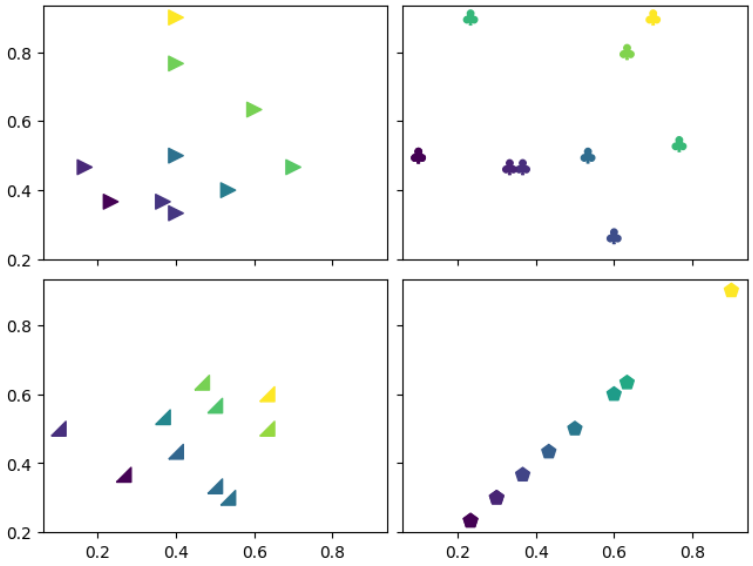

Fig 16. The normalization matrix.

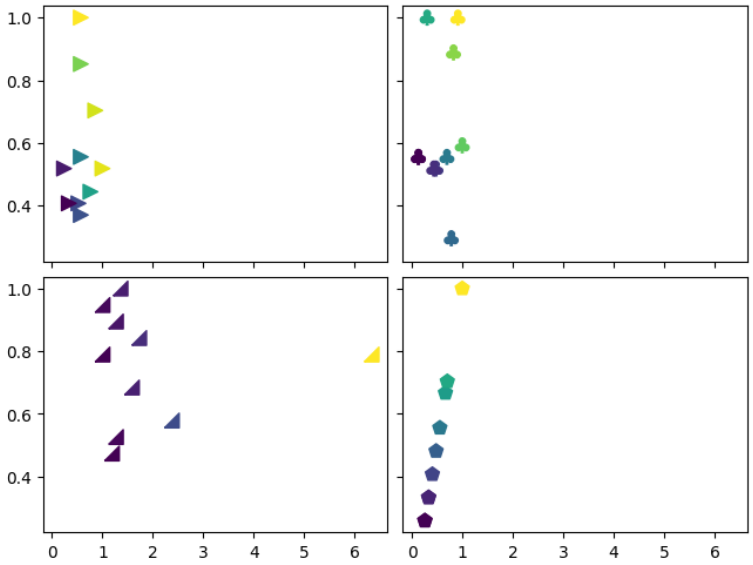

Fig 17. The weighted matrix.

Then we finally ranked the alternatives as shown in Fig 18.

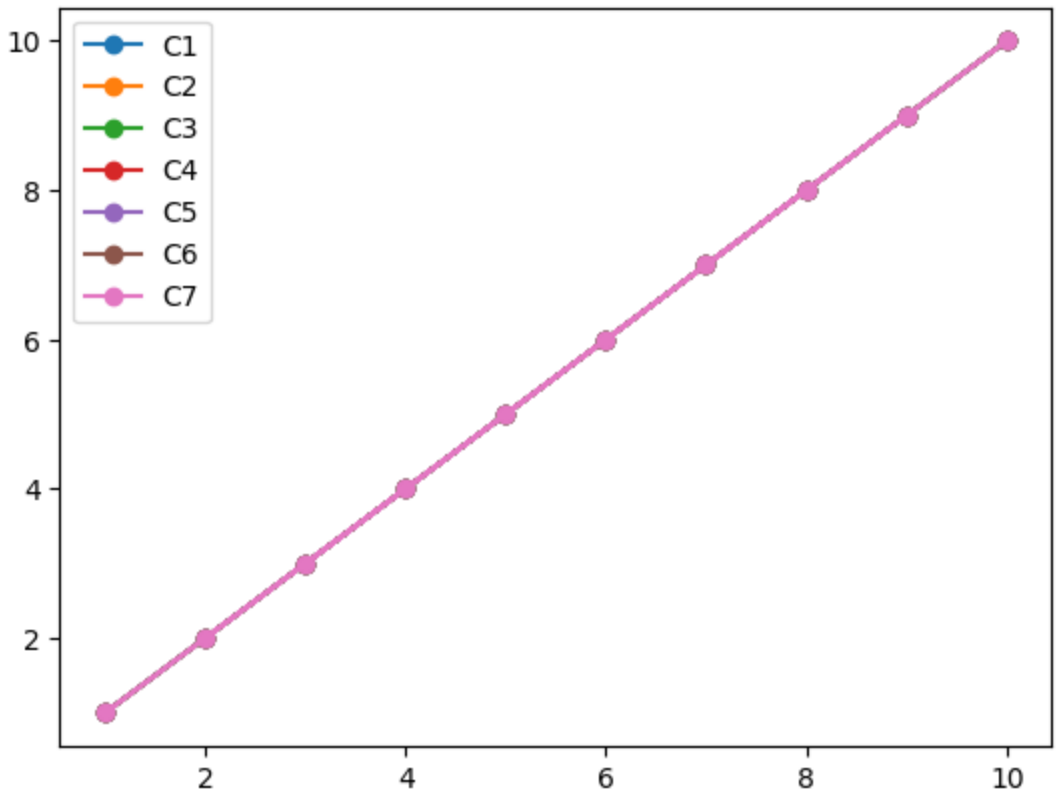

Fig 18. The final ranks.

4. Analysis

This section shows the comparative analysis between the proposed approach and other MCDM methods to show the effectiveness of the proposed approach. We compared our model with four MCDM methods as shown in Fig 19. The results showed that all models show alternative 2 is the best and alternative 3 is the worst. So, the results showed that our model is effective compared to other MCDM methods.

MARCOS method is effective compared to other MCDM methods because it deals with positive and negative criteria by computing the ideal and anti-ideal solution for negative and positive criteria. It has the utility degrees and utility functions of all alternatives. The utility function is used to rank the alternatives.

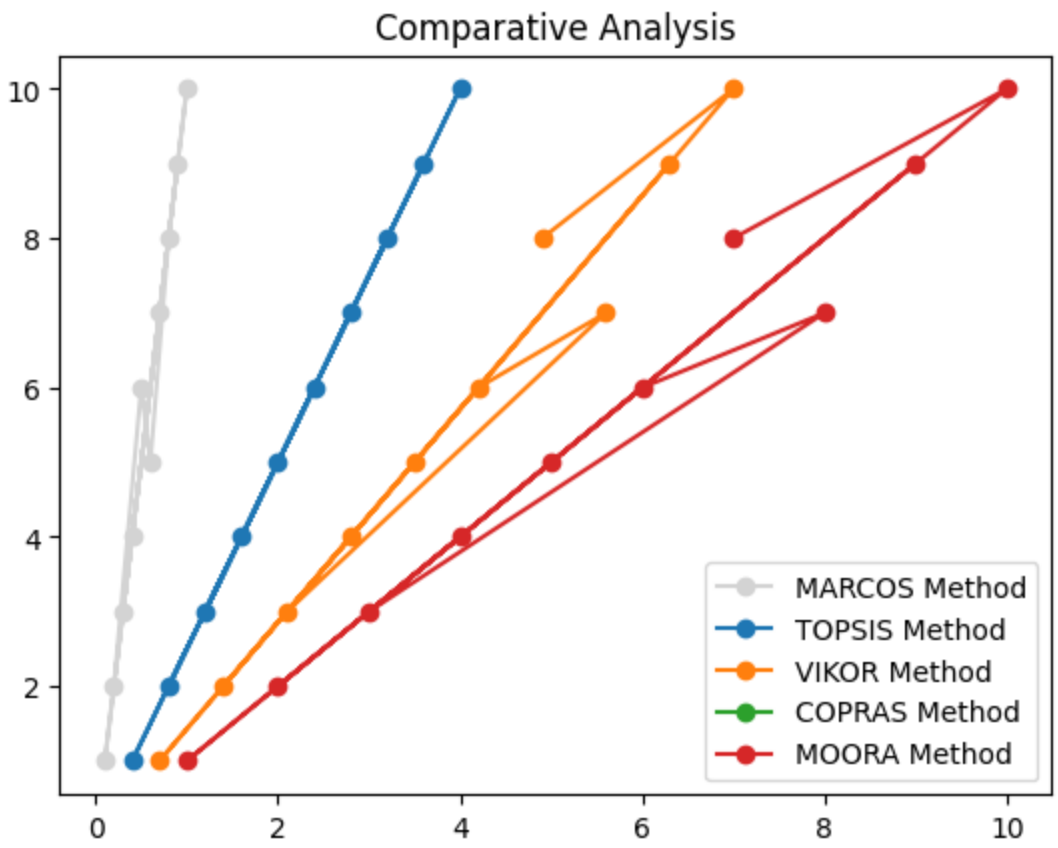

Fig 19. Comparative results

5. Conclusions

The IndetermSoft set is used in this study to treat the indeterminacy in the values of the criteria. IndetermSoft set is combined with two MCDM methods such as the Entropy method to compute the criteria weights and the MARCOS method to rank the alternatives. In this study, we have indeterminacy in the first criterion with three values such as strong, moderate, and weak. We change these values by seven cases. Then we ranked the alternatives under these cases. The results show alternative 2 is the best and alternative 3 is the worst. We compared the proposed approach with the different MCDM methods. The results show the proposed approach is effective compared to other MCDM methods.

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Multineutrosophic Analysis of Financial Statements as a Basis for Strategic Decision-Making in Uncertain Environments

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Abstract. The analysis of financial statements has been essential for managerial decision-making, guiding strategies in areas such as budgeting, investments, and risk management. However, traditional models have faced challenges due to the uncertainty and incomplete information inherent in real financial data. Therefore, this study focused on analyzing the key strategic guidelines to strengthen organizational resilience in volatile environments by integrating predictive and proactive financial analysis along with risk management. To achieve this, tools such as the Multineutrosophic Set and the ARAS multineutrosophic method were employed to evaluate the strategic guidelines and their impact. The results have indicated that the most relevant guidelines are the strengthening of predictive financial analysis and risk management, with an emphasis on organizational innovation. In conclusion, it has been observed that the integration of these guidelines enhances organizational capacity to adapt to unforeseen changes, facilitating resilient decision-making in dynamic environments.

Keywords: Financial analysis, dynamic environments, Multineutrosophic Set, ARAS multineutrosophic method**.**

1 Introduction

The analysis of financial statements plays a crucial role in managerial decision-making by providing a clear and detailed view of an organization's economic situation [1, 2]. These documents not only reflect the company's past performance but also serve as fundamental instruments for strategic planning, allowing managers to assess profitability, solvency, liquidity, and operational efficiency [3]. Therefore, in an increasingly volatile and uncertain business environment, financial statements face challenges related to uncertainty when identifying emerging trends. Moreover, they contribute to the timely adjustment of strategies, thereby ensuring long-term sustainability and growth [4].

In fact, the ability to interpret and effectively use financial statements facilitates key decision-making in areas such as budgeting, resource allocation in investments, and risk management [5, 6]. These analyses enable executives not only to assess the viability of projects and investments but also to anticipate the effects of various decisions under different economic scenarios. In this way, financial information becomes an essential component for informed decision-making that optimizes organizational performance and promotes adaptability in dynamic markets [7].

Based on the aforementioned information, this study focuses on analyzing key strategic guidelines for strengthening organizational resilience in volatile environments through the integration of predictive and proactive financial analysis, as well as financial and operational risk management. Additionally, it examines the impact of financial statement analysis on managerial decision-making by evaluating its influence on budgeting, investment management, and risk mitigation [8]. This approach aims to optimize business strategies in commercial environments characterized by uncertainty and constant change. To achieve this, tools such as the Multineutrosophic Set and the modeling of the multineutrosophic ARAS method are integrated for the selection of key strategic guidelines in dynamic and uncertain environments.

In this context, the present study seeks to expand the understanding of how the implementation of neutrosophic analysis can strengthen strategic decision-making in highly volatile financial environments. Specifically, it examines how these methods can optimize financial evaluation, reduce biases in analysis, and enhance organizational adaptive capacity in situations characterized by uncertainty and complexity, thus consolidating business resilience and sustainability.

2 Preliminaries

2.1 MultiNeutrosophic Set

Definition 1. The *Neutrosophic set N* is characterized by three membership functions [9], which are the truth-membership function T_A, indeterminacy-membership function I_A, and falsehood-membership function F_A, where U is the Universe of Discourse and $\forall x \in U$, $T_A(x), I_A(x), F_A(x) \subseteq]^-0, 1^+[$, and $^-0 \leq \inf T_A(x) + \inf I_A(x) + \inf F_A(x) \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$ [10].

See that according to the definition, $T_A(x), I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$ and hence, $T_A(x), I_A(x)$ and $F_A(x)$ can be sub-intervals of $[0, 1]$. $^-0$ and 1^+ belong to the set of hyperreal numbers.

Definition 2[10,11]. The *Single-Valued Neutrosophic Set* (SVNS) A over U is $A = \{< x, T_A(x), I_A(x), F_A(x) > : x \in U\}$, where $T_A: U \to [0, 1]$, $I_A: U \to [0, 1]$ and $F_A: U \to [0, 1]$. $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

The SVNS emerged with the idea of applying neutrosophic sets for practical purposes. Some operations between SVNN are expressed below:

Given $A_1 = (a_1, b_1, c_1)$ and $A_2 = (a_2, b_2, c_2)$ two SVNN, the sum between A_1 and A_2 is defined as:

$$A_1 \ A_2 = (a_1 + a_2 - a_1 a_2, b_1 b_2, c_1 c_2) \quad (1)$$

Given $A_1 = (a_1, b_1, c_1)$ and $A_2 = (a_2, b_2, c_2)$ two SVNN, the multiplication between A_1 and A_2 is defined as:

$$A_1 \ A_2 = (a_1 a_2, b_1 + b_2 - b_1 b_2, c_1 + c_2 - c_1 c_2) \quad (2)$$

The product of a positive scalar with an SVNN, A = (a, b, c) is defined as:

$$A = (1 - (1 - a), b, c) \quad (3)$$

The *Single-Valued Neutrosophic Number* (SVNN) is symbolized by
$N = (t, i, f)$, such that $0 \leq t, i, f \leq 1$ and $0 \leq t + i + f \leq 3$.

Definition 3 [11]. The Subset Refined Neutrosophic Set (SRNS).

Let $\mathcal{U}$ be a universe of discourse, and a set $R \subset \mathcal{U}$. Then a Subset Refined Neutrosophic R is defined as follows:

$R = \{x, x(T, I, F), x \in U\}$, where T is refined/split into p sub-truths, $T = \langle T_1, T_2, \dots T_p \rangle, T_j \subseteq [0,1], 1 \leq j \leq p$; I is refined/split into r sub-indeterminacies, $I = \langle I_1, I_2, \dots I_r \rangle, I_k \subseteq [0,1], 1 \leq k \leq r$, and F is refined/split into s sub-falsehoods, $F = \langle F_1, F_2, \dots F_l \rangle, F_s \subseteq [0,1], 1 \leq l \leq s$, where $p, r, s \geq 0$ are integers, and $p + r + s = n \geq 2$, and at least one of $p, r, s \ is \geq 2$ in order to ensure the existence of refinement (splitting).

Definition 4 ([12])**.** The MultiNeutrosophic Set (or Subset MultiNeutrosophic Set SMNS).

Let $\mathcal{U}$ be a universe of discourse and M a subset of it. Then, a MultiNeutrosophic Set is: $M = \{x, x(T_1, T_2, \dots, T_p; I_1, I_2, \dots, I_r; F_1, F_2, \dots, F_s)\}, x \in U$,

where p, r, s are integers $\geq 0, p + r + s = n \geq 2$ and at least one of $p, r, s \ is \geq 2$, in order to ensure the existence of multiplicity of at least one neutrosophic component: truth/membership, indeterminacy, or falsehood/non-membership; all subsets $T_1, T_2, \dots, T_p; I_1, I_2, \dots, I_r; F_1, F_2, \dots, F_s \subseteq [0,1]$;

$0 \leq \sum_{j=1}^{p} inf\ T_j + \sum_{k=1}^{r} inf\ I_k + \sum_{l=1}^{s} inf\ F_l \leq \sum_{j=1}^{p} sup\ T_j + \sum_{k=1}^{r} sup\ I_k + \sum_{l=1}^{s} sup\ F_l \leq n$.

No other restrictions apply on these neutrosophic multicomponents.

$T_1, T_2, \dots, T_p$ are multiplicities of the truth, each one provided by a different source of information (expert).

Similarly, $I_1, I_2, \dots, I_r$ are multiplicities of the indeterminacy, each one provided by a different source.

And $F_1, F_2, \dots, F_s$ are multiplicities of the falsehood, each one provided by a different source.

The Degree of MultiTruth (MultiMembership), also called *MultiDegree of Truth*, of the element x with respect to the set M are $T_1, T_2, \dots, T_p$.

the Degree of MultiIndeterminacy (MultiNeutrality), also called *MultiDegree of Indeterminacy*, of the element x with respect to the set M are $I_1, I_2, \dots, I_r$.

and the Degree of MultiFalsehood (MultiNonmembership), also called *MultiDegree of Falsehood*, of element x with respect to the set M are $F_1, F_2, \dots, F_s$.

All these $p + r + s = n \geq 2$ are assigned by n sources (experts) that may be:

- either totally independent;
- or partially independent and partially dependent;
- or totally dependent; according or as needed to each specific application.

A generic element x with regard to the MultiNeutrosophic Set A has the form:

$x(T_1, T_2, \dots, T_p;$	$I_1, I_2, \dots, I_r;$	$F_1, F_2, \dots, F_s)$
multi-truth	multi-indeterminacy	multi-falsehood

In many particular cases $p = r = s$, and a source (expert) assigns all three degrees of truth, indeterminacy, and falsehood T_j, I_j, F_j for the same element.

Definition 5 [12]. Ranking of n-valued MultiNeutrosophic types of the same (p, r, s)-form.

$(T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s)$, where p, r, s are integers ≥ 0, and $p + r + s = n \geq 2$, and at least one of $p, r, s \geq 2$ to be sure that it has multiplicity for at least one neutrosophic component (either truth, or indeterminacy, or falsehood).

It offers an easier n-ranking, but this is rather an approximation. Let's compute the following.

Average positivity (4).

$$\frac{\sum_{j=1}^{p} T_j + \sum_{k=1}^{r}(1 - I_k) + \sum_{e=1}^{s}(1 - F_e)}{p + r + s} \tag{4}$$

Average (Truth-Falsehood) (5)

$$\frac{\sum_{j=1}^{p} T_j + \sum_{e=1}^{s}(1 - F_e)}{p + s} \tag{5}$$

Average Truth (6).

$$\frac{\sum_{j=1}^{p} T_j}{p} \tag{6}$$

Definition 6 [12]. Ranking n-valued MultiNeutrosophic tuples of different (p, r, s)–forms.

Let's consider two n-valued multi neutrosophic tuples of the forms (p_1, r_1, s_1) and respectively (p_2, r_2, s_2), where $p_1, r_1, s_1, p_2, r_2, s_2$ are integers ≥ 0, and $p_1 + r_1 + s_1 = n_1 \geq 2$, and at least one of $p_1, r_1, s_1 is \geq 2$, to be sure that there is multiplicity for at least one neutrosophic component (either truth, or indeterminacy, or falsehood); similarly $p_2 + r_2 + s_2 = n_2 \geq 2$, and at least one of $p_2, r_2, s_2\ is \geq 2$.

Let's take the following Single-Valued Multi Neutrosophic Tuples (SVMNT):

$SVMNT = (T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s)$ of $(p_1, r_1, s_1) - form$, and

$SVMNT' = (T'_1, T'_2, \ldots, T'_p; I'_1, I'_2, \ldots, I'_r; F'_1, F'_2, \ldots, F'_s)$ of $(p_1, r_1, s_1) - form$.

It makes the classical averages of truth (T_a), indeterminacies (I_a) and falsehood (F_a), respectively for $SVMNT = (T_a, I_a, F_a)$ and the averages of truths (T_a), indeterminacies (I_a) and falsehood (F_a) respectively for: $SVMNT = (T'_a, I'_a, F'_a)$. And then it applies the Score (S), Accuracy (A), and Certainty (C) Functions, as for the single valued neutrosophic set:

Compute the Score Function (average of positiveness) (7).

$$S(T_a, I_a, F_a) = \frac{T_a + (1 - I_a) + (1 - F_a)}{3}$$
$$S(T'_a, I'_a, F'_a) = \frac{T'_a + (1 - I'_a) + (1 - F'_a)}{3} \tag{7}$$

i. If $S(T_a, I_a, F_a) \geq S(T'_a, I'_a, F'_a)$ then $SVMNT \geq SVMNT'$,
ii. If $S(T_a, I_a, F_a) \leq S(T'_a, I'_a, F'_a)$ then $SVMNT \leq SVMNT'$,
iii. And if $S(T_a, I_a, F_a) = S(T'_a, I'_a, F'_a)$ then $SVMNT = SVMNT'$, then go to the second step.

Compute the Accuracy Function (difference between the truth and falsehood) (8).

$$A(T_a, I_a, F_a) = T_a - F_a$$
$$A(T'_a, I'_a, F'_a) = T'_a - F'_a \tag{8}$$

i. If $A(T_a, I_a, F_a) \geq A(T'_a, I'_a, F'_a)$ then $SVMNT \geq SVMNT'$,
ii. If $A(T_a, I_a, F_a) \leq A(T'_a, I'_a, F'_a)$ then $SVMNT \leq SVMNT'$,
iii. And if $A(T_a, I_a, F_a) = A(T'_a, I'_a, F'_a)$ then $SVMNT = SVMNT'$, then go to the third step.

3. Compute the Certainty Function (truth) (9).

$$C(T_a, I_a, F_a) = T_a$$
$$C(T'_a, I'_a, F'_a) = T'_a \tag{9}$$

i. If $C(T_a, I_a, F_a) \geq C(T'_a, I'_a, F'_a)$ then $SVMNT \geq SVMNT'$,
ii. If $C(T_a, I_a, F_a) \leq C(T'_a, I'_a, F'_a)$ then $SVMNT \leq SVMNT'$,
iii. And if $C(T_a, I_a, F_a) = C(T'_a, I'_a, F'_a)$ then $SVMNT = SVMNT'$ are multi-neutrosophically equal, i.e. $T_a = T'_a, I_a = I'_a, F_a = F'_a$, or their corresponding truth, indeterminancy, and falsehood averages are equal.

Definition 7 [12]. In cases some sources have a greater weight in evaluation than others, one uses the weighted averages, indexed as T_{wa}, I_{ua}, F_{va} and $T'_{wa}, I'_{ua}, F'_{va}$, respectively. Because the sources may be independent or partially independent, the sum of weights should not necessarily be equal to 1. As such:

i. $w_1, w_2, \ldots, w_p \in [0,1]$, while the sum $w_1 + w_2 + \cdots + w_p$ may be $< 1, or\ = 1, or\ > 1$.
ii. $u_1, u_2, \ldots, u_p \in [0,1]$, while the sum $u_1 + u_2 + \cdots + u_p$ may be $< 1, or\ = 1, or\ > 1$.
iii. $v_1, v_2, \ldots, v_p \in [0,1]$, while the sum $v_1 + v_2 + \cdots + v_p$ may be $< 1, or\ = 1, or\ > 1$.

And, similarly, one applies the Score, Accuracy, and Certainty Functions on these weighted averages to rank them.

In 2013, Smarandache [13] refined and split the neutrosophic components (T, I, F) into more detailed neutrosophic subcomponents ($T1, T2, \ldots$; $I1, I2, \ldots$; $F1, F2, \ldots$), allowing for greater granularity in modeling uncertainty and imprecision. This refinement is based on the need to more accurately represent the levels of truth, indeterminacy, and falsehood in complex systems.

This refined structure has been key in the development of new applications in artificial intelligence, decision-making, and computational modeling, as it enables a more detailed analysis of uncertain information. Additionally, it has been demonstrated that the MultiNeutrosophic Set is isomorphic to the Refined Neutrosophic Set [12], implying that both models can represent the same underlying mathematical structure, providing different approaches to managing uncertainty.

3. Materials and Methods

The ARAS (Additive Ratio Assessment) method is a multi-criteria decision-making technique used to select the best option from a set of alternatives [15]. In this case, the study establishes among its objectives a series of strategic guidelines aimed at enhancing decision-making in financial analysis. To this end, an extension of the traditional method is proposed through evaluation using multineutrosophic sets. Consequently, it is reformulated as the multineutrosophic ARAS method to determine the complex relative efficiency of each strategic guideline. This involves evaluating each strategic guideline through multiple sources (experts) based on the corresponding criteria(Figure 1).

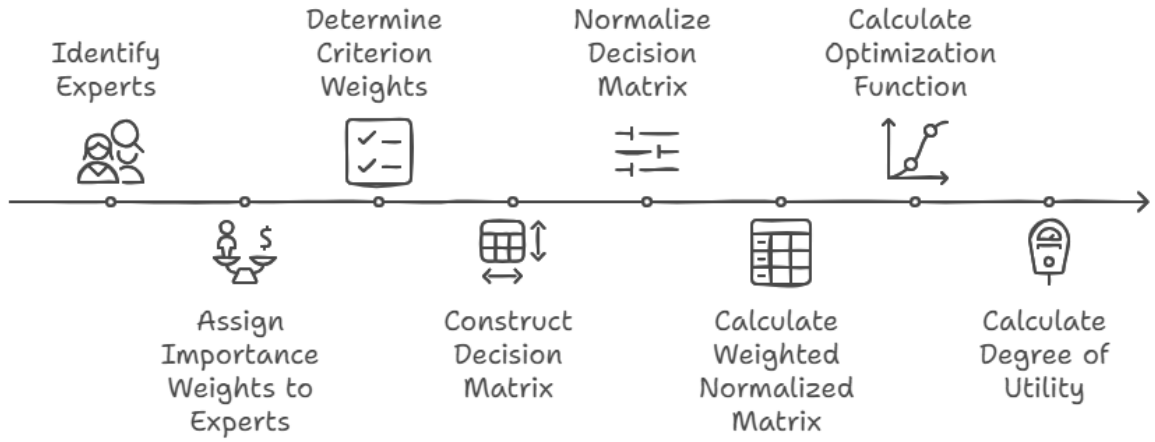

Figure 1. Multicriteria Decision Making Process

By integrating the analysis of multineutrosophic sets into the ARAS method, the following steps are defined:

Step 1: Identify multiple sources (experts) for the multi-criteria evaluation and assign an importance weight to each expert based on their knowledge and contribution to financial statement analysis (according to Definition 7 in Section 2.1). For this purpose, the neutrosophic AHP method by Saaty is applied (following procedures referenced in bibliographic sources [13,15]).

Step 2: Determine the importance weights of each criterion in decision-making for each source (expert).

Step 3: Construct the decision matrix L_{ij} (see Figure 2), where the element L_{ij} represents each strategic guideline (SG) evaluated by multiple sources (experts (Exp.), according to Definitions 5 and 6 in Section 2.1) based on an identified criterion (C).

$$\begin{bmatrix} l_{11} & l_{12} & \ldots & l_{1j} & \ldots & l_{1n} \\ l_{21} & l_{22} & \ldots & l_{2j} & \ldots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ l_{i1} & l_{i2} & \ldots & l_{ij} & \ldots & l_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ l_{m1} & l_{m2} & \ldots & l_{mj} & \ldots & l_{mn} \end{bmatrix}$$

Figure 2: Decision matrix L_{ij} for the multineutrosophic ARAS method. Source: Own elaboration.

Step 4: The normalized decision matrix $\bar{L}_{ij}$, considering the benefit and cost values, is calculated using Equation (10) and (11):

$$\bar{L}_{ij} = \frac{l_{ij}}{\sum_{i=0}^{m} l_{ij}} \tag{10}$$

$$L_{ij} = \frac{1}{l_{ij}^{*}} \tag{11}$$

Step 5: The weighted normalized decision matrix is calculated using Equation (12).

$$\hat{L}_{ij} = \bar{L}_{ij} \cdot W_j \tag{12}$$

The weight values $W_j$are determined by the Entropy method. Where W_j is the weight of criterion j and $\bar{L}_{ij}$ is the normalized ranking of each criterion.

Step 6: Calculation of the optimization function S_i using Equation (13).

$$G_i = \sum \hat{L}_{ij} \tag{13}$$

Where G_i is the value of the optimization function of the alternative i. This calculation has a directly proportional relationship with the process of the values $\hat{L}_{ij}$ and weights W_j of the investigated criteria and their relative influence on the result.

Step 7: Calculation of the degree of utility. This degree is determined by comparing the variant being analyzed with the best one G_o, according to Equation (14).

$$K_i = \frac{G_i}{G_o} \tag{14}$$

Where G_i and G_o are the values of the optimization function. These values range from 0 to 100%, therefore, the alternative with the highest K_i is the best of the alternatives analyzed.

4 Case Study

4.1 Impact of financial information on strategic decision-making.

The quality and relevance of financial information are crucial for effective managerial decision-making. An analysis of financial statements must evaluate three key aspects: relevance, reliability, and timeliness. Relevance ensures that the information influences strategic and operational decisions by providing meaningful data aligned with corporate objectives, such as key performance indicators including EBITDA (Earnings Before Interest, Taxes, Depreciation, and Amortization), ROE (Return on Equity), and net profit margin. In contrast, reliability guarantees that the data accurately reflect the company's financial situation by preventing biases and material errors, which is essential for financial stability. Meanwhile, timeliness ensures that information is available promptly, enabling proactive decision-making in response to threats or opportunities and preventing outdated decisions that could impact organizational performance.

Additionally, financial analysis is critical for budgeting and investment planning, as it allows for assessing an organization's economic performance and making informed strategic decisions. In budgeting, it facilitates the realistic projection of revenues, expenses, and capital needs while ensuring control and identifying deviations. In investment planning, financial analysis evaluates the feasibility and profitability of projects using tools such as Net Present Value (NPV) and Internal Rate of Return (IRR), while managing risks related to liquidity, solvency, and capital structure. Key profitability, liquidity, solvency, and operational efficiency indicators guide strategic and operational decisions, optimizing resource allocation and maximizing organizational profitability.

Similarly, financial analysis is essential in corporate risk management, as it helps identify potential threats and adopt proactive measures to mitigate risks. By analyzing key indicators such as liquidity, solvency, profitability, and operational efficiency, financial, operational, and strategic risks can be detected. Early detection of these risks facilitates the implementation of preventive and corrective strategies. Furthermore, financial analysis enables forecasting future risks through simulations and projections, optimizing resource allocation and enhancing organizational stability. This continuous process, supported by monitoring tools and predictive analysis, ensures the resilience and adaptability of the organization in a changing environment.

Moreover, integrating expert judgment, financial analysis, and MultiNeutrosophy optimizes strategic decision-making by combining the rigor of financial metrics with contextual experience and a comprehensive analysis of uncertainty. Thus, it enables the processing of multiple sources of information by balancing quantitative and qualitative dimensions, strengthening organizational resilience and proactive risk management in dynamic environments. Its practical application is evident in the combined evaluation of profitability and risks, the implementation of dynamic dashboards, scenario simulations, comprehensive opportunity identification, and multidisciplinary participation, all supported by continuous monitoring and iterative feedback. This approach enables organizations to make more accurate decisions, avoiding those based solely on numerical data or unsubstantiated intuition.

4.2 Strategic guidelines: Integration of MultiNeutrosophy in financial analysis.

The integration of financial analysis with multi-expert judgment provided by the MultiNeutrosophic Set and advanced technological tools enables companies to develop a more informed, adaptable, and resilient decision-making process. By adopting these strategic guidelines, organizations not only strengthen their ability to face uncertainty and constant change but also promote continuous improvement in business management, ensuring sustainable and competitive growth (see Table 1).

No	Strategic guideline	Objective	Strategies	Impact
LE1	Strengthening predictive and proactive financial analysis	Anticipate scenarios and potential risks through financial projections adjusted to the variability of possible scenarios.	Implement predictive analysis models based on time series and machine learning to project revenue, expenses, and cash flows while accounting for uncertainty. Develop advanced multineutrosophic sensitivity analyses and scenario simulations (optimistic, moderate, pessimistic, and neutrosophic), evaluated by multi-experts. Conduct quarterly projections adjusted according to market evolution and internal variations.	Enhances the ability to make informed, adaptive, and resilient decisions in the face of economic uncertainties and variable scenarios.
LE2	Encouraging a culture of innovation and financial adaptability	Promote organizational flexibility through agile and adaptive financial practices.	Establish flexible budgets that allow dynamic adjustments in response to macroeconomic changes. Encourage continuous training of the financial team in advanced tools and multineutrosophic approaches for decision-making. Promote the adoption of emerging technologies, such as financial process automation and artificial intelligence.	Increases organizational innovation and adaptability by maintaining financial stability amid unpredictable market changes.
LE3	Prioritizing key financial and non-financial performance indicators (KPIs)	Comprehensively monitor organizational performance using both quantitative and qualitative metrics.	Define key financial KPIs such as EBITDA, profit margin, asset turnover, and cash conversion cycle, considering uncertain scenarios. Complement financial KPIs with non-financial indicators such as customer satisfaction, operational efficiency, and organizational commitment, incorporating neutrosophic variations in their analysis. Automate real-time data collection and analysis, adapting it to projected scenarios.	Improves organizational visibility and flexibility by optimizing resource allocation and operational efficiency through quantitative and qualitative metrics.
LE4	Integrating expert judgment into financial evaluation	Complement quantitative analysis with the contextualized expertise of professionals to maximize evaluation accuracy.	Establish flexible multidisciplinary committees to evaluate investment projects, combining quantitative analysis with expert judgment from key areas (operations, marketing, legal). Incorporate feedback from industry experts and external consultants to integrate perspectives on uncertainty. Develop interactive dashboards that combine financial indicators with qualitative interpretations by integrating multineutrosophic evaluation from multiple sources.	Enhances decision accuracy and reduces cognitive biases, promoting more comprehensive and adaptive choices in uncertain contexts.
LE5	Promoting continuous improvement through financial audits and feedback	Ensure the accuracy, transparency, and relevance of financial information for long-term decision-making.	Implement periodic internal and external audits to validate the reliability of information, incorporating variability analysis. Establish continuous feedback mechanisms in which financial results are analyzed and adjusted according to uncertain scenarios. Develop dynamic financial reports accessible at all management levels, adjusting them in real time.	Strengthens organizational transparency by aligning decisions with evolving business environments.
LE6	Focus on financial and operational risk management	Identify, mitigate, and manage financial and operational risks through a multisource (multineutrosophic) analysis that considers	Create a financial risk map that considers variables such as exchange rate exposure and credit risk while accounting for uncertainties. Implement early warning alerts based on key metrics adjustable to internal and external uncertainties. Design contingency plans by simulating adverse	Mitigates negative impacts from unforeseen events by improving resilience against complex financial and operational risks.

		environmental uncertainties.	scenarios to reformulate strategies in the face of potential crises.	
LE7	Implementing a dynamic working capital management approach	Optimize organizational liquidity and solvency through a flexible working capital management approach.	Establish collection and payment policies aligned with cash flow projections adjusted for uncertainty analysis. Continuously monitor inventory turnover and adjust purchasing decisions through demand simulations, integrating multineutrosophic analysis. Apply financial analysis to identify cost-reduction opportunities without compromising quality.	Increases organizational liquidity and enhances responsiveness to uncertain scenarios with a flexible approach.

Table 1: Strategic guidelines for continuous improvement in business decision-making. Source: own elaboration.

3.3 Multineutrosophic ARAS Modeling. Selection of the strategic guideline.

Step 1: Selection of multiple sources (experts) in the multicriteria evaluation. Modeling of the neutrosophic Saaty AHP method.

To provide an effective interpretation of financial statements, it is crucial to involve experts from various disciplines who can offer a comprehensive and detailed view of an organization's financial situation (see Table 2). Below, eight expert professions are evaluated to assess each strategic guideline.

Expert	Exp-1	Exp-2	Exp-3	Exp-4	Exp-5	Exp-6	Exp-7	Exp-8
Profession	Certified Public Accountant	Internal Auditor	Economist	Financial risk manager	Financial Analyst	Tax advisor	Financial Consultant	Corporate Lawyer

Table 2: Multi-sources for the neutrosophic evaluation (experts). Source: Own elaboration.

To assign an importance weight to each expert in relation to their knowledge and contribution to the analysis of financial statements, several factors must be considered, such as technical specialization, the ability to interpret financial data, and experience in the practical application of such knowledge. Below, a weight distribution is proposed, based on the modeling of Saaty's neutrosophic AHP method (see Tables 3 and 4). Therefore, it should be noted that some roles have a more direct impact on the interpretation and analysis of financial statements, while others provide complementary perspectives.

Source	Exp-1	Exp-2	Exp-3	Exp-4	Exp-5	Exp-6	Exp-7	Exp-8
Exp-1	(0.54,0.51,0.52)	(0.64,0.41,0.42)	(0.94,0.11,0.12)	(0.94,0.11,0.12)	(0.54,0.51,0.52)	(0.94,0.11,0.12)	(0.74,0.31,0.32)	(0.84,0.21,0.22)
Exp-2	(0.44,0.61,0.62)	(0.54,0.51,0.52)	(0.74,0.31,0.32)	(0.74,0.31,0.32)	(0.24,0.81,0.82)	(0.74,0.31,0.32)	(0.54,0.51,0.52)	(0.74,0.31,0.32)
Exp-3	(0.14,0.91,0.92)	(0.34,0.71,0.72)	(0.54,0.51,0.52)	(0.64,0.41,0.42)	(0.34,0.71,0.72)	(0.64,0.41,0.42)	(0.44,0.61,0.62)	(0.64,0.41,0.42)
Exp-4	(0.14,0.91,0.92)	(0.34,0.71,0.72)	(0.44,0.61,0.62)	(0.54,0.51,0.52)	(0.24,0.81,0.82)	(0.64,0.41,0.42)	(0.44,0.61,0.62)	(0.54,0.51,0.52)
Exp-5	(0.54,0.51,0.52)	(0.84,0.21,0.22)	(0.74,0.31,0.32)	(0.84,0.21,0.22)	(0.54,0.51,0.52)	(0.84,0.21,0.22)	(0.64,0.41,0.42)	(0.84,0.21,0.22)
Exp-6	(0.14,0.91,0.92)	(0.34,0.71,0.72)	(0.44,0.61,0.62)	(0.44,0.61,0.62)	(0.24,0.81,0.82)	(0.54,0.51,0.52)	(0.54,0.51,0.52)	(0.54,0.51,0.52)
Exp-7	(0.34,0.71,0.72)	(0.54,0.51,0.52)	(0.64,0.41,0.42)	(0.64,0.41,0.42)	(0.44,0.61,0.62)	(0.54,0.51,0.52)	(0.54,0.51,0.52)	(0.74,0.31,0.32)
Exp-8	(0.24,0.81,0.82)	(0.34,0.71,0.72)	(0.44,0.61,0.62)	(0.54,0.51,0.52)	(0.24,0.81,0.82)	(0.54,0.51,0.52)	(0.34,0.71,0.72)	(0.54,0.51,0.52)

Table 3: Neutrosophic AHP paired matrix. Source: Own elaboration.

Source	A x Weight	Weight	Approximate Eigenvalues
Exp-1	2.93	0.32	9.151851
Exp-2	1.21	0.13	9.227916
Exp-3	0.53	0.06	8.537562
Exp-4	0.33	0.04	8.504645
Exp-5	2.87	0.30	9.535476
Exp-6	0.30	0.03	8.657509
Exp-7	0.88	0.10	9.121778

The analysis of the consistency of the method showed that its eigenvalue is 8.90, IC=0.13 and RC=0.09, which confirms that the exercise was consistent.

Exp-8	0.26	0.03	8.470680
Eigenvalue = 8.900927018			

Table 4: Analysis of the consistency of the paired matrix. Source: Own elaboration.

The results of the modeling of Saaty's AHP method highlight two experts with the greatest weight in the multineutrosophic evaluation of each strategic guideline, based on their experience in dynamic environments and financial and operational risks. Among them, the *certified public accountant* stands out, ensuring the accuracy and compliance of financial reports with a weight of 0.32. Meanwhile, the financial analyst complements this role through projections and strategic analysis, with a weight of 0.30.

Step 2: Selection and evaluation of criteria by each multi-source (expert).

The evaluation criteria focus on the ability of strategic guidelines to influence managerial decisions and achieve organizational objectives. These include:

- Relevance for decision-making (C1): Measures the impact on the quality of decisions.
- Improvement of transparency and accuracy (C2): Values the reliability and clarity of the resulting decisions.
- Flexibility and agility (C3): Determines the ability to respond to new situations
- Ability to adapt to the economic environment (C4): Evaluates flexibility in the face of changes and risks.
- Contribution to organizational stability (C5): Analyzes the effectiveness in strengthening financial stability.

Therefore, once each criterion is defined, the next step is to evaluate the weight of importance based on evaluations from multiple sources. As shown in Table 5, the highest-weighted criterion is C4, followed by C1. Since the scores in the score function are different, there is no need to apply the Accuracy and Certainty functions in the multineutrosophic analysis.

$(\{T_1,T_2,T_3\},\{I_1,I_2\},\{F_1,F_2,F_3,F_4\})$	(T_a,I_a,F_a)	Weight	Score (S)
C1({0.9,0.5,0.6},{0.6,0.1},{0.5,0.6,0.7,0.9})	(0.67,0.35,0.68)	0.25	0.55
C2({0.2,0.4,0.1},{0.2,0.1},{0.3,0.5,0.3,0.4})	(0.23,0.35,0.38)	0.09	0.50
C3({0.4,0.4,0.6},{0.4,0.1},{0.8,0.8,0.4,0.5})	(0.47,0.45,0.63)	0.17	0.46
C4({0.8,0.7,0.9},{0.2,0.1},{0.3,0.6,0.3,0.2})	(0.8,0.45,0.35)	0.30	0.67
C5({0.2,0.9,0.5},{0.7,0.1},{0.2,0.4,0.4,0.8})	(0.53,0.6,0.45)	0.20	0.49

Table 5: Multineutrosophic evaluation of each criterion. Source: Own elaboration.

Step 3 to 7: Calculation of the optimization function S_i, from the formation of the multineutrosophic decision matrix L_{ij}.

Once the weight of the set of criteria and the evaluation sources is defined, the multineutrosophic ARAS method is applied to evaluate each strategic guideline. First, the multineutrosophic decision matrix L_{ij} must be defined, and then the resulting optimization function S_i of each strategic guideline is determined based on its impact on managerial decision-making (see Tables 6 and 7). For the modeling, it was determined that all criteria have been classified as "B" (Benefit) in the evaluation of each guideline.

$(\{T_1,T_2,T_3\},\{I_1,I_2\},\{F_1,F_2,F_3,F_4\})$	$l_{ij}=(T_a,I_a,F_a)$	Score (S)
Criterion: C1		
LE1({0.8,0.5,0.4},{0.8,0.9},{0.5,0.6,0.2,0.2})	(0.6,0.84,0.34)	0.47
LE2({0.5,0.5,0.4},{0.7,0.9},{0.1,0.3,0.3,0.9})	(0.48,0.7,0.53)	0.42
LE3({0.6,0.7,0.1},{0.4,0.9},{0.5,0.1,0.5,0.2})	(0.51,0.29,0.25)	0.66
LE4({0.8,0.3,0.9},{0.3,0.9},{0.2,0.9,0.1,0.7})	(0.66,0.34,0.6)	0.57
LE5({0.3,0.1,0.8},{0.4,0.9},{0.4,0.9,0.1,0.8})	(0.36,0.29,0.65)	0.47
LE6({0.9,0.4,0.2},{0.6,0.9},{0.1,0.3,0.5,0.3})	(0.56,0.53,0.33)	0.57
LE7({0.7,0.8,0.8},{0.6,0.9},{0.5,0.9,0.9,0.6})	(0.76,0.53,0.75)	0.49
Criterion: C2		
LE1({0.7,0.6,0.3},{0.2,0.9},{0.8,0.7,0.8,0.5})	(0.57,0.43,0.65)	0.50
LE2({0.5,0.6,0.3},{0.3,0.9},{0.5,0.6,0.1,0.9})	(0.48,0.45,0.61)	0.47
LE3({0.2,0.6,0.7},{0.8,0.9},{0.7,0.3,0.7,0.6})	(0.46,0.69,0.53)	0.41
LE4({0.6,0.5,0.6},{0.1,0.9},{0.4,0.8,0.7,0.7})	(0.57,0.33,0.71)	0.51
LE5({0.4,0.6,0.9},{0.7,0.9},{0.4,0.8,0.3,0.5})	(0.59,0.74,0.54)	0.44
LE6({0.7,0.9,0.2},{0.2,0.9},{0.7,0.2,0.4,0.7})	(0.64,0.28,0.48)	0.63
LE7({0.1,0.1,0.8},{0.1,0.9},{0.3,0.4,0.7,0.9})	(0.28,0.4,0.66)	0.41
Criterion: C3		
LE1({0.2,0.7,0.4},{0.5,0.9},{0.3,0.5,0.5,0.1})	(0.43,0.35,0.35)	0.58
LE2({0.3,0.1,0.9},{0.9,0.9},{0.3,0.2,0.2,0.1})	(0.43,0.5,0.2)	0.58

LE3({0.4,0.7,0.9},{0.2,0.9},{0.9,0.3,0.3,0.7})	(0.67,0.45,0.55)	0.56
LE4({0.7,0.2,0.8},{0.8,0.9},{0.5,0.8,0.1,0.5})	(0.57,0.65,0.48)	0.48
LE5({0.6,0.6,0.1},{0.9,0.9},{0.4,0.9,0.2,0.8})	(0.43,0.65,0.58)	0.40
LE6({0.6,0.5,0.2},{0.3,0.9},{0.9,0.6,0.5,0.2})	(0.43,0.55,0.55)	0.44
LE7({0.7,0.8,0.6},{0.3,0.9},{0.8,0.2,0.8,0.4})	(0.7,0.35,0.55)	0.60
Criterion: C4		
LE1({0.2,0.3,0.2},{0.8,0.9},{0.7,0.5,0.3,0.5})	(0.23,0.45,0.5)	0.43
LE2({0.9,0.9,0.4},{0.1,0.9},{0.6,0.5,0.2,0.8})	(0.73,0.1,0.53)	0.70
LE3({0.2,0.7,0.3},{0.8,0.9},{0.1,0.3,0.9,0.7})	(0.4,0.55,0.5)	0.45
LE4({0.4,0.8,0.6},{0.7,0.9},{0.2,0.9,0.1,0.2})	(0.6,0.55,0.35)	0.57
LE5({0.5,0.4,0.3},{0.4,0.9},{0.8,0.4,0.4,0.8})	(0.4,0.45,0.6)	0.45
LE6({0.5,0.9,0.8},{0.5,0.9},{0.8,0.3,0.5,0.4})	(0.73,0.55,0.5)	0.56
LE7({0.5,0.3,0.1},{0.5,0.9},{0.5,0.5,0.1,0.9})	(0.3,0.3,0.5)	0.50
Criterion: C5		
LE1({0.7,0.4,0.6},{0.5,0.9},{0.4,0.2,0.9,0.4})	(0.57,0.35,0.48)	0.58
LE2({0.7,0.4,0.8},{0.1,0.9},{0.1,0.8,0.8,0.6})	(0.63,0.35,0.58)	0.57
LE3({0.6,0.2,0.4},{0.1,0.9},{0.7,0.7,0.1,0.6})	(0.4,0.5,0.53)	0.46
LE4({0.7,0.1,0.1},{0.1,0.9},{0.1,0.3,0.4,0.7})	(0.3,0.35,0.38)	0.52
LE5({0.9,0.2,0.7},{0.7,0.9},{0.5,0.3,0.3,0.1})	(0.6,0.8,0.3)	0.50
LE6({0.8,0.5,0.7},{0.3,0.9},{0.7,0.5,0.4,0.2})	(0.67,0.45,0.45)	0.59
LE7({0.5,0.3,0.4},{0.2,0.9},{0.1,0.5,0.3,0.5})	(0.4,0.5,0.35)	0.52

Table 6: Multineutrosophic ARAS decision matrix of each strategic guideline. Source: Own elaboration.

	C1	C2	C3	C4	C5	Gi	Ki	
Weight	**0.25**	**0.09**	**0.17**	**0.30**	**0.20**			
LE1	0.0322	0.0134	0.0271	0.1045	0.0310	0.2082	100.00%	
LE2	0.0288	0.0126	0.0271	0.0452	0.0305	0.1440	69.19%	
LE3	0.0452	0.0109	0.0262	0.0247	0.0246	0.1317	63.24%	$G_0 = 0.2082$
LE4	0.0390	0.0136	0.0224	0.0371	0.0278	0.1400	67.25%	
LE5	0.0322	0.0118	0.0187	0.0247	0.0267	0.1141	54.81%	
LE6	0.0390	0.0168	0.0205	0.0452	0.0316	0.1531	73.55%	
LE7	0.0336	0.0109	0.0280	0.0186	0.0278	0.1189	57.11%	

Table 7: Optimization function G_i from the assignment of the criterion weight W_j. Source: Own elaboration.

Through the application of the multineutrosophic ARAS method, it is determined that the strategic guidelines with the highest weight of importance are the strengthening of predictive and proactive financial analysis and the focus on managing financial and operational risks. These guidelines should be prioritized due to their direct impact on making informed and resilient decisions, especially in volatile environments. Meanwhile, fostering a culture of financial innovation and adaptability also stands out as a significant guideline to promote organizational agility.

5 Discussion.

The results obtained through the Multineutrosophic ARAS method align with the existing literature on the priority of predictive financial analysis and risk management as key factors in organizational decision-making. In fact, in an environment characterized by economic uncertainties and rapid market changes, organizations that have implemented a proactive approach to financial analysis manage to stay ahead.

On the other hand, the research has shown that the ability to adapt to changes, combined with constant innovation, not only improves organizational agility but also increases resilience to potential crises. These results support the need for organizations to cultivate a culture of innovation in all areas, particularly in the financial sector, to remain competitive in the long term.

Finally, the integration of these elements into organizational strategy, as observed in the study's results, allows decisions to have a broader reach and be less vulnerable to external fluctuations. The interrelationship between predictive analysis, risk management, strategic innovation, and the use of tools like multineutrosophic analysis constitutes an essential factor that not only improves decision-making but also contributes to the sustainable growth of organizations. Therefore, it enables companies to not only respond to current challenges but also anticipate future trends with characteristics of uncertainty and market opportunities, ensuring their stability and long-term success.

In line with these findings, recent studies have demonstrated that the application of neutrosophic methods effectively addresses the inherent uncertainty in supplier evaluation, organizational performance, and corporate strategies. For instance, Sallam and Mohamed[16] highlight the utility of neutrosophic sets for evaluating supplier quality under uncertain environments, providing a robust tool for managing uncertainty in a structured manner.

Similarly, Alarcón et al. [17] employed sentiment analysis combined with NeutroAlgebra to evaluate organizational strategies and performance levels among basic education teachers, emphasizing how this methodology facilitates informed strategic decisions in complex situations.

Furthermore, other investigations have reinforced the practical applicability of neutrosophic approaches in critical areas such as healthcare and engineering. Abdullah [18] demonstrated that integrating neutrosophic sets with deep learning models significantly improves accuracy in breast cancer classification, enabling more precise and earlier clinical decisions. In the same vein, Cevallos-Torres et al .[19]developed neutrosophic control charts to detect cardiac arrhythmias in electrocardiograms, demonstrating the efficacy of these methods in highly sensitive contexts. Additionally, Mohamed, Smarandache, and Voskoglou [20] implemented a neutrosophic decision tree model to evaluate obstacles in electric mobility adoption, offering a clear analytical framework in the face of multiple uncertainties.

Conclusion

The analysis conducted through the multineutrosophic ARAS method has shown that strengthening predictive and proactive financial analysis, along with solid financial and operational risk management, are essential for informed and resilient organizational decision-making, especially in volatile economic environments. Furthermore, it has been confirmed that promoting a culture of innovation and financial adaptability significantly improves organizational agility, enabling companies to respond quickly to market changes. Therefore, the use of tools derived from the MultiNeutrosophic Set, integrated between financial analysis, risk management, and organizational culture, has demonstrated its capacity to enhance business resilience and sustainability. This opens the door to future research that delves into its application and effectiveness across various sectors immersed in financial uncertainty.

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Sustainability Impact of Online Consumption Behavior from the Perspective of Digital Empowerment: IndetermSoft Set with Application

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Abstract: This study uses decision-making methods for the Sustainability Evaluation of Online Consumption Behavior from the Perspective of Digital Empowerment. Two methods are used such as the CRITIC method to compute the criteria weights and the MULTIMOORA method to rank the alternatives. We use the IndetermSoft Set to deal with the indeterminacy of criteria and their values. This study uses six criteria and seven alternatives. The fifth criterion has two different sub-criteria, so the IndetermSoft Set is used to deal with this indeterminacy. The ranks of alternatives are obtained in three cases. The case study is applied to show the validation of the proposed two decision-making methods.

Keywords: Sustainability; Consumption Behavior; Digital Empowerment; MCDM Methodology; IndetermSoft Set.

1. Introduction

The rapid advancement of digital technologies has transformed consumer behavior, enabling individuals to make more informed and sustainable purchasing decisions. As online consumption continues to rise, there is a growing need to evaluate its sustainability impact, particularly from the perspective of digital empowerment[1], [2]. Digital tools, including AI-driven recommendations, blockchain for supply chain transparency, and e-commerce sustainability ratings, provide consumers with greater access to information about ethical and eco-friendly products. This shift has encouraged businesses to adopt greener digital practices, such as offering carbon-neutral shipping, digital receipts, and sustainable packaging options. However, assessing the sustainability of online consumption behavior requires a structured evaluation framework that integrates environmental awareness, digital literacy, and ethical decision-making[3], [4]. Digital empowerment plays a crucial role in fostering responsible

consumerism by equipping individuals with the knowledge and tools needed to make eco-conscious choices. Consumers with higher digital literacy can analyze product sustainability claims, compare ethical brand certifications, and engage in circular economic practices, such as buying second-hand goods or renting instead of purchasing. Moreover, the rise of subscription-based digital services, such as streaming platforms and cloud storage, has reduced the demand for physical products, leading to lower carbon footprints. However, the effectiveness of these sustainable digital behaviors depends on consumers' trust in online sustainability information, as misinformation and greenwashing can undermine responsible purchasing decisions.

Despite its advantages, the sustainability of online consumption is challenged by issues such as e-waste generation, energy-intensive digital activities, and fast fashion e-commerce expansion. The growing reliance on digital platforms has increased electricity consumption in data centers, and frequent device upgrades contribute to electronic waste[5], [6]. Additionally, while online shopping provides convenience, the rise of fast delivery services and single-use packaging has led to a surge in carbon emissions. To address these challenges, businesses must implement sustainable digital solutions, such as optimizing AI-driven logistics for lower emissions, promoting recyclable packaging, and encouraging responsible product disposal programs. Evaluating the sustainability of online consumption behavior from a digital empowerment perspective requires a multi-criteria decision-making (MCDM) approach that balances environmental, economic, and social factors[7], [8]. By assessing key criteria such as environmental awareness, ethical consumption behavior, and digital literacy, researchers and policymakers can develop strategies to enhance sustainable consumption practices. As digital platforms continue to evolve, their role in shaping sustainable consumer behavior will become even more significant, making it essential to integrate technology-driven solutions with responsible purchasing habits. By fostering greater digital empowerment, individuals and businesses can collectively contribute to a more sustainable and eco-friendly digital economy[9], [10]. One important characteristic shared by nearly all the above discussed approaches is the moderate to high complexity of their mathematical models. These methods appear to be challenging to use and call very sophisticated mathematical understanding. Consequently, decision makers may find that an undemanding MCDM approach is a true boon. The mathematics behind Brauers and Zavadskas' multi-objective optimization based on ratio analysis (MOORA) approach is straightforward. As a result, it may be used for material selection with ease and effectiveness. A more thorough version of the MOORA methodology is the MULTIMOORA method. The findings of the MULTIMOORA methodology can be more reliable than those of typical MCDM techniques, which provide a single rank, because the final rank is produced by integrating three subordinate values[11], [12].

This study proposes a MCDM approach for the Sustainability Evaluation of Online Consumption Behavior from the Perspective of Digital Empowerment with the IndetermSoft set to deal with indeterminacy in the values of the criteria. Two MCDM methods are used in this study such as

the CRITIC method to compute the criteria weights and the MULTIMOORA method to rank the alternatives.

2. IndetermSoft Set

It was introduced by Smarandache in 2022. Let U be a universe of discourse, P(H) be the powerset of H, and H be a non-empty subset of U. Assume that A is a collection of the values of an attribute, a.
Then, an IndetermSoft is defined as F: A→ P(H). Set if: i) the set A contains some indeterminacy; ii) the set P(H) contains some indeterminacy; iii) at least one attribute-value v∈A exists such that F(v) = indeterminate (unclear, incomplete, conflicting, or not unique); iv) any two or all three of the previously specified conditions.

The NeutroFunction, described in 2014-2015, is a function that is only partially well-defined (inner-defined), partially indeterminate, and partially outer-defined because of its degree of indeterminacy. The IndetermSoft Set is a particular instance of this function. The NeutroFunction is a fully specified extension of the classical function[13], [14], [15].

IndetermSoft Set, an extension of the classical (determinate) Soft Set, handles indeterminate data since certain sources cannot provide accurate or thorough information on the sets A, H, or P(H), as well as on the function F. We did not invent indeterminacy; we found it in the actual world. We still must deal with these types of situations since many sources offer information that is imprecise, confusing, incomplete, and conflicting rather than accurate like in the Soft Set.

3. MCDM Model

We use two MCDM methods to evaluate the Sustainability Evaluation of Online Consumption Behavior from the Perspective of Digital Empowerment. This section shows the steps of two MCDM methods.

CRITIC Method

This part shows the steps of the CRITIC method to compute the criteria weights.

Create the decision matrix.

$$R = \begin{bmatrix} r_{11} & r_{21} & r_{31} & \cdots & r_{1n} \\ r_{21} & r_{22} & r_{32} & \cdots & r_{2n} \\ r_{31} & & & & r_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{m1} & r_{m2} & \cdots & \vdots & r_{mn} \end{bmatrix}_{m\times n} ; i = 1, \dots, m; j = 1, \dots, n \quad (1)$$

Normalize the decision matrix

We can normalize the decision matrix for beneficial and non-beneficial criteria such as:

$$y_{ij} = \frac{r_{ij} - r_i^-}{r_i^+ - r_i^-}; i = 1, \dots, m; j = 1, \dots, n \quad (2)$$

$$y_{ij} = \frac{r_{ij} - r_i^+}{r_i^- - r_i^+}; i = 1, \dots, m; j = 1, \dots, n \quad (3)$$

$$r_i^- = \min(x_{ij}) \quad (4)$$

$$r_i^+ = \max(x_{ij}) \quad (5)$$

Obtain the correlation coefficient

We can obtain the correlation matrix between the criteria and the alternatives.

Obtain the standard deviation.

Compute the index c

$$C_j = \sigma_j \sum_{k=1}^{n} (1 - U_{jk}) \quad (6)$$

Where U_{jk} refers to the value of the correlation coefficient and σ_j refers to the standard deviation.

Compute the criteria weights.

$$w_j = \frac{c_j}{\sum_{j=1}^{n} c_j} \quad (7)$$

MULTIMOORA Method

We use this method to rank the alternatives.

Normalize the decision matrix

$$r_{ij}^* = \frac{r_{ij}}{\left(\sum_{i=1}^{m} r_{ij}^2\right)^{0.5}} \quad (8)$$

Compute the value of ratio system

$$q_i = \sum_{j=1}^{g} w_j r_{ij}^* - \sum_{j=g+1}^{n} w_j r_{ij}^* \quad (9)$$

We rank the alternative by the max value in the q_i

Then we rank the alternatives based on the reference point as:

$$z_i = \max_j \left| w_j p_j - w_j r_{ij}^* \right| \quad (10)$$

We rank the alternatives based on the min value of z_i

$$p_j = \begin{cases} \max\limits_i \ r_{ij}^* \ for\ positive\ criteria \\ \min\limits_i \ r_{ij}^* \quad\quad for\ cost\ criteria \end{cases} \quad (11)$$

Then we rank the alternatives based on the full multiplicative form

We obtain the values of full multiplicative such as:

$$D_i = \frac{\prod_{j=1}^{g}\left(r_{ij}^{*}\right)^{w_j}}{\prod_{j=g+1}^{n}\left(r_{ij}^{*}\right)^{w_j}} \tag{12}$$

The rank of alternatives is obtained by the max value of D_i. Then we apply the dominance theory to obtain the final ranks.

4. Case Study

This section presents a case study on the Sustainability Evaluation of Online Consumption Behavior from the Perspective of Digital Empowerment to validate the proposed approach. Four experts, with experience in the Multi-Criteria Decision-Making (MCDM) field, assessed the criteria and alternatives using a scale ranging from 0.1 to 0.8. Their evaluations were based on their expertise and professional judgment.

The study examines six key criteria and seven alternatives, structured as follows:

- Criteria:
  - Trust in Digital Sustainability Information (High)
  - Environmental Awareness (High)
  - E-Waste Reduction Practices (Medium)
  - Ethical Consumption Behavior (Strong)
  - Digital Literacy (Advanced, Basic)
  - Preference for Green Digital Services (Strong)
- Alternatives:
  - Conscious Online Food Delivery Choices
  - Ethical Brand Loyalty
  - Digital Subscription Services
  - Green E-commerce Shopping
  - Smart Energy Consumption in Digital Activities
  - Responsible Use of Fast Fashion Online Platforms
  - Second-Hand and Circular Economy Participation

Results of the CRITIC Method

To assess the relationship between the criteria and alternatives, Equation (1) was used to construct the decision matrix. The four experts rated the criteria on a scale between 0.8 and 0.9. The process involved the following steps:

1. Normalization of the decision matrix using Equation (2) (illustrated in Figure 1).
2. Calculation of the correlation coefficient (shown in Figure 2).
3. Determination of the standard deviation.
4. Computation of the index (c) using Equation (6).
5. Derivation of criteria weights using Equation (7) (results displayed in Figure 3).

Fig 1. Normalization values by CRITIC method.

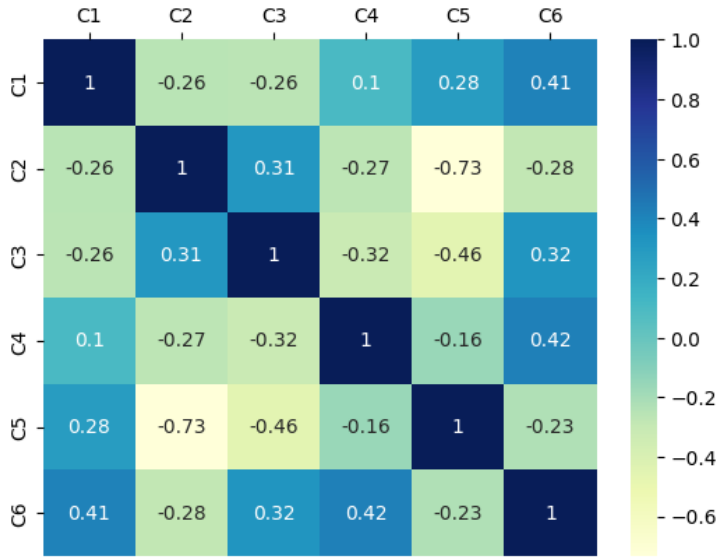

Fig 2. The correlation matrix.

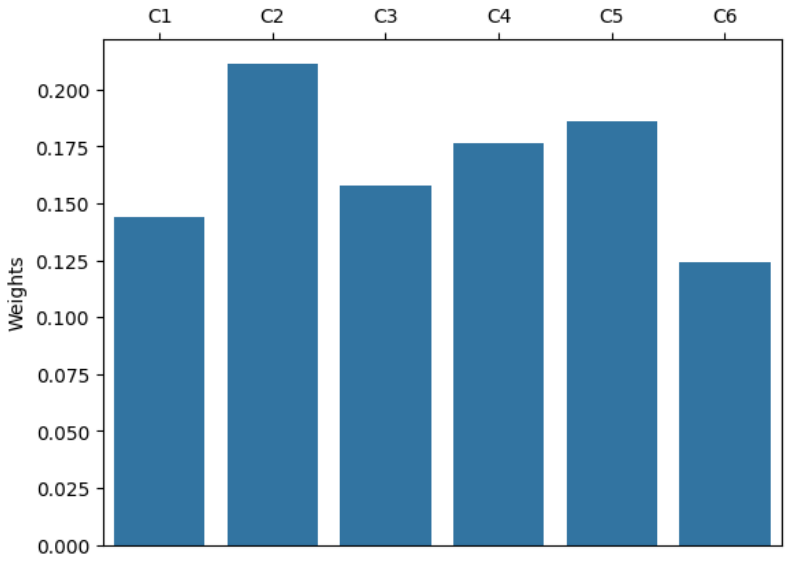

Fig 3. Criteria weights.

Results of the MULTIMOORA Method

When ranking the alternatives, we encountered indeterminacy in selecting the value for the fifth alternative, which had two possible values: basic and advanced. To handle this uncertainty, we applied the IndetermSoft Set, which allows for a flexible approach in dealing with indeterminate values. Consequently, we implemented the MULTIMOORA method under three different scenarios: one using the basic value, another using the advanced value, and a third that considers both values simultaneously.

In the first case, where the basic value was applied, we normalized the decision matrix using Equation (8), as illustrated in Figure 4. Following this, we computed the ratio system value using Equation (9) (see Figure 5). Next, we determined the reference point with the help of Equation (10), which is visualized in Figure 6. Finally, we obtained the full multiplicative values by applying Equation (12), as demonstrated in Figure 7.

A similar process was followed in the second case, where we substituted the basic value with the advanced value. In the third case, both basic and advanced values were considered simultaneously, allowing us to evaluate the effect of indeterminacy on the ranking process. By systematically applying the MULTIMOORA method in all three cases, we ensured a robust and comprehensive assessment of the alternatives.

Fig 4. Normalization matrix.

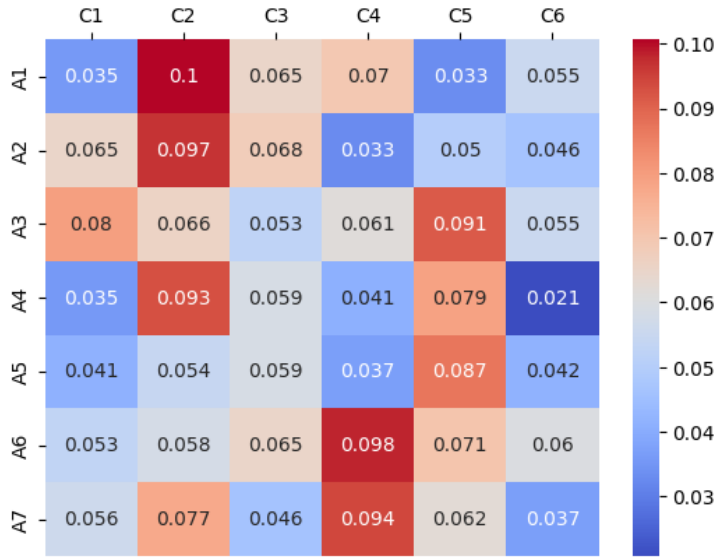

Fig 5. The ratio system values.

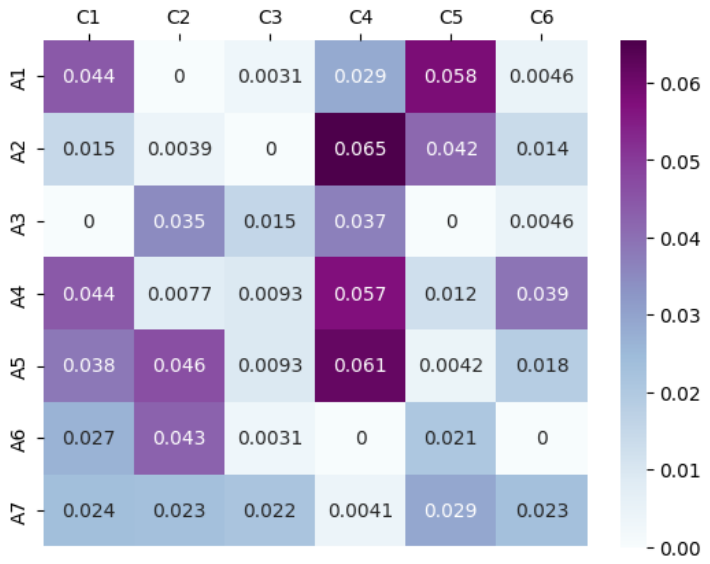

Fig 6. Reference point matrix.

Fig 7. Full multiplicative matrix.

Results for the Second and Third Cases

In the second case, where we applied the advanced value, the normalized decision matrix was generated, as shown in Figure 8. Subsequently, we calculated the ratio system values, which are illustrated in Figure 9. The reference point was determined and is displayed in Figure 10. Finally, we computed the full multiplicative values, as represented in Figure 11.

In the third case, which considered both basic and advanced values simultaneously, we followed the same procedure. The normalized decision matrix was obtained and shown in Figure 12. We then computed the ratio system values displayed in Figure 13. The reference point was

determined and is presented in Figure 14. Lastly, the full multiplicative values were calculated and are illustrated in Figure 15.

To finalize the rankings, we applied the dominance theory across all three cases to establish a definitive ranking of the alternatives. The results, as shown in Figure 16, indicate that Alternative 3 emerged as the best choice, while Alternative 5 ranked the lowest.

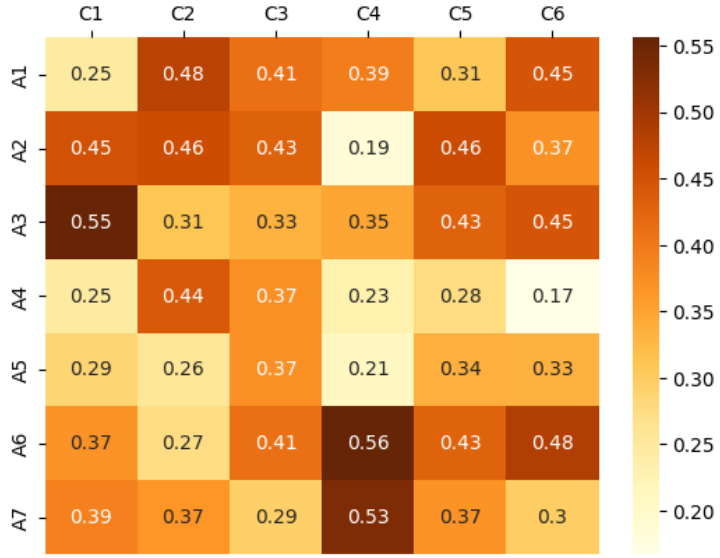

Fig 8. Normalization matrix.

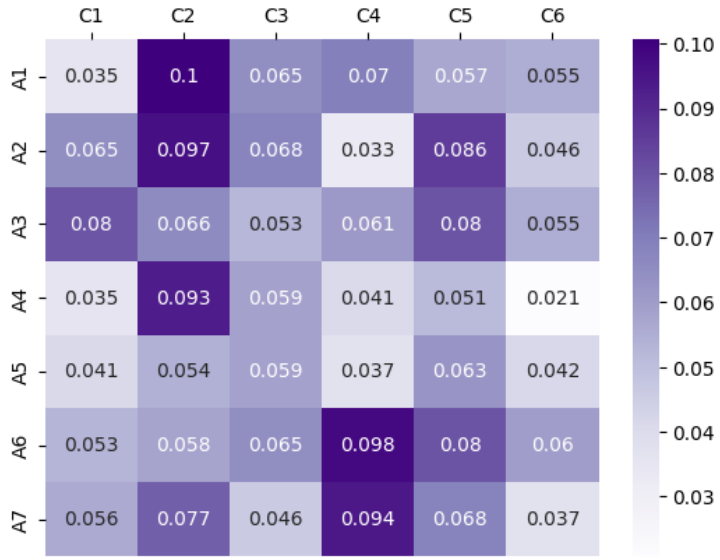

Fig 9. The ratio system values.

Fig 10. Reference point matrix.

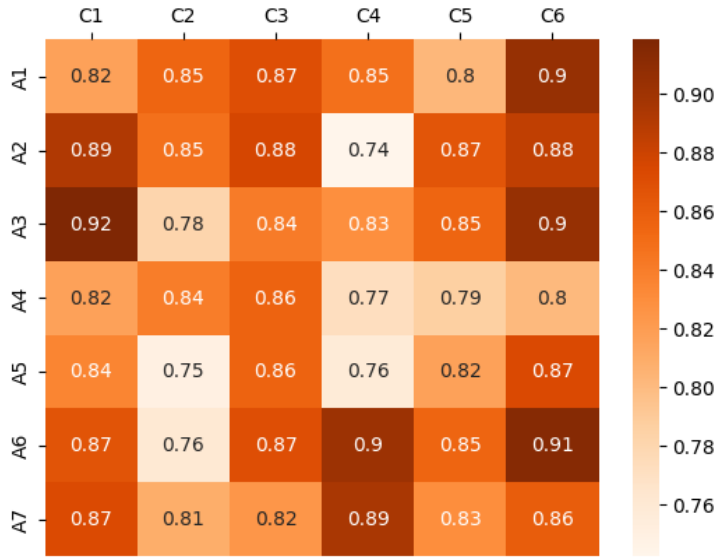

Fig 11. Full multiplicative matrix.

Fig 12. Normalization matrix.

Fig 13. The ratio system values.

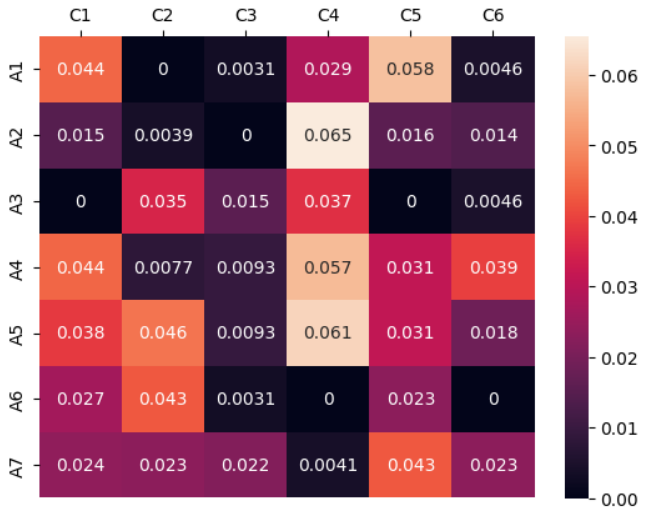

Fig 14. Reference point matrix.

Fig 15. Full multiplicative matrix.

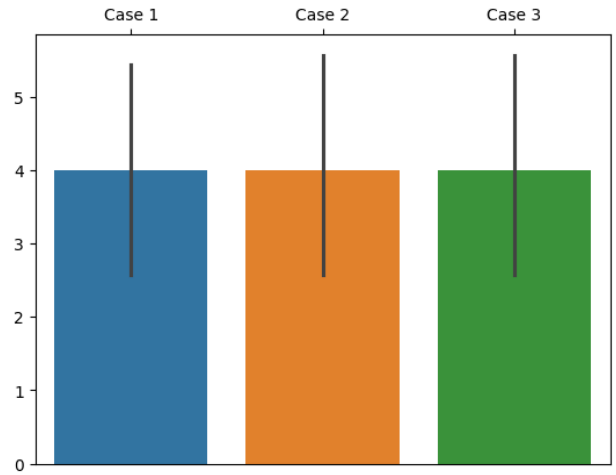

Fig 16. Final ranks of three cases.

4.1 Theoretical Implications

This study contributes to the growing body of research on digital empowerment and sustainable online consumption by introducing a multi-criteria decision-making (MCDM) framework that incorporates IndetermSoft Sets. The proposed approach enhances traditional MCDM models by addressing indeterminacy and uncertainty in consumer behavior data, which is a critical challenge in evaluating sustainability initiatives.

By integrating CRITIC and MULTIMOORA methods, this research offers a structured methodology to assess the effectiveness of digital sustainability initiatives, such as green e-commerce, responsible digital consumption, and eco-conscious online behaviors. Furthermore, the application of the IndetermSoft Set in handling ambiguous consumer data introduces a novel computational tool for decision-making in sustainability studies.

The findings suggest that digital literacy and trust in online sustainability information play a significant role in influencing consumer behavior. This highlights the need for policymakers, businesses, and researchers to focus on improving transparency in online sustainability claims and promoting digital education programs to enhance eco-friendly consumer decision-making.

4.2 Practical Implications

The study's findings have several practical implications for businesses, policymakers, and consumers seeking to improve sustainability in digital commerce.

Companies can leverage the insights from this study to enhance their sustainability strategies, such as providing transparent sustainability ratings, integrating blockchain for supply chain traceability, and promoting carbon-neutral shipping. The findings also suggest that improving digital literacy among consumers can enhance their engagement with eco-friendly products, leading to higher brand loyalty and sustainable purchasing behaviors.

The results highlight the need for regulatory frameworks that ensure credible and verifiable sustainability claims in digital marketplaces. Governments can implement policies to promote responsible e-commerce practices, such as mandating standardized sustainability labels and encouraging tax incentives for green digital initiatives.

The study emphasizes the importance of digital empowerment in making informed purchasing decisions. By enhancing their understanding of sustainability metrics, environmental impact assessments, and ethical consumption choices, consumers can contribute to a more sustainable digital economy.

By applying the IndetermSoft Set approach, businesses and decision-makers can develop more flexible and data-driven strategies to support sustainable consumer behaviors in an increasingly digitalized world.

4.3 Limitations of the Study

Despite its contributions, this study has some limitations that should be considered in future research:

1. The reliance on expert evaluations introduces subjectivity in the weight assignment process. While the CRITIC method mitigates this to some extent, future studies could explore machine learning techniques for more objective weight determination.
2. The study's case analysis is based on a specific number of criteria and alternatives. Expanding the dataset to include more diverse consumer behaviors and industry sectors could strengthen the generalizability of the findings.
3. Although the IndetermSoft Set effectively manages uncertainty, its computational complexity may pose challenges for scalability in large datasets.

5. Conclusions

In this research, we applied the MULTIMOORA approach with the IndetermSoft Set to handle the Sustainability Evaluation of Online Consumption Behavior from the Perspective of Digital Empowerment problem by using CRITIC weight. The weighted and CRITIC-weighted MULTIMOORA techniques are the two scenarios that make up the extended model. The CRITIC method is used to compute the criteria weights and the MULTIMOORA method is used to rank the alternatives. Decision makers' opinions, which are based on their understanding and their experiences, are the direct source of subjective weight. However, the CRITIC concept is used to determine the objective weight. IndetermSoft Set is used to deal with indeterminacy in the sub criteria. The IndetermSoft Set is applied with three cases. The results show alternative 3 is the best and alternative 5 is the worst.

5.1 Future Research Directions

While this study provides valuable insights into the sustainability of online consumption behavior, several areas remain open for further investigation:

I. Future studies could incorporate additional sustainability criteria, such as carbon footprint analysis, water usage, and social sustainability metrics, to provide a more comprehensive evaluation.
II. The impact of digital empowerment on sustainable consumer behavior may vary across different regions and cultures. Future research can conduct comparative studies across multiple countries to assess regional variations in sustainable digital consumption.
III. With the rise of AI, blockchain, and IoT, future studies could explore how these technologies enhance transparency and trust in digital sustainability information.
IV. Since consumer behavior evolves over time, conducting long-term studies can provide deeper insights into how digital literacy and sustainability awareness change consumer preferences over extended periods.

V. Future research could collaborate with businesses, policymakers, and sustainability organizations to implement the proposed model in real-world settings and measure its practical impact on consumer decision-making.

Acknowledgment

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Rule-Based Neutrosophic Triplet Refined Interval-Valued Data Partition Analyzing The Movement of Stock Market Price

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Abstract:

This study introduces an innovative approach to analysing stock market index movements through a Rule-Based Neutrosophic Triplet Refined Interval-Valued Data Partition (NTRIVDP) model. This method segments data into intervals and applies Neutrosophic statistical measures to convert it into Neutrosophic triplets. These triplets are then processed through a defined rule-base, calculating the Neutrosophic triplet count to derive the average predicted output. The stock market poses significant challenges for accurate prediction, as traditional methods often fall short in capturing the intricacies of market behaviour. The proposed NTRIVDP model leverages Neutrosophic logic, effectively handling uncertain and vague information to offer a more nuanced understanding of market dynamics. This provides valuable insights for investors and financial analysts aiming to analyse stock market index behaviour more accurately. To assess the efficacy of the proposed model, performance metrics such as Mean Squared Error (MSE) and Mean Absolute Percent Error (MAPE) are utilized. For the parameter 'α = 0.1, 0.2, 0.3', the model yields promising results, with an MSE of 25,969.68 and a MAPE of 6.570417. These findings indicate that the proposed model surpasses existing methods, demonstrating its effectiveness in forecasting stock market movements. This research highlights the potential of Neutrosophic logic in enhancing the accuracy of stock market predictions, presenting a significant advancement in financial forecasting methodologies.

Keywords: Neutrosophic Triplet, Rule-base, interval valued data partition, Mean Absolute Percent Error, Mean Squared Error.

1. Introduction

In the constantly evolving and volatile financial markets, accurate prediction and analysis of stock market indices are crucial for investors, financial analysts, and policymakers. Traditional forecasting methods often fall short in capturing the intricate, dynamic, and uncertain nature of market behaviour. This shortcoming has spurred the development of advanced computational models capable of managing uncertainty and delivering more reliable predictions. A notable advancement in this regard is the application of Neutrosophic Set Theory, developed by Florentin Smarandache in 1995. This theory extends fuzzy logic, intuitionistic fuzzy logic, and paraconsistent logic to address uncertainty and enhance forecasting accuracy.

Neutrosophic Set Theory is a powerful extension of fuzzy set theory, incorporating three-valued logic to manage indeterminate and inconsistent information. This theory is well-suited for tackling real-world problems characterized by uncertainty, incompleteness, and contradiction. In the realm of stock market analysis, Neutrosophic logic provides a more nuanced and comprehensive framework for understanding market dynamics compared to traditional methods.

Fuzzy set theory was initially introduced in [1], and its application across various domains was detailed in [2]. The foundational concepts of fuzzy set theory, introduced by Zadeh in 1965, have seen widespread application. Fuzzy set theory represents uncertain and vague information by assigning membership grades to elements, offering a more flexible approach than classical binary logic. Researchers have explored its application in time series forecasting, developing fuzzy relations on time series data with linguistic values to predict future values [3], and refining these methods with simplified computational approaches based on the highest membership grade [4]. Despite these advancements, fuzzy set theory has limitations in handling the full spectrum of uncertainties in stock market analysis.

To overcome these limitations, researchers have turned to Neutrosophic logic. For instance, [5] introduced a method utilizing a Neutrosophic environment and a Fuzzy CETD matrix to assess stock market fluctuations. Comprehensive discussions on Neutrosophic logic, defining propositional connectives and examining relationships with other uncertainty reasoning frameworks like intuitionistic fuzzy logic, were provided by [6]. The potential of Neutrosophic logic in forecasting stock index trends for investment decisions was demonstrated by [7]. Moreover, [8] highlighted Neutrosophic logic's superiority over Interval Statistics in handling various indeterminacies, offering a detailed probability framework. The concepts of Neutrosophic measure, Neutrosophic integral, and Neutrosophic probability were introduced by [9], while [10] extended classical statistics to handle set values instead of crisp values.

The objective of this study is to address the challenges in stock market forecasting by proposing a Rule-Based Neutrosophic Triplet Refined Interval-Valued Data Partition (NTRIVDP) model. This innovative approach segments data into intervals and applies Neutrosophic statistical measures to convert the data into Neutrosophic triplets. These triplets are processed through a defined rule-base, calculating the Neutrosophic triplet count to derive the average predicted output. The proposed NTRIVDP model leverages the strengths of Neutrosophic logic to handle the inherent uncertainty and vagueness in stock market data. By converting data into Neutrosophic triplets, the model provides a more nuanced representation of market behavior, capturing the indeterminate and

inconsistent aspects that traditional methods often overlook. This enhanced representation allows for more accurate and reliable predictions, offering valuable insights for investors and financial analysts.

To evaluate the efficacy of the proposed NTRIVDP model, performance metrics such as Mean Squared Error (MSE) and Mean Absolute Percent Error (MAPE) were utilized. The experimental results demonstrate that the proposed model is more effective and reliable in forecasting stock market movements compared to existing methods, making it a valuable tool for financial forecasting.

The main significance of the NTRIVDP model lies in its ability to effectively handle the uncertainty and vagueness inherent in stock market data. By leveraging Neutrosophic logic, the model provides a more comprehensive representation of market behaviour, capturing the subtle nuances that traditional methods often miss. This enhanced representation allows for more accurate predictions, offering valuable insights for investors and financial analysts. The proposed methodology also highlights the potential of Neutrosophic logic in advancing the field of financial forecasting. By incorporating three-valued logic to handle indeterminate and inconsistent information, Neutrosophic logic provides a powerful framework for dealing with the complexities of real-world problems. The success of the NTRIVDP model demonstrates the effectiveness of this approach in addressing the challenges of stock market forecasting, paving the way for further research and development in this area.

The proposed methodology is detailed in section 3, with experimental results and discussions presented in section 4. Finally, a brief conclusion is provided in section 5, summarizing the findings and implications of the study. This structured approach underscores the objective of the study and highlights the potential of Neutrosophic logic in advancing financial forecasting methodologies.

2. Basic concepts

2.1 Definition of Neutrosophic set: [9]

Let U be a universe of discourse, and M a subset of U. An element x from U is characterized with respect to the set M as $x(T, I, F)$, where it is $t\%$ true in the set, $i\%$ indeterminate in the set, and $f\%$ false. Here, t varies in T, i varies in I, and f varies in F.

2.2 Definition of Neutrosophic Logic triplet:[9]

The Neutrosophic Logic (NL) truth value of a proposition P is represented as NL(P) = (T, I, F), where T is the degree of truth, I is the indeterminate degree (the extent to which it is unknown whether P is true or false), and F is the degree of falsehood of the proposition P.

Consider <A> as a notion, attribute, idea, proposition, theorem, theory, etc. Let <antiA> be the opposite of <A>, and <neutA> be neither <A> nor <antiA>, but the neutral (indeterminate or unknown) aspect related to <A>.

For example:

- If <A> = positive, then <antiA> = negative, and <neutA> = zero.
- If <A> = win, then <antiA> = lose, and <neutA> = tie-game.

2.3 Definition of Neutrosophic Statistics: [10]

Neutrosophic statistics encompasses the handling of indeterminate data, characterized by varying degrees of ambiguity, vagueness, partial knowledge, contradictions, and incompleteness. While classic statistics uses crisp data represented through charts, diagrams, and algorithms, Neutrosophic statistics accommodate the inherent uncertainty and imprecision found in real-world data. This approach provides a more comprehensive and flexible framework for analyzing and interpreting data with inherent uncertainties.

3. Methodology:

The step-by-step procedure for the proposed Rule-Based Neutrosophic Triplet Refined Interval-Valued Data Partition (NTRIVDP) model is meticulously designed to handle the uncertainties inherent in stock market data. The block diagram of the proposed model is shown in Figure 1. The detailed steps are as follows:

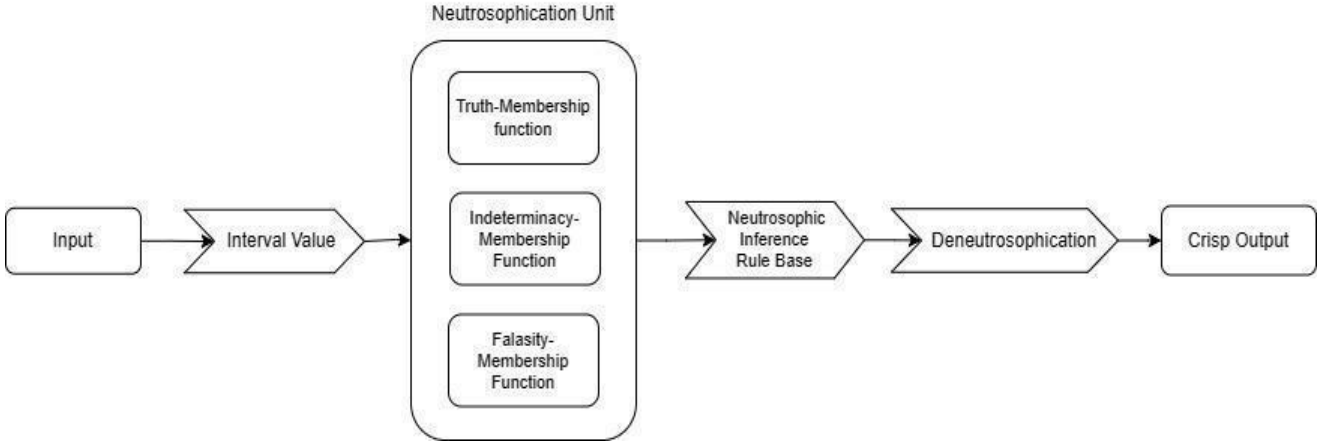


Figure 1: Block diagram of NTRIVDP

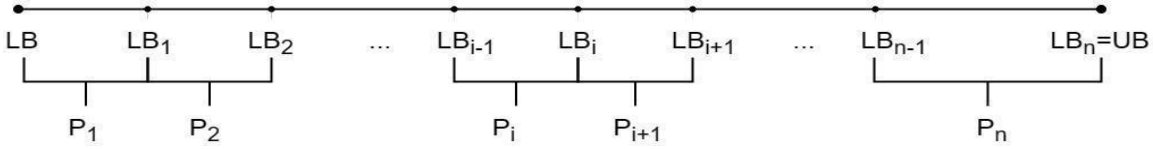
Step1: Define the Universe of Discourse

- **Universe Definition**: Define the universe of discourse 'U' in a column vector as, U = $\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{ij} \end{pmatrix}$

- **Bounds and Increment Calculation**: Choose the lower bound 'LB' and upper bound 'UB' for the column vector. Calculate the incremental value 'h' using the formula: h = $\frac{UB-LB}{n}$, where 'n' is the number of data points in the column vector.

- **Data Conversion**: Convert the data into Actual Value Interval-Valued Data Partition (AV_{IVDP}) as illustrated in Figure 2 and the Actual Value of Interval-Valued Data Partition (AV_{IVDP}) is given by,

$$AV_{IVDP} = \begin{pmatrix} a_{11}, b_{11} \\ a_{21}, b_{21} \\ a_{31}, b_{31} \\ \vdots \\ a_{ij}, b_{ij} \end{pmatrix}$$



$$where\ LB_1 = LB + h,\ \ LB_2 = LB_1 + h, \ldots\ldots, LB_{i-1} + h, \ldots\ldots, UB = LB_n = LB_{n-1} + h$$

Figure 2: Data partitioning

Step 2: Neutrosophic Statistical Computations

- **Average and Standard Deviation**: Compute the average and standard deviation within the Neutrosophic Statistical environment using the following formulas:

$$(\mu_1, \mu_2)_{IVDP} = \left(\sum_{i=1}^{n} \frac{a_{ij}}{n}, \sum_{i=1}^{n} \frac{b_{ij}}{n}\right), where\ j\ is\ fixed$$

$$(\sigma_1, \sigma_2)_{IVDP} = \left(\sqrt{\frac{\sum(a_{ij} - \mu_1)^2}{n}}\ , \sqrt{\frac{\sum(b_{ij} - \mu_2)^2}{n}}\right), where\ j\ is\ fixed$$

Here, $(\mu_1, \mu_2)_{IVDP}$ denotes the average, and $(\sigma_1, \sigma_2)_{IVDP}$ denotes the standard deviation of interval-valued data partition.

Step 3: Neutrosophic Triplet Conversion

- **Conversion to Neutrosophic Triplets**: Transform the statistical measurements such as average and standard deviation into an appropriate Neutrosophic triplet refined interval valued data partitions denoted as <A>, <antiA>, <neutA> using the IF-THEN Neutrosophic refined rule-base outlined below.

If $(a_{ij}, b_{ij})_{IVDP} < [\ (\mu_1, \mu_2)_{IVDP} - \alpha\ (\sigma_1, \sigma_2)_{IVDP}\ ]$ then (-1,-1) i.e., <antiA, antiA>

If $(a_{ij}, b_{ij})_{IVDP} = [\ (\mu_1, \mu_2)_{IVDP} - \alpha\ (\sigma_1, \sigma_2)_{IVDP}\ ]$ then (0,0) i.e., <neutA, neutA>

If $(a_{ij}, b_{ij})_{IVDP} > [\ (\mu_1, \mu_2)_{IVDP} - \alpha\ (\sigma_1, \sigma_2)_{IVDP}\ ]$ then (1,1) i.e., <A, A>

This step, known as Neutrosophication, involves the parameter 'α' which is in the range [0,1].

Step 4: Average Predicted Value Calculation

- **Neutrosophic Inference Engine**: To compute the average predicted value of AV_{IVDP} for the Neutrosophic triplets, denoted as 'AP_{IVDP}', the Neutrosophic inference engine is used. This

computation can be performed in Excel using COUNTIF, SUMIF, and AVERAGEIF functions as given below.

- **COUNTIF Rule Syntax**:

  = COUNTIF (range, "criteria")

- **SUMIF Rule Syntax**:

  = SUMIF (range, criteria, [sum range])

- **AVERAGEIF Rule Syntax**:

  = AVERAGEIF (range, criteria, [average range])

This step produces a crisp output, a process known as Deneutrosophication, which provides a definitive value for the average predicted output.

Step 5: Error Calculation

- **Error Calculation**: The error, denoted by $Error_{IVDP}$, is calculated as the difference between the actual value of interval- valued data partition (AV_{IVDP}) and the average predicted value of the interval-valued data partition (AP_{IVDP}). The formula is given by:

$$Error_{IVDP} = AV_{IVDP} - AP_{IVDP}$$

Step 6: Midpoint Error Computation

- **Midpoint Error (MPE)**: To convert the data partition into a single value, compute the Midpoint Error (MPE), denoted as MPE. This is calculated using the following formula:

$$MPE = \frac{\text{Error}_{IVDP}}{2}$$

This step helps in simplifying the error measurement by providing a single representative value.

Step 7: Sum of Squared Errors (SSE)

- **SSE Calculation**: Compute the Sum of Squared Errors (SSE) by summing the squares of the Midpoint Errors (MPE):

$$SSE = \sum(MPE)^2$$

This step aggregates the squared errors, providing a comprehensive measure of the total deviation.

Step 8: Average Predicted Value Representation

- **Average Predicted Value (AP)**: Find the average of the lower bound and upper bound in the interval of the average predicted value (AVIVDP). This is represented as the average predicted value (AP).

Step 9: Performance Error Measures

- **Performance Metrics**: Measure the forecasting accuracy using performance error metrics such as Mean Squared Error (MSE) and Mean Absolute Percent Error (MAPE). These are defined as follows:
  - **Mean Squared Error (MSE)**:

$$MSE = \frac{\sum_{i=1}^{n}((AV_{IVDP})_i - (AP_{IVDP})_i)^2}{n}$$

  - **Mean Absolute Percent Error (MAPE)**:

$$MAPE = \frac{1}{n}\sum_{i=1}^{n}\frac{(AP_{IVDP})_i - (AV_{IVDP})_i}{(AV_{IVDP})_i} \; X\,100$$

4. Results and discussion:

In this section, the proposed method is applied to predict the share price of the State Bank of India (SBI) at the Bombay Stock Exchange (BSE). For the experimental study, the annual report of SBI for the fiscal year 2009–2010 is utilized. Specifically, share price data for SBI at BSE from April 2009 to March 2010 is analysed, as detailed in Table 1.

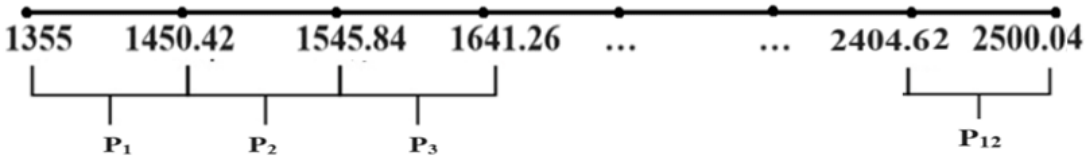
Table 1: Actual value of SBI shares at BSE.

Months	Actual Value
Apr-2009	1355.00
May-2009	1891.00
Jun-2009	1935.00
Jul-2009	1840.00
Aug-2009	1886.90
Sep-2009	2235.00
Oct-2009	2500.00
Nov-2009	2394.00
Dec-2009	2374.75
Jan-2010	2315.25
Feb-2010	2059.95
Mar-2010	2120.05

The Experiments results are discussed in the following steps:

Step 1: Based on the data in Table 1, the universe of discourse U is defined as a column vector for the dataset under consideration: $U = \begin{pmatrix} 1355.00 \\ 1891.00 \\ 1935.00 \\ \vdots \\ 2120.05 \end{pmatrix}$

Here, we choose a lower bound 'LB' = 1355 and an upper bound 'UB' = 2500. With n=12 (for instance), the incremental height is computed for the column vector as h = 95.42. The data partitioning for the dataset is performed accordingly.



The actual value interval-valued data partition (AV_{IVDP}) is subsequently obtained and presented in Table 2.

Table 2: Conversion of Actual Value to Actual Value Interval Valued Data partition (AV_{IVDP}).

Actual value	**Actual value Interval valued Data partition (AV_{IVDP})**	**Data Partition**
1355.00	(1355, 1450.42)	P_1
1891.00	(1832.1, 1927.52)	P_6
1935.00	(1927.52, 2022.94)	P_7
1840.00	(1832.1, 1927.52)	P_6
1886.90	(1832.1, 1927.52)	P_6
2235.00	(2213.78, 2309.2)	P_{10}
2500.00	(2404.62, 2500.04)	P_{12}
2394.00	(2309.2, 2404.62)	P_{11}
2374.75	(2309.2, 2404.62)	P_{11}
2315.25	(2309.2, 2404.62)	P_{11}
2059.95	(2022.94, 2118.36)	P_8
2120.05	(2118.36, 2213.78)	P_9

Step 2: Calculating the average and standard deviation of 'AV_{IVDP}' using the formulas within the Neutrosophic statistical environment, we obtain:

$$(\mu_1, \mu_2)_{IVDP} = \left(\frac{1355+1832.1+\cdots+2118.36}{12}, \frac{1450.42+1927.52+\cdots+2213.78}{12}\right) = (2038.843, 2134.263)$$

$$(\sigma_1, \sigma_2)_{IVDP} = \left(\sqrt{\left(\frac{(1355-2038.843)^2+(1832.1-2038.843)^2+\ldots+(2118.36-2038.843)^2}{12}\right)}, \sqrt{\left(\frac{(1450.42-2134.263)^2+(1927.52-2134.263)^2+\cdots+(2213.78-2134.263)^2}{12}\right)}\right) = (288.460, 288.460)$$

The results obtained are summarized in Table 3.

Table 3: Average and standard Deviation of AV_{IVDP}

Average '$(\mu_1, \mu_2)'_{IVDP}$	(2038.8433, 2134.263)
Standard Deviation '$(\sigma_1, \sigma_2)_{IVDP}$'	(288.460, 288.460)

Step 3: The statistical measurements obtained in Step 2 are then transformed into Neutrosophic triplet refined interval-valued data partitions, denoted as <A>, <antiA>, and <neutA>. This conversion is performed using the IF-THEN Neutrosophic rule base for values of α ranging from 0.1 to 0.9. The results are detailed in Table 4.

Table 4: NTRIVDP for α = 0.1, 0.2 ,0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9.

AV_{IVDP}	NTRIVDP for α= 0.1, 0.2, 0.3	NTRIVDP for α= 0.4, 0.5, 0.6, 0.7	NTRIVDP for α= 0.8,0.9
(1355, 1450.42)	<antiA, antiA>	<antiA, antiA>	<antiA,antiA>
(1832.1, 1927.52)	<antiA, antiA>	<antiA, antiA>	<A, A>
(1927.52, 2022.94)	<antiA, antiA>	<A, A>	<A, A>
(1832.1, 1927.52)	<antiA, antiA>	<antiA, antiA>	<A, A>
(1832.1, 1927.52)	<antiA, antiA>	<antiA, antiA>	<A, A>
(2213.78, 2309.2)	<A, A>	<A, A>	<A, A>
(2404.62, 2500.04)	<A, A>	<A, A>	<A ,A>
(2309.2, 2404.62)	<A, A>	<A, A>	<A, A>
(2309.2, 2404.62)	<A, A>	<A, A>	<A, A>
(2309.2, 2404.62)	<A, A>	<A, A>	<A, A>
(2022.94, 2118.36)	<A, A>	<A, A>	<A, A>
(2118.36, 2213.78)	<A, A>	<A, A>	<A, A>

Step 4: For α values of 0.1, 0.2, and 0.3, the COUNTIF function is used to determine the number of cells categorized as <antiA, antiA> and <A, A> across the columns of the Neutrosophic Triplet Refined Interval-Valued Data Partition (NTRIVDP). The results are depicted in Figure 3.

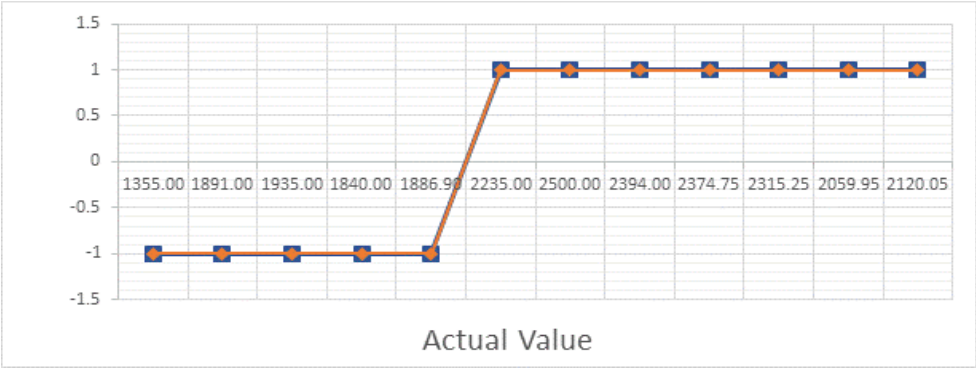


Figure 3: NTRIVDP for (α= 0.1, 0.2, 0.3)

Figure 3 shows that the share prices (1355.00, 1891.00, 1935.00, 1840.00, 1886.90), which fall into the <antiA, antiA> category, yield a count of 5. Conversely, the share prices (2235.00, 2500.00, 2394.00, 2374.75, 2315.25, 2059.95, 2120.05) that fall into the <A, A> category result in a count of 7.

Similarly, for α values of 0.4, 0.5, 0.6, and 0.7, the results from the COUNTIF function are illustrated in Figure 4.

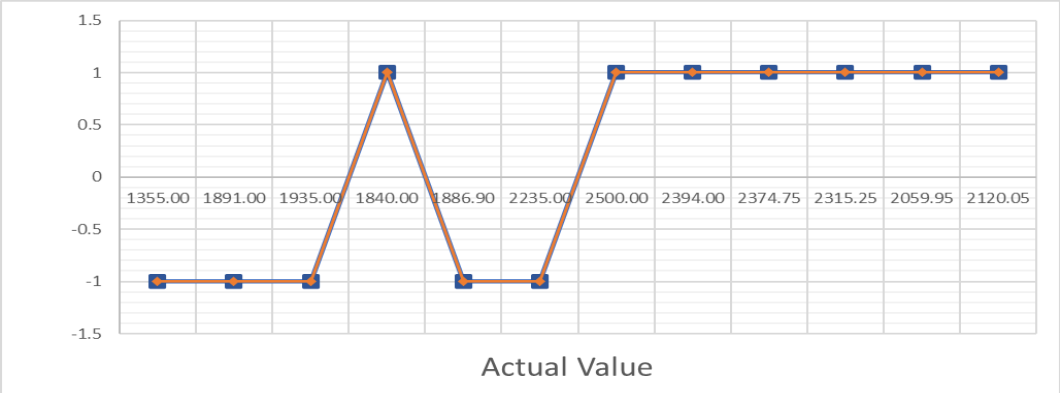


Figure 4: NTRIVDP for (α = 0.4, 0.5, 0.6, 0.7)

Figure 4 reveals that the first three share prices fall into the <antiA, antiA> category, the fourth share price is categorized as <A, A>, the fifth and sixth share prices are again in the <antiA, antiA> category, while the remaining six share prices are classified as <A, A>. This results in a count of 5 for <antiA, antiA> and 7 for <A, A>.

Also, for α values of 0.8 and 0.9, the COUNTIF results are depicted in Figure 5.

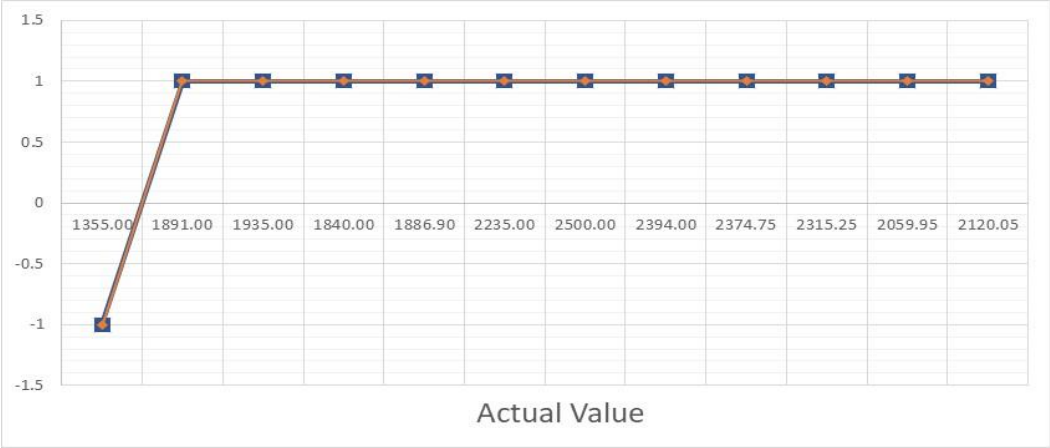


Figure 5: NTRIVDP for (α = 0.8, 0.9)

Figure 5 demonstrates that the first share price falls into the <antiA, antiA> category, resulting in a count of 1, while the remaining share prices are classified as <A, A>, yielding a count of 11.

- The SUMIF results for α values of 0.1, 0.2, and 0.3, computed in Excel, are displayed below.

$$\text{SUMIF } of\ AV_{IVDP}\ for\ < antiA, antiA > = (1355,1450.42) + (1832.1,1927.52) + (1927.52,2022.94) + (1832.1.1927.52) + (1832.1,1927.52)$$

$$\text{SUMIF } of\ AV_{IVDP}\ for\ < A, A > = (2213.78,2309.18) + (2404.62,2500.04) + (2309.20,2404.62) + (2309.20,2404.62) + (2309.20,2404.62) + 2022.94,2118.36) + (2118.36,2213.78)$$

- The AVERAGEIF results for α values of 0.1, 0.2, and 0.3, computed in Excel, are presented below.

AVERAGEIF of AV_{IVDP} for <antiA, antiA> = (8778.82, 9255.92) / (5,5)

AVERAGEIF of AV_{IVDP} for <A, A> = (15687.22, 16355.14) / (7,7)

- The computed COUNTIF, SUMIF, and AVERAGEIF values of AV_{IVDP} for α = 0.1, 0.2 and 0.3 are shown in Table 5.

Table 5: Actual value and predicted value for α = 0.1, 0.2, 0.3

NTR $_{IVDP}$	COUNTIF of AV_{IVDP}	SUMIF of AV_{IVDP}	AVERAGEIF of AV_{IVDP} (AP_{IVDP})
<antiA,antiA>	5	(8778.82, 9255.92)	(1755.764,1851.184)
<A,A>	7	(15687.30, 16355.24)	(2241.043, 2336.463)

- Similarly, the computed COUNTIF, SUMIF, and AVERAGEIF values of AVIVDP for α = 0.4, 0.5, 0.6, and 0.7 are shown in Table 6.

Table 6: Actual value and predicted value for α = 04, 0.5, 0.6, 0.7

NTR $_{IVDP}$	COUNTIF of AV_{IVDP}	SUMIF of AV_{IVDP}	AVERAGEIF of AV_{IVDP} (AP_{IVDP})
<antiA,antiA>	4	(6851.3,7239.98)	(1712.825, 1808,245)
<A,A>	8	(17614.7418378.08)	(2201.8425,2297.26)

- Additionally, the computed COUNTIF, SUMIF, and AVERAGEIF values of AVIVDP for α = 0.8 and 0.9 are presented in Table 7.

Table 7: Actual value and predicted value for α = 0.8, 0.9

NTR $_{IVDP}$	COUNTIF of AV_{IVDP}	SUMIF of AV_{IVDP}	AVERAGEIF of AV_{IVDP} (AP_{IVDP})
<antiA,antiA>	1	(1355,145042)	(1355,1450,42)
<A,A>	11	(23111.04,24160.64002)	(2101.003636,2196.42182)

Step 5: The values of AP_{IVDP}, AP, $Error_{IVDP}$, MPE, $(MPE)^2$, and SSE are calculated and detailed in Table 8 for α=0.1 ,0.2, and 0.3.

Table 8: The result of AP $_{IVDP}$, AP, Error $_{IVDP}$, MPE ,$(MPE)^2$, for α = 0.1 ,0.2, 0.3.

AV $_{IVDP}$	AP $_{IVDP}$	AP	Error $_{IVDP}$	MPE	$(MPE)^2$
(1355, 1450.42)	(1755.764,1851.184)	1803.47	(-400.764, -400.764)	-400.764	160611.7837
(1832.1, 1927.52)	(1755.764,1851.184)	1803.47	(76.336 , 76.366)	76.336	5857.184896
(1927.52, 2022.94)	(1755.764,1851.184)	1803.47	(171.756 , 171.756)	171.756	29397.06994
(1832.1, 1927.52)	(1755.764,1851.184)	1803.47	(76.336 , 76.366)	76.336	5857.184896
(1832.1, 1927.52)	(1755.764,1851.184)	1803.47	(76.336 , 76.366)	76.336	5857.184896
(2213.78, 2309.2)	(2241.0314285714, 2336.4485714285)	2288.74	(-27.251, -27.268)	-27.2595	743.0803403
(2404.62, 2500.04)	(2241.0314285714, 2336.4485714285)	2288.74	(163.569, 163.572)	163.5705	26755.30847
(2309.2, 2404.62)	(2241.0314285714, 2336.4485714285)	2288.74	(68.149, 68.152)	68.1505	4644.49065
(2309.2, 2404.62)	(2241.0314285714, 2336.4485714285)	2288.74	(68.149, 68.152)	68.1505	4644.49065
(2309.2, 2404.62)	(2241.0314285714, 2336.4485714285)	2288.74	(68.149, 68.152)	68.1505	4644.49065
(2022.94, 2118.36)	(2241.0314285714, 2336.4485714285)	2288.74	(-218.091, -218.088)	-218.0895	47563.03001
(2118.36, 2213.78)	(2241.0314285714, 2336.4485714285)	2288.74	(-122.671, -122.668)	-122.6695	15047.80623
					SSE = 31163.15888575

Similarly, the values of AP_{IVDP}, AP, Error $_{IVDP}$, MPE, $(MPE)^2$, and SSE are calculated and presented in Table 9 for α = 0.4 , 0.5, 0.6, and 0.7

Table 9: The result of AP $_{IVDP}$, AP, Error $_{IVDP}$, MPE ,$(MPE)^2$, for α = 0.4 , 0.5, 0.6, 0.7

AV $_{IVDP}$	AP $_{IVDP}$	AP	Error $_{IVDP}$	MPE	$(MPE)^2$
(1355, 1450.42)	(1712.825, 1808.245)	1760.535	(-357.825, -357.825)	-357.825	128,038.730625
(1832.1, 1927.52)	(1712.825, 1808245)	1760.535	(119.275, 119.275)	119.275	14226.525625
(1927.52, 2022.94)	(2201.8425, 2297.26)	2249.55125	(- 274.3225, -274.3225)	- 274.3225	75251.4624
(1832.1, 1927.52)	(1712.825, 1808.245)	1760.535	(119.275, 119.275)	119.275	14226.525625
(1832.1, 1927.52)	(1712.825, 1808.245)	1760.535	(119.275, 119.275)	119.275	14226.525625
(2213.78, 2309.2)	(2201.8425, 2297.26)	2249.55125	(119.275, 119.275)	119.275	14226.525625
(2404.62, 2500.04)	(1712.825, 1808.245)	2249.55125	(202.7575, 202.7575)	202.7575	41111.6176

AV $_{IVDP}$	AP $_{IVDP}$	AP	Error $_{IVDP}$	MPE	$(MPE)^2$
(2309.2, 2404.62)	(1712.825, 1808245)	2249.55125	(107.34, 107.34)	107.34	11521.8756
(2309.2, 2404.62)	(2201.8425, 2297.26)	2249.55125	(107.34, 107.34)	107.34	11521.8756
(2309.2, 2404.62)	(1712.825, 1808.245)	2249.55125	(107.34, 107.34)	107.34	11521.8756
(2022.94, 2118.36)	(1712.825, 1808.245)	2249.55125	(-178.9, 178.9)	-178.9	32005.21
(2118.36, 2213.78)	(2201.8425, 2297.26)	2249.55125	(83.48, 83.48)	83.48	6968.9104
					SSE = 360,763.2211

Additionally, the values of AP_{IVDP}, AP, Error_{IVDP}, MPE, $(MPE)^2$, and SSE are calculated and presented in Table 10 for $\alpha = 0.8$ and 0.9

Table 10: The result of AP $_{IVDP}$, AP, Error_{IVDP}, MPE, $(MPE)^2$, for $\alpha = 0.8, 0.9$

AV $_{IVDP}$	AP $_{IVDP}$	AP	Error $_{IVDP}$	MPE	$(MPE)^2$
(1355, 1450.42)	(1355, 1450.42)	1402.71	(0,0)	0	0
(1832.1, 1927.52)	(2101.0036, 2196.42182)	2148.72171	(-268.9036, -268.9018)	268.9027	72308.662067
(1927.52, 2022.94)	(2101.0036, 2196.42182)	2148.72171	(-173.4836, -173.5118)	173.4977	30101.4519
(1832.1, 1927.52)	(2101.0036, 2196.42182)	2148.72171	(-268.9036, -268.9018)	268.9027	72308.662067
(1832.1, 1927.52)	(2101.0036, 2196.42182)	2148.72171	(-268.9036, -268.9018)	268.9027	72308.662067
(2213.78, 2309.2)	(2101.0036, 2196.42182)	2148.72171	(112.7764, 112.75818)	112.76729	12716.4617
(2404.62, 2500.04)	(2101.0036, 2196.42182)	2118.746917	(303.5964, 303.59818)	303.59729	92171.31449
(2309.2, 2404.62)	(2101.0036, 2196.42182)	2148.72171	(208.1764, 208.17818)	208.17729	4337.7841
(2309.2, 2404.62)	(2101.0036, 2196.42182)	2148.72171	(208.1764, 208.17818)	208.17729	4337.7841
(2309.2, 2404.62)	(2101.0036, 2196.42182)	2148.72171	(208.1764, 208.17818)	208.17729	4337.7841
(2022.94, 2118.36)	(2101.0036, 2196.42182)	2148.72171	(-78.0636, -78.06182)	78.06271	6093.78669
(2118.36, 2213.78)	(2101.0036, 2196.42182)	2148.72171	(17.3564, 17.35818)	17.35729	301.2755161441
					SSE = 569329.69116258

Figure 6, shown below, illustrates the actual values and the average predicted values of the interval-valued data partition (IVDP).

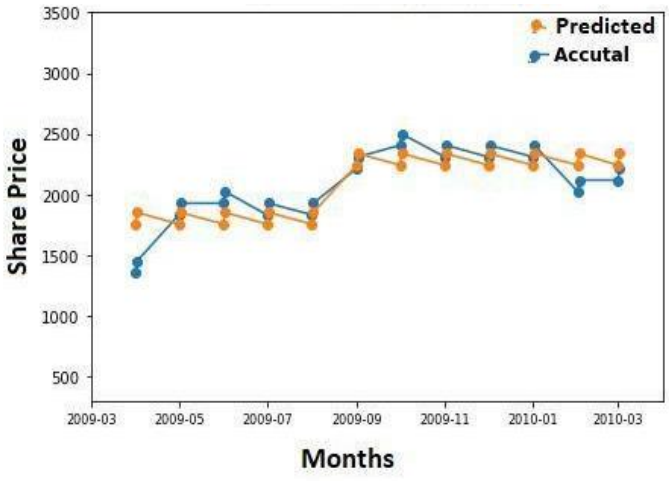


Figure 6: Actual and Average Predicted values of IVDP

Step 6: For $\alpha = 0.1, 0.2, 0.3$, the performance error measures for the proposed method are calculated as follows: Mean Squared Error (MSE) = 25,969.68 and Mean Absolute Percent Error (MAPE) = 6.570417.

Table 11 and Figure 7 present a comparative analysis of the actual versus predicted values, contrasting the results of the proposed model with those obtained using existing methods, such as Singh's method and B.P. Joshi's method.

Table 11**:** Comparison of AP of the proposed model with the other existing models.

MONTHS	Singh's method	B.P.Joshi method	Average Predicted Value of 'AP' of NTRIVDP
Apr-2009	2000	1750	1803.47
May-2009	2000	1850	1803.47
Jun-2009	2000	1850	1803.47
Jul-2009	2000	1850	1803.47
Aug-2009	2000	1850	1803.47
Sep-2009	2400	2250	2288.74
Oct-2009	2000	2250	2288.74
Nov-2009	2000	2250	2288.74
Dec-2009	2000	2250	2288.74
Jan-2010	2000	2250	2288.74
Feb-2010	2000	2250	2288.74
Mar-2010	2000	2250	2288.74

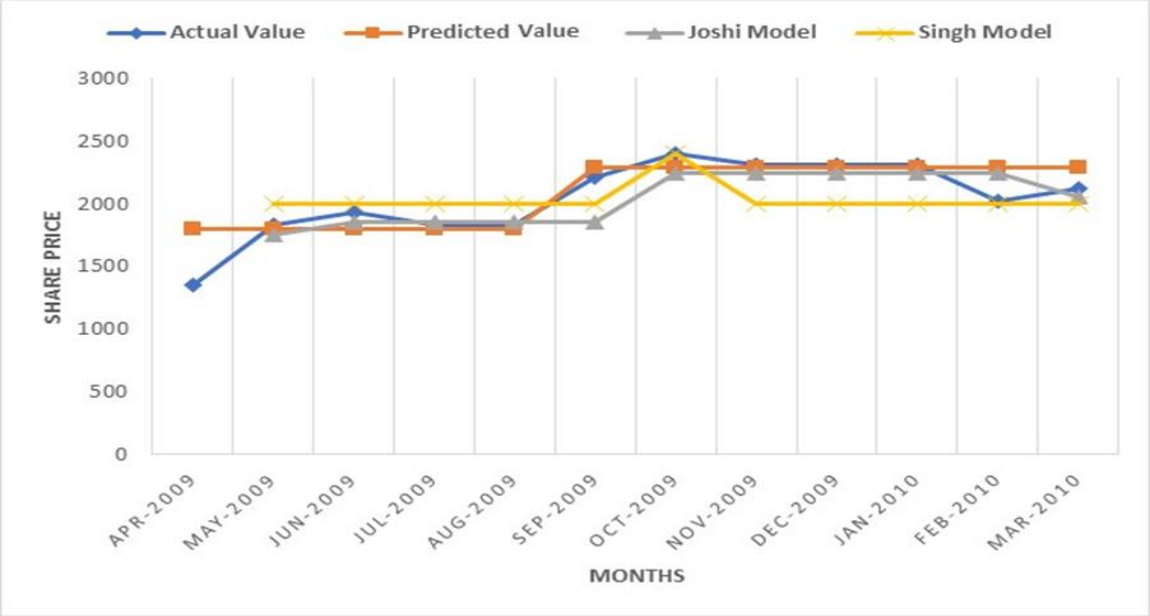


Figure 7: Comparison of proposed model with existing model.

Table 12 provides a comparative analysis of various performance measures, evaluating the proposed model against existing models.

Table 12. Comparison of MSE and MAPE 0f NTRIVDP with existing methods.

Performance Measure	B.P. Joshi Model	Singh Model	Proposed Model
Mean Square Error (MSE)	29170.43	48434.98	25,969.68
Mean Absolute percentage Error (MAPE)	6.199161	8.432273	6.570417

Table 13 presents the Mean Squared Error (MSE) for various values of α: 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, and 0.9. The results indicate that α = 0.1, 0.2, and 0.3 yield the most favourable outcomes compared to the other α values.

Table 13: Neutrosophic triplet to average predicted share price.

Parameter ‘α’	MSE
0.1, 0.2, 0.3	25,969.68
0.4, 0.5, 0.6, 0.7	33,711.02
0.8, 0.9	47,444.14

5. Conclusion:

This study introduces a novel approach to analyzing stock market index movements through a rule-based Neutrosophic Triplet Refined Interval-Valued Data Partition (NTRIVDP) model. The model has demonstrated notable efficacy, yielding a Mean Squared Error (MSE) of 25,969.68 and a Mean Absolute Percent

Error (MAPE) of 6.570417 when α = 0.1, 0.2, and 0.3. These performance metrics reflect the model's superior ability to predict stock market fluctuations with high accuracy. The NTRIVDP model effectively addresses the complexities and uncertainties inherent in stock market dynamics, leveraging Neutrosophic logic to capture subtle nuances that traditional methods often overlook. Its flexibility in adjusting α values enables fine-tuning to different market conditions, enhancing its adaptability and responsiveness in the ever-changing financial landscape. The study's results underscore the model's advancement over existing forecasting techniques, highlighting its potential to provide more precise predictions and better alignment with actual market movements. This improvement in accuracy facilitates more informed decision-making for investors and financial analysts, ultimately aiding in risk mitigation and return maximization. By pushing the boundaries of conventional forecasting methods, this research emphasizes the value of integrating innovative methodologies like the NTRIVDP in financial analysis. The promising outcomes not only validate the effectiveness of Neutrosophic-based approaches but also pave the way for further exploration of these techniques in broader financial forecasting applications. This advancement represents a significant step forward in developing robust and reliable frameworks for navigating the complexities of investment decision-making in a dynamic market environment.

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Structured Entrepreneurship Education Assessment in Vocational Colleges: A HyperSoft Set Model with an MCDM-Based Framework

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Abstract: We develop a multi-criteria decision making (MCDM) approach for Innovation and Entrepreneurship Education Evaluation in Vocational Colleges. Two MCDM methods are used in this study such as MEREC methodology to compute the criteria weights and the VIKOR methodology to rank the alternatives. We use the HyperSoft set to deal with criteria and sub criteria. This study uses eight criteria and eight alternatives. Then we use the best sub criteria to evaluate and rank the alternatives. We conduct sensitivity and comparative in this study. The results show the proposed approach is effective compared to MCDM methods and the sensitivity analysis shows the rank of alternatives are stable under different cases.

Keywords: HyperSoft Set Model; MCDM Approach; Innovation and Entrepreneurship; Education; Vocational Colleges.

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1. Introduction

In the modern knowledge-based economy, innovation and entrepreneurship education (IEE) plays a pivotal role in enhancing students' employability and fostering economic development. Vocational colleges, which are designed to equip students with practical skills and industry-oriented knowledge, have increasingly integrated entrepreneurial education into their curricula to meet the demands of an evolving job market. The objective of IEE in vocational education is not only to instill business acumen but also to develop students' creativity, problem-solving skills, and adaptability in competitive industries[1], [2]. However, evaluating the effectiveness of these programs requires a structured approach that considers curriculum relevance, institutional support, industry collaboration, and student engagement. The integration of innovation and entrepreneurship education in vocational colleges faces various challenges, such as limited financial resources, inadequate faculty training, and weak industry partnerships. Effective evaluation frameworks must assess how well the curriculum aligns with real-world business challenges, the extent to which practical training opportunities are provided, and the level of institutional backing available for student startups. Moreover, factors such as digital transformation, technological integration, and mentorship availability significantly impact the

success of entrepreneurship education[3], [4]. Without a comprehensive evaluation framework, vocational colleges may struggle to identify areas for improvement and optimize their programs for better student outcomes. To systematically evaluate innovation and entrepreneurship education, Multi-Criteria Decision-Making (MCDM) methods provide a structured approach to ranking and assessing different educational models. Methods can be employed to measure the effectiveness of various vocational colleges in terms of curriculum design, faculty expertise, student engagement, and startup success rates. These methodologies enable policymakers and educators to make data-driven decisions that enhance the quality of entrepreneurial training, ensuring that students receive the necessary skills to thrive in dynamic job markets[5], [6]. Furthermore, the use of big data analytics and artificial intelligence in evaluating IEE programs can offer deeper insights into student performance and institutional effectiveness.

The evaluation of innovation and entrepreneurship education in vocational colleges is crucial for enhancing program efficiency, fostering industry collaboration, and ensuring student success. A well-structured evaluation framework helps identify the best practices, refine teaching methodologies, and bridge the gap between education and industry needs. By adopting advanced MCDM techniques and digital evaluation tools, vocational colleges can continually improve their entrepreneurship education models, ultimately contributing to a more innovative and competitive workforce[7], [8]. Moving forward, it is essential to integrate technology, research-based evaluation methods, and industry-driven insights to shape a robust and future-ready entrepreneurial education system in vocational training institutions.

Innovation and entrepreneurship education in vocational colleges is a MCDM problem, which is a problem in which decision alternatives are evaluated according to their performance values in each evaluation criteria, which are weighted according to their importance to the decision makers (DMs). Many MCDM methods have been developed to solve decision problems, and they can be divided into compensatory and non-compensatory techniques[9]. Compensatory techniques are those in which a lower performance value of an alternative in one criterion can be offset by a higher performance value in another criterion.

Innovations in sustainable engineering and technology are essential for cutting carbon emissions. The active involvement of businesses in tackling carbon reduction as a systemic strategy is a crucial component of this. Chinese businesses have created carbon reduction systems in response to government incentives in the People's Republic of China, which help them match their organizational objectives with long-term national policies. Six firms' carbon reduction plans are assessed by Esangbedo and Tang [10] as MCDM dilemma. To do this, they suggested the grey-MEREC-MAIRCA approach, a novel hybrid MCDM technique. Their approach combined the multi-attribute ideal-real comparative analysis (MAIRCA) based on the grey system theory with the recently established method based on the removal effects of criteria (MEREC) for weighting.

The added advantage of their hybrid approach is that it takes decision-making uncertainty into consideration. According to the decision-maker scores, their study's noteworthy conclusions

include the importance of energy-saving efficiency and the reduction of direct carbon emissions. On the other hand, making a commitment to corporate social responsibility through information transparency and carbon public welfare is regarded as a lower priority. Furthermore, the ranking results generated using this method are contrasted with those from the conventional weighted sum model and the methodology for order preference by similarity to ideal solution (TOPSIS), verifying the pick of the top organization. Even if their suggested approach has drawbacks and requires extra steps for evaluation, it created the possibility for future studies to create more straightforward MCDM techniques in the face of ambiguity.

1.1 Why HyperSoft Set Was Used in This Study

The HyperSoft Set is a powerful mathematical tool for solving complex decision-making problems. Unlike traditional soft sets, which assign only one value to each parameter, the HyperSoft Set allows each parameter to have multiple values, making it more suitable for real-world applications where criteria often overlap.

It helps in managing uncertain and vague data, making it highly effective in fields like education, industry, healthcare, and artificial intelligence. By providing a structured way to group and classify information, it ensures that decision-making processes are more accurate and adaptive.

The HyperSoft Set also enhances multi-criteria decision-making (MCDM) by offering a more detailed and flexible evaluation model. It improves the ranking and selection of alternatives by considering all possible influences and dependencies between different factors.

Because of these advantages, the HyperSoft Set is widely used in educational assessment, industrial decision-making, medical diagnostics, and intelligent systems. Its ability to handle complex and uncertain data makes it an essential tool for modern decision-making challenges.

1.2 Importance of HyperSoft Set in the Proposed Work

In this study, the HyperSoft Set is essential for handling the complex decision-making process involved in evaluating entrepreneurship education in vocational colleges. Traditional methods often struggle with multi-level criteria and overlapping factors, making decision-making less precise. The HyperSoft Set offers a structured way to deal with multiple criteria and sub-criteria, improving accuracy and clarity in ranking alternatives.

By using the HyperSoft Set, this research ensures a more flexible and systematic evaluation framework. It helps in categorizing different educational factors effectively, considering the relationships between various criteria. The integration of HyperSoft Set with the MEREC method allows for precise calculation of criteria weights, while its combination with the VIKOR method enhances the ranking process, ensuring a data-driven and reliable decision-making approach.

This approach is particularly useful in entrepreneurship education, where multiple elements such as curriculum relevance, faculty expertise, student engagement, and industry collaboration must

be evaluated together. The HyperSoft Set makes it easier to handle these interconnected factors, leading to more accurate and insightful results.

2. HyperSoft Set-MEREC-VIKOR

This section shows the steps of the proposed approach. We use the MEREC method to compute the criteria weights and the VIKOR method to rank the alternatives. HyperSoft set is used in this study to deal with the criteria and sub criteria[11].

HyperSoft Set [12], [13]

We can define the soft set as:

Let's have $n \geq 2$ Soft set

$$\begin{pmatrix} F_1: A_1 \to P(H), \\ F_2: A_2 \to P(H), \\ F_3: A_3 \to P(H) \\ F_4: A_4 \to P(H) \\ F_5: A_5 \to P(H) \\ F_6: A_6 \to P(H) \\ F_7: A_7 \to P(H) \\ \vdots \\ F_n: A_n \to P(H) \end{pmatrix} \tag{1}$$

Where $a_1, a_2, \ldots\ldots a_n$ are the attributes and the $A_1, A_2, \ldots., A_n$ are the corresponding sets of attributes' values such that

$$A_i \cap A_j = \emptyset \; for \; i \neq j \; and \; i, j \; \in \{1,2, \ldots, n\} \tag{2}$$

The Soft set product computed such as:

$$F_1 \times F_2 \ldots \times F_n: A_1 \times A_2 \times \ldots A_n \to P(H) \times P(H) \times \ldots \times P(H) \tag{3}$$

Let we have $F = \; F_1 \times F_2 \ldots \times F_n$

Then we can define the HyperSoft Set as:

$$F: A_1 \times A_2 \times \ldots \times A_n \to P(H) \text{ such that for each } (e_1, e_2, \ldots e_n) = F_1(e_1) \cap F_2(e_2) \cap \ldots \cap F_n(e_n) \tag{4}$$

MEREC Model

This method is used to compute the criteria weights[10].

Construct the decision matrix

The decision matrix is built in the first step to show the opinions of the experts and decision makers for each criterion and alternatives.

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{mn} \end{pmatrix}$$

(5)

Combine the decision matrix

We combine the decision matrix into a single matrix.

Normalize the decision matrix

The decision matrix is normalized for the positive and negative criteria such as:

$$n_{ij} = \frac{\min_i x_{ij}}{x_{ij}} \tag{6}$$

$$n_{ij} = \frac{x_{ij}}{\max_i x_{ij}} \tag{7}$$

Compute the overall performance of the alternatives

$$T_i = \ln\left(1 + \left(\frac{1}{m}\sum_i \left|\ln(n_{ij})\right|\right)\right) \tag{8}$$

Compute the overall performance by deleting each criterion

$$T_{ij}^{/} = \ln\left(1 + \left(\frac{1}{m}\sum_{k,k\neq j} \left|\ln(n_{ij})\right|\right)\right) \tag{9}$$

Compute the removal effect of the criterion

$$U_i = \sum_i \left|T_{ij}^{/} - T_i\right| \tag{10}$$

Compute the criteria weights

$$w_j = \frac{U_i}{\sum_i U_i} \tag{11}$$

VIKOR Model

This method is used to rank the alternatives based on a set of criteria and sub criteria[14], [15]. The steps of the VIKOR method are shown as follows:

Compute the min and max values in the decision matrix.

$$z_i^* = \max_i x_{ij} \tag{12}$$

$$z_i^- = \min_i x_{ij} \tag{13}$$

For the non-beneficial criteria

$$z_i^* = \min_i x_{ij} \tag{14}$$

$$z_i^- = \max_i x_{ij} \tag{15}$$

Normalize the decision matrix

$$y_{ij} = \frac{(z_i^* - x_{ij})}{(z_i^* - z_i^-)} \tag{16}$$

Compute the weighted decision matrix

$$q_{ij} = w_j y_{ij} \tag{17}$$

Compute the S and R indexes

$$S_i = \sum_{j=1}^{n} q_{ij} \tag{18}$$

$$R_i = \max_j q_{ij} \tag{19}$$

Compute the VIKOR index

$$V_i = \varphi \times \left(\frac{(S_i - S^+)}{(S^- - S^+)}\right) + (1 - \varphi) \times \left(\frac{(R_i - R^+)}{(R^- - R^+)}\right) \tag{20}$$

$$S^+ = \min_i S_i \tag{21}$$

$$S^- = \max_i S_i \tag{22}$$

$$R^+ = \min_i R_i \tag{23}$$

$$R^- = \max_i R_i \tag{24}$$

Rank the alternatives

3. Results and Discussion

To analyze the effectiveness of the proposed model, we conducted a case study that evaluates the criteria weights and ranks the available alternatives. The MEREC method was used to determine the importance of each criterion, while the VIKOR method was applied to rank the alternatives based on their performance. These methods provided a structured approach to decision-making, ensuring a clear and objective evaluation process.

For this study, three experts and decision-makers were invited to assess the criteria and alternatives. The evaluation was based on eight key criteria, each representing a significant aspect of entrepreneurship education in vocational colleges. Each criterion was assigned three levels to allow for a more detailed assessment. The criteria used in this study were:

1. Student Engagement in Innovation (High, Medium, Low)
2. Technological Integration (Advanced, Moderate, Basic)
3. Employment and Startup Success Rate (High, Medium, Low)
4. Industry Collaboration (Strong, Moderate, Weak)

5. Institutional Support (Strong, Moderate, Weak)
6. Faculty Competency (Highly Skilled, Moderately Skilled, Basic Skills)
7. Curriculum Relevance (High, Medium, Low)
8. Practical Training Opportunities (Extensive, Moderate, Limited)

Each of these criteria plays a vital role in assessing the quality of entrepreneurship education programs and their impact on student success. The study aimed to identify strengths and weaknesses within these programs, helping educational institutions make informed improvements.

To begin the evaluation process, we constructed the decision matrix using Equation (5). The decision matrix included values ranging from 0.1 to 0.7, representing the relative performance of each alternative against the selected criteria. These values were then combined into a single matrix for further analysis.

To ensure consistency and comparability, the decision matrix was normalized using Equation (6), as illustrated in Figure 1. Normalization ensures that all values are scaled appropriately, allowing for a fair comparison between different criteria.

Once the normalization was completed, we calculated the overall performance of the alternatives using Equation (8). This step helped determine which alternatives performed better relative to the set criteria. To further refine the results, we computed the overall performance after removing each criterion individually using Equation (9), as shown in Figure 2. This approach helped us understand the impact of each criterion on the final ranking.

Additionally, we calculated the removal effect of each criterion using Equation (10). This allowed us to measure how much influence each criterion had on the overall decision-making process. Finally, we determined the criteria weights using Equation (11), as presented in Figure 3. These weights provided valuable insights into which criteria were the most influential in ranking the alternatives.

By following this structured evaluation process, we ensured that the ranking method was objective, data-driven, and reliable. The results obtained from this case study demonstrate the effectiveness of integrating HyperSoft Set, MEREC, and VIKOR in decision-making. This approach not only improves the accuracy of evaluations but also provides a solid framework for enhancing entrepreneurship education programs in vocational colleges.

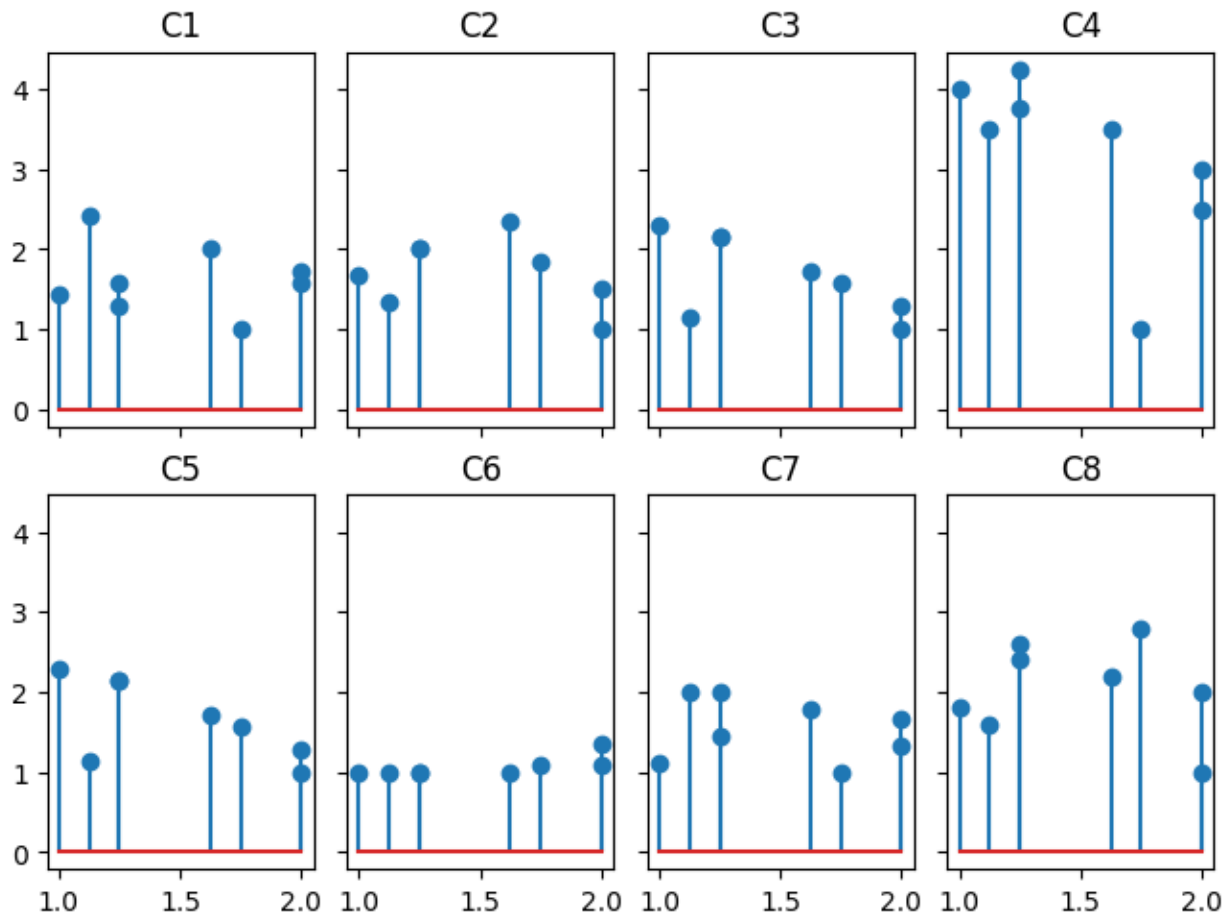

Fig 1. The normalization matrix.

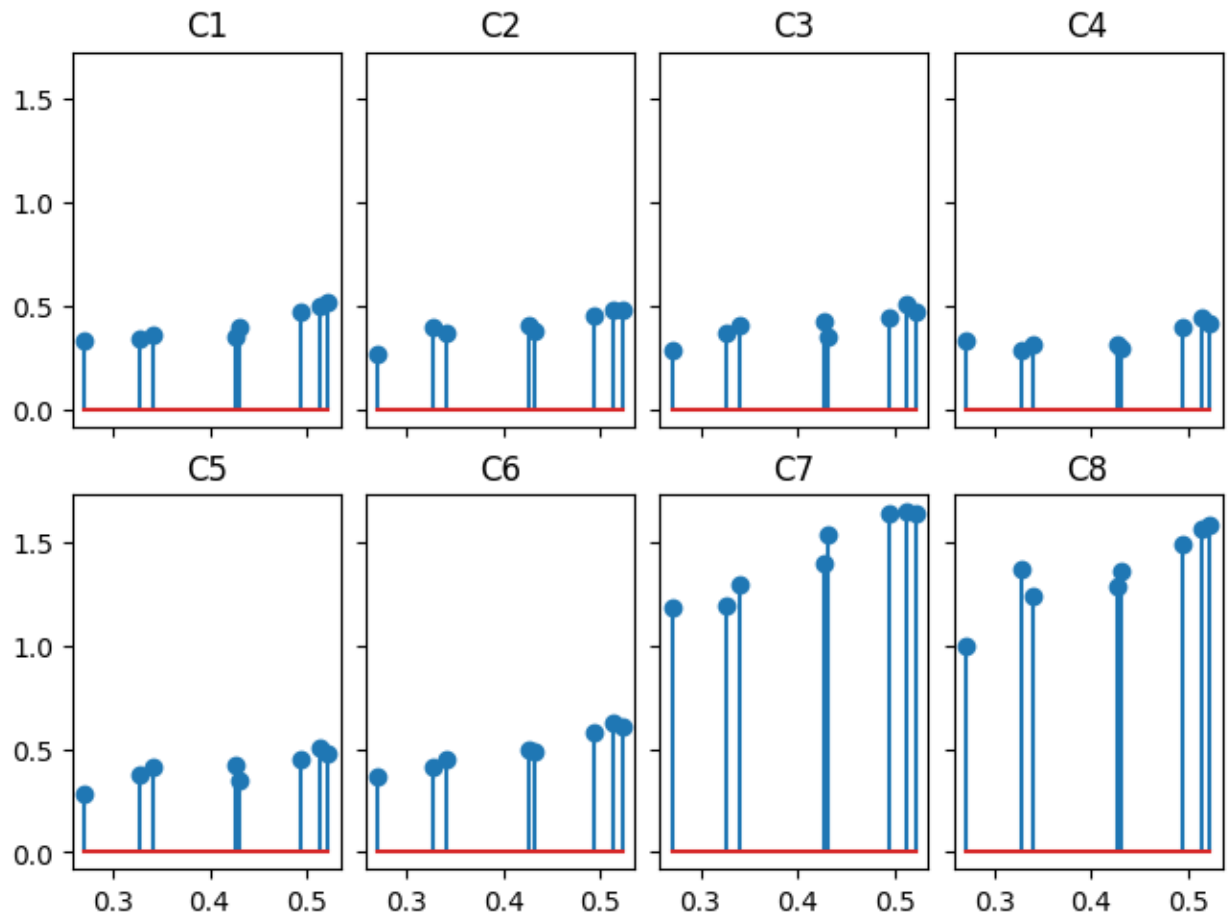

Fig 2. The overall performance by deleting each criterion.

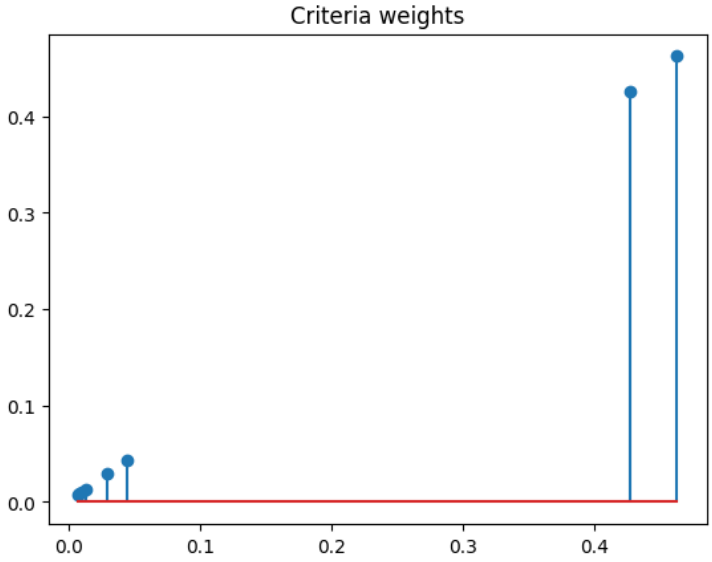

Fig 3. The criteria weights.

4. Results of the VIKOR Model

To rank the alternatives, we used the HyperSoft Set method to handle the criteria and sub-criteria. The sub-criteria considered in this study were:
{High, Advanced, Medium, Moderate, Strong, Highly Skilled, High, Moderate}. After defining these sub-criteria, we applied the VIKOR method to evaluate the alternatives.

Calculation Steps:

1. Finding the Minimum and Maximum Values:

   We used Equations (12) and (13) to determine the lowest and highest values in the decision matrix.

2. Normalizing the Decision Matrix:

   Equation (16) was applied to standardize the values in the matrix, making it easier to compare different alternatives. The normalized matrix is shown in Figure 4.

3. Calculating the Weighted Decision Matrix:

   Using Equation (17), we assigned weights to the decision matrix based on the importance of each criterion. The result is shown in Figure 5.

4. Computing S and R Indexes:

   We calculated these indexes using Equations (18) and (19). These values represent the performance of each alternative, as shown in Figure 6.

5. Computing the VIKOR Index:

   After obtaining the previous values, we used Equation (20) to calculate the VIKOR index, which determines the final ranking of the alternatives. This is illustrated in Figure 7.

   In this study, we used $\varphi = 0.5$ as a balancing factor.

6. Ranking the Alternatives:

   Finally, based on the calculated values, we ranked the alternatives using the VIKOR method. The rankings are shown in Figure 8.

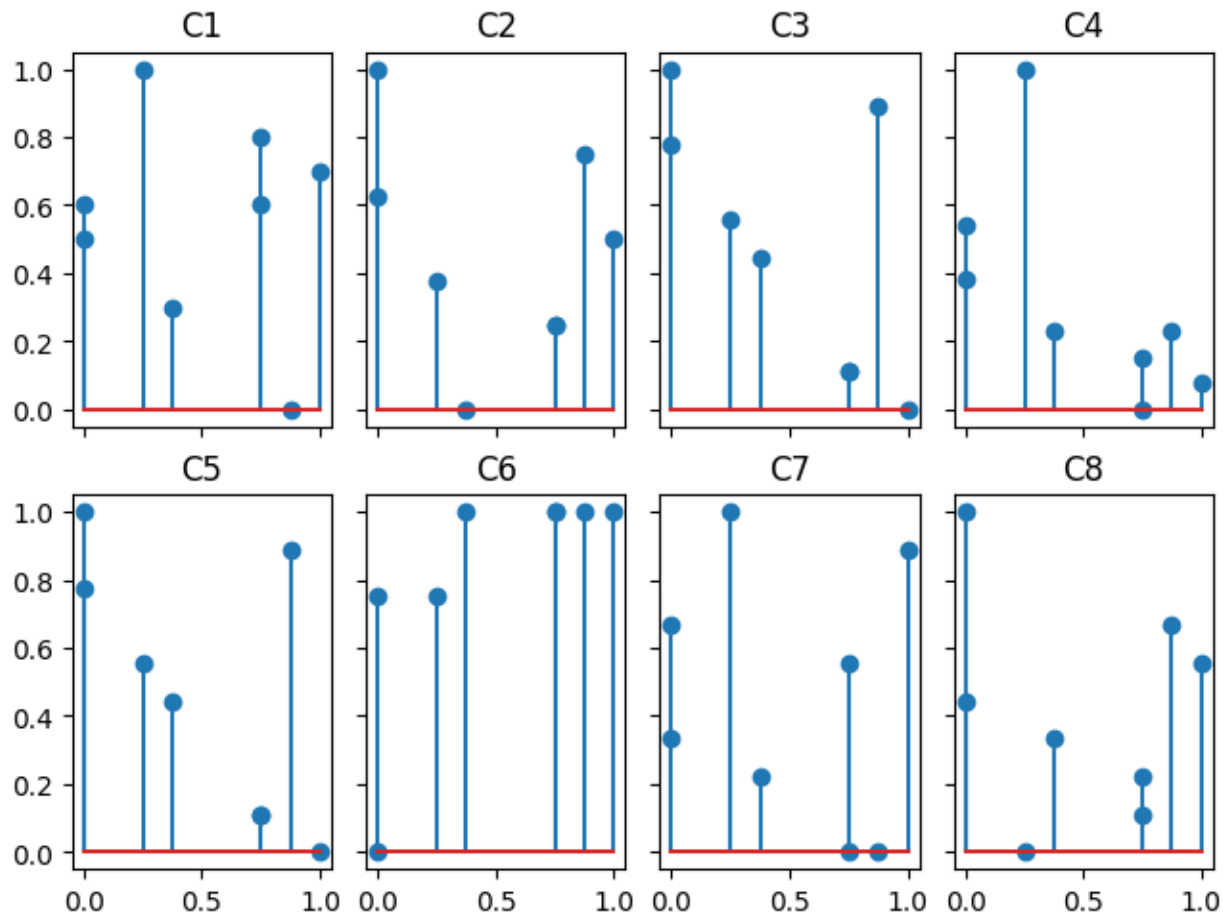

Fig 4. The normalization values by the VIKOR method.

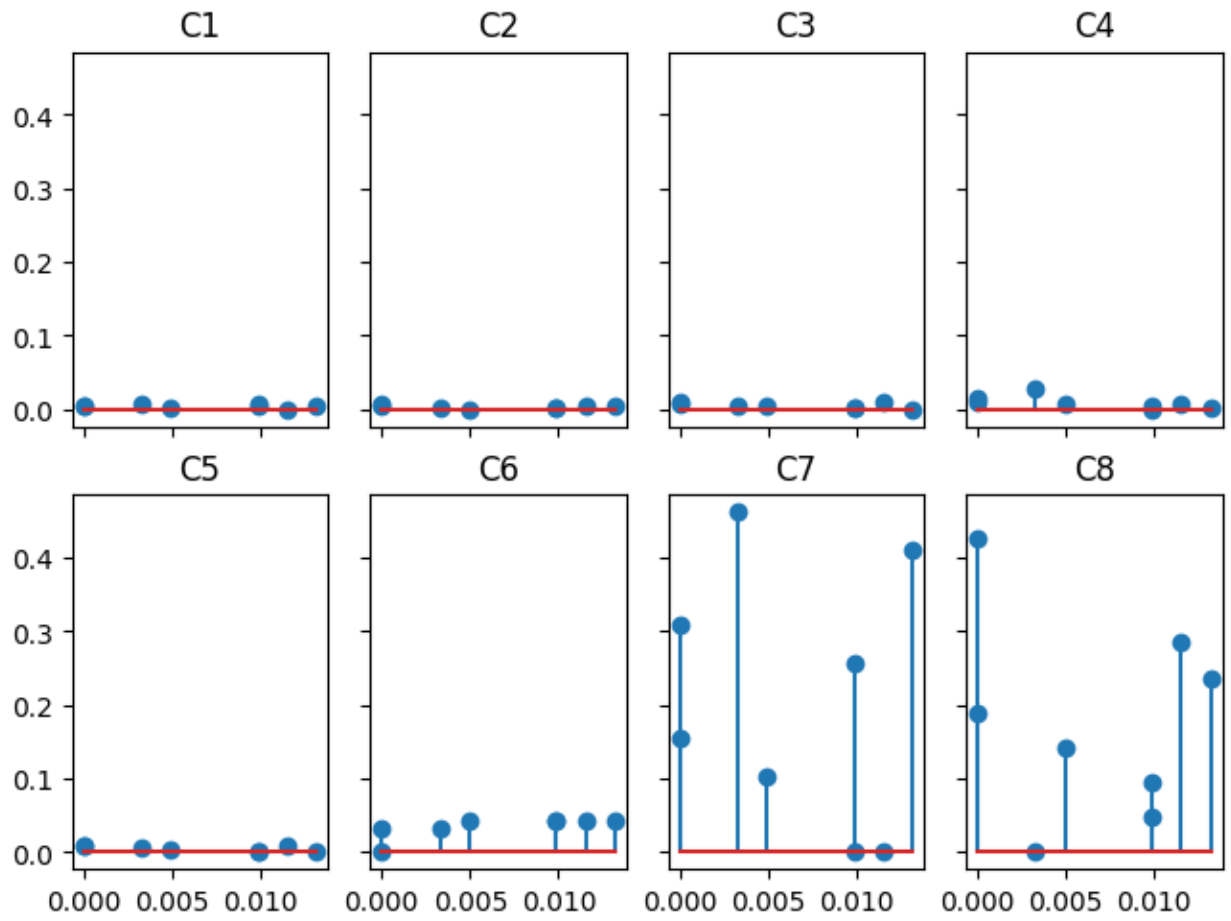

Fig 5. The weighted decision matrix.

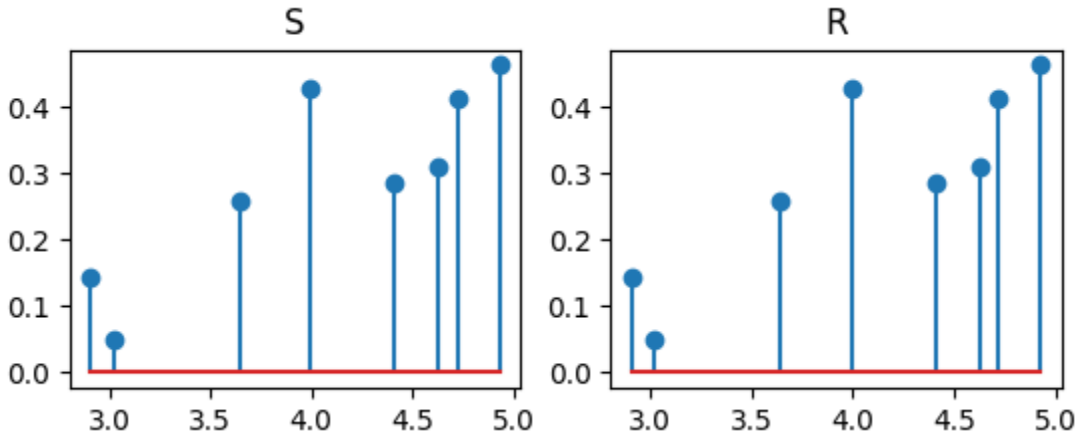

Fig 6. The values of S and R indexes.

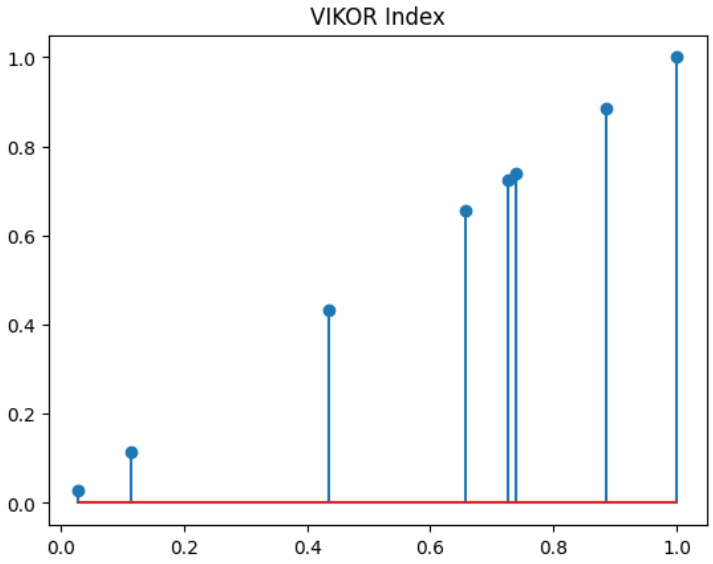

Fig 7. The VIKOR index.

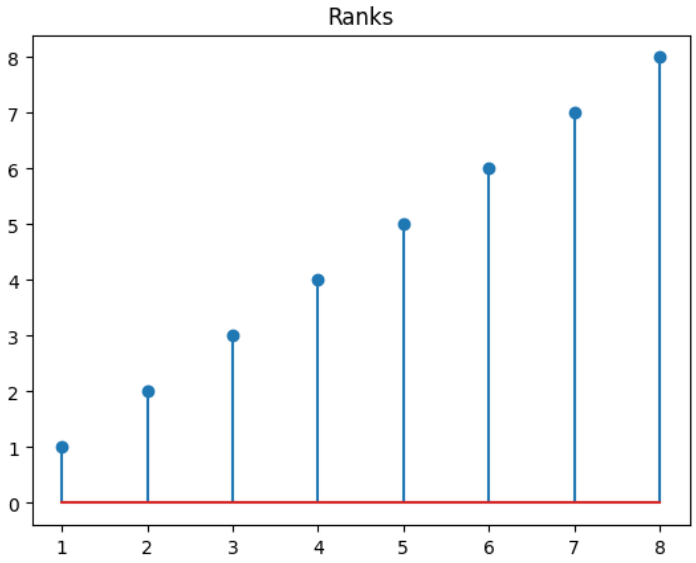

Fig 8. The rank of alternatives.

4. Investigation

This section shows the comparative and sensitivity analysis of the proposed approach. In the comparative analysis, we compered the proposed approach with different MCDM methods to show the validation and effectiveness of the proposed approach. We compared it by four different methods. The results show alternative 1 is the best and alternative 4 is the worst. The results show the proposed approach is effective compared to other MCDM methods. The results of comparative analysis are shown in Fig 9. The proposed approach can deal with the positive and negative criteria in the decision-making process and ranking the alternatives.

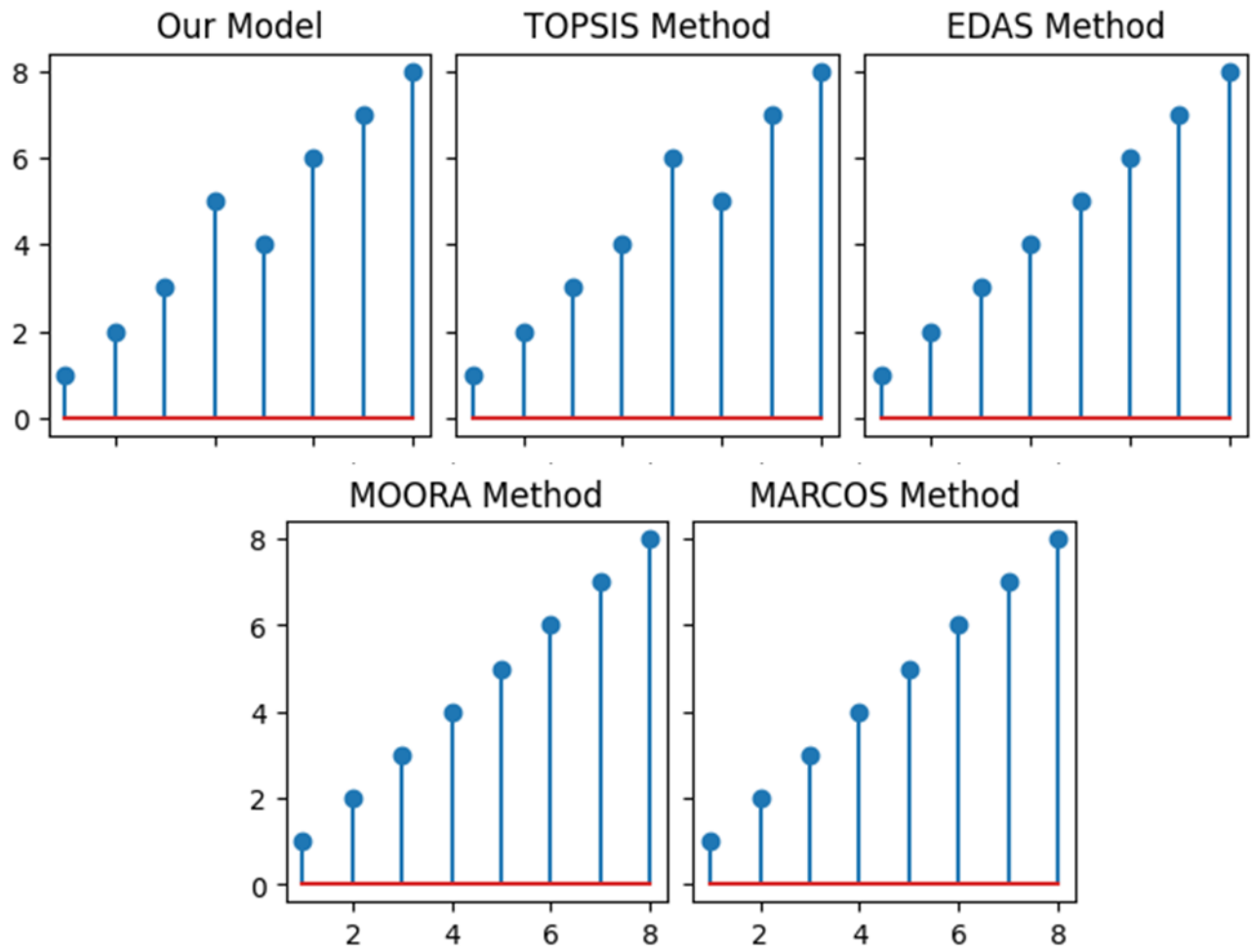

Fig 9. Comparative analysis.

Sensitivity Analysis

In this section, we examine how the ranking of alternatives changes under different conditions. To do this, we adjust the φ value, varying it between 0 and 1, and then observe how the rankings are affected.

Findings*:*

1) After testing different values of φ, the results show that Alternative 1 consistently ranks as the best option, while Alternative 4 remains the worst.
2) The stability of rankings across different scenarios indicates that the proposed model is robust and reliable in decision-making.
3) Figure 10 presents the results of the sensitivity analysis, showing that the rankings do not significantly change under different conditions.

This analysis confirms that the VIKOR method produces stable and reliable rankings, ensuring that the decision-making process is consistent and unaffected by minor changes in parameters.

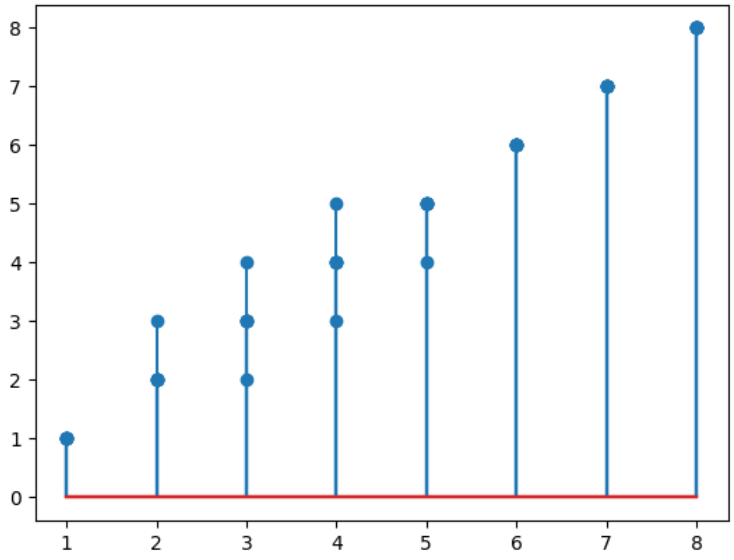

Fig 10. The ranks of alternatives under different cases.

5. Conclusion and Future Work

5.1 Conclusion

This study developed a decision-making model to evaluate Innovation and Entrepreneurship Education in vocational colleges. We used the MEREC method to calculate the importance of each criterion and the VIKOR method to rank the alternatives. Three experts evaluated the options using a scale from 0 to 0.7.

The results showed that Alternative 1 was the best, while Alternative 4 ranked the lowest. A sensitivity analysis was done with ten different cases, and the rankings remained stable, proving the reliability of the model. The comparative analysis with four other MCDM methods confirmed that this approach is effective and can be used to improve decision-making in vocational education.

5.2 Future Work

- Test the model on more vocational colleges and different types of alternatives
- Use artificial intelligence and machine learning to improve accuracy and provide better recommendations
- Compare the results with other MCDM methods like TOPSIS and AHP to check if there are better alternatives
- Study the effect of industry partnerships and hands-on training on the success of innovation and entrepreneurship education
- Conduct long-term research to see how the rankings change over time and how policies impact education
- Develop a digital tool based on this model to help educators and policymakers make better decisions

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Sombor Index in Neutrosophic Graphs

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Abstract: One of the most fundamental ideas in chemical graphs is topological indices. Numerous applications of the topological index, a wholly pictorial notion, may be found in the domains of nanotechnology, chemistry, materials science, medicines, and other things. Neutrosophic graphs are three-valued logic-based graphs. Although in some circumstances it is challenging to optimize and model such utilizing fuzzy graphs, they may be regarded as a fuzzy graph. In this study, the kinds of Sombor indices on neutrosophic graphs are first defined. A degree-based indicator called the Sombor index was developed by Gutman. The formula for this index is $SO(G) = \sum_{uv\in E(G)} \sqrt{deg(u)^2 + deg(v)^2}$. The following triad is the definition of the Sombor index for neutrosophic graphs.

$$So(G) = \big(So_T(G), So_I(G), So_F(G)\big), \qquad |So(G)| = \frac{4 + 2So_T(G) - So_I(G) - 2So_F(G)}{6}.$$

Keywords: Sombor index, Reduced Sombor index, Average Sombor index, Neutrosophic Graphs, Topological indices, Complete neutrosophic graph, Regular neutrosophic graph.

1. Introduction

According to graph theory, the graph is an ordered pair $G = (V, E)$, such that V is the set of vertices and E is a relation on V when called edges. Here V and E both accept values belonging to the set of natural numbers $(\mathbb{N})$, that is, $|V| = n$, and $|E| = m$ such that $n, m \epsilon \mathbb{N}$. In 1965, Zadeh [1] revolutionized modern mathematics by defining fuzzy sets. This development was able to create a new and practical branch of graph theory. Accordingly, m and n do not necessarily belong to natural numbers $(\mathbb{N})$, they can derive their values from real numbers $(\mathbb{R})$. Thus fuzzy graphs were born and human beings have witnessed many applications of this theory in recent years. Finally, in 1995, new collections called neutrosophic collections were identified and introduced by Smarandach [16]. These collections quickly spread to other areas of mathematical research. Those interested in graph theory, like other researchers, generalized the concept of neutrosophy in graph theory and thus led to the birth of a new subset of graphs called neutrosophic graphs. Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational. It is the base of neutrosophic logic, a multiple-value logic that generalizes fuzzy logic and deals with paradoxes, contradictions, antitheses, and antinomies [3, 21-23].

The neutrosophic graph is an ordered pair $G = (V, E)$, such that V is the set of vertices with membership function N, when N is a neutrosophic function, and E is edges set, that defined with neutrosophic function M as triad (T, I, F). For further study on the neutrosophic and neutrosophic sets of graphs, see References [2,3,24-27].

Topological indices are quantities that, by attributing a numerical value to a graph or a network according to its properties, help us greatly in analyzing, inferring, and comparing different graphic structures. These indices are divided into two general types indices based on degree and indices based on distance. Each of these types has its own applications. The Sombor index is a degree-based index introduced by Gutman in 2020 [4]. After the introduction of this index, many researchers tried to introduce and use this index and in a short time, several articles were presented about this index. Here are some articles published in this field for further reading by those who are interested [5-15].

In short, it can be said that a topological index is a real number that is attributed to a molecular graph and is stable with respect to the uniformity of the graph. They are used as a tool to determine the chemical and physical properties of molecules. Also, currently, these indicators are widely used in networks. The Wiener index, abbreviated as W, is the first topological index used in chemistry. This index was introduced by Harold Wiener in 1947 to show the relationship between the physical and chemical properties of organic compounds and their topological structure.

After the introduction of fuzzy graphs, the generalization of topological indices on this type of graph and the investigation of its applications were very interesting for mathematical and computer science researchers. In recent years, very beautiful and practical results have been obtained from this research, so readers can refer to references [28-31] to learn more about a corner of this valuable research.

In previous articles, we have thoroughly examined the connectivity index [17-19], and the Wiener index [20] in neutrosophic graphs, and in this paper, we define the Sombor index on neutrosophic graphs.

2. Preliminaries

In the section provides some definitions and theorems.

Definition 1. [3] Let $G = (N, M)$ is an single-valued Neutrosophic graph, where N is a Neutrosophic set on V and, M is a Neutrosophic set on E, which satisfy the following

$$T_M(u,v) \le min\big(T_N(u), T_N(v)\big),$$
$$I_M(u,v) \ge max\big(I_N(u), I_N(v)\big),$$
$$F_M(u,v) \ge max\big(F_N(u), F_N(v)\big),$$

Where u and v are two vertices of G, and $(u,v) \in E$ is an edge of G.

Definition 2. [3] Given $G = (N, M)$ be a single-valued neutrosophic graph and P is a path in G. P is a collection of different vertices, $v_0, v_1, v_2, \dots, v_n$ such that $\big(T_M(v_{i-1}, v_i), I_M(v_{i-1}, v_i), F_M(v_{i-1}, v_i)\big) > 0$ for $0 \le i \le n$. P is a neutrosophic cycle if $v_0 = v_n$ and $n \ge 3$.

Definition 3. [3] Given $G = (N, M)$ is a single-valued neutrosophic graph, and $v \in V$ is vertex of G. the degree of the vertex v is the sum of the truth membership values, the sum of the indeterminacy membership values, and the sum of the falsity membership values of all the edges that are adjacent to vertex v, and is denoted by $d(v)$, that

$$d(v) = \big(d_T(v), d_I(v), d_F(v)\big) = \left(\sum_{\substack{v \in V \\ v \ne u}} T_M(v,u), \sum_{\substack{v \in V \\ v \ne u}} I_M(v,u), \sum_{\substack{v \in V \\ v \ne u}} F_M(v,u)\right).$$

$$Td(v) = \left(Td_T(v), Td_I(v), Td_F(v)\right)$$
$$= \left(\sum_{\substack{v\in V\\ v\neq u}} T_M(v,u) + T_N(v), \sum_{\substack{v\in V\\ v\neq u}} I_M(v,u) + I_N(v), \sum_{\substack{v\in V\\ v\neq u}} F_M(v,u) + F_N(v)\right).$$

Definition 4. [3] Given $G = (N, M)$ is a single-valued neutrosophic graph, and the d_m–degree of any vertex v in G is denoted as $d_m(v)$ where

$$d_m(v) = \left(\sum_{u\neq v\in V} T_M^m(u,v), \sum_{u\neq v\in V} I_M^m(u,v), \sum_{u\neq v\in V} F_M^m(u,v)\right)$$

Here, the path $v = v_0, v_1, v_2, \dots, v_n = u$ is the shortest path between the vertices v and u, when the length of this path is m.

Definition 5. [3] Given $G = (N, M)$ is a single-valued neutrosophic graph, G is called a regular neutrosophic graph if any vertex has the same degree, that is, for all u in $V(G)$ we have $d_G(u) = (d_1, d_2, d_3)$.

Definition 6. [3] Given $G = (N, M)$ is a single-valued neutrosophic graph, G is called a totally regular neutrosophic graph if any vertex has the same degree, that is, for all u in $V(G)$ we have $Td_G(u) = (d_1, d_2, d_3)$.

Definition 7. [3] Given $G = (N, M)$ is a single-valued neutrosophic graph, G is a complement regular neutrosophic graph if it satisfies the following,

$$\sum_{v\neq u} T_M(v,u) = c, \qquad \sum_{v\neq u} I_M(v,u) = c, \qquad \sum_{v\neq u} F_M(v,u) = c,$$

Where c is a constant value.

Definition 8. [3] A neutrosophic graph $G = (N, M)$ is called a *complete neutrosophic graph* if the following conditions are satisfied:

$$T_M(uv) = \min\{T_N(u), T_N(v)\},$$
$$I_M(uv) = \min\{I_N(u), I_N(v)\},$$
$$F_M(uv) = \max\{F_N(u), F_N(v)\},$$

For all $u, v \in V$.

3. Mane results

This section provides some definitions and theorems.

Definition 9. let $G = (N, M)$ be a neutrosophic graph. the Sombor index (SO) of the graph G is defined by,

$$So_T(G) = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)d_{2_T}(u) + T_N^2(v)d_{2_T}(v))^{\frac{1}{2}},$$
$$So_I(G) = \frac{1}{2}\sum_{uv\in E(G)} (I_N^2(u)d_{2_I}(u) + I_N^2(v)d_{2_I}(v))^{\frac{1}{2}},$$
$$So_F(G) = \frac{1}{2}\sum_{uv\in E(G)} (F_N^2(u)d_{2_F}(u) + F_N^2(v)d_{2_F}(v))^{\frac{1}{2}}.$$

$$So(G) = \big(So_T(G), So_I(G), So_F(G)\big), \qquad |So(G)| = \frac{4 + 2So_T(G) - So_I(G) - 2So_F(G)}{6}.$$

Too, we have:

$$d_2(u) = \Big(d_{2_T}(u), d_{2_I}(u), d_{2_F}(u)\Big) = \left(\sum_{u\neq v\in V} T_M^2(u,v)\,, \sum_{u\neq v\in V} I_M^2(u,v)\,, \sum_{u\neq v\in V} F_M^2(u,v)\right).$$

Definition 10. Sombor index whit the totally degree of the graph G is defined as:

$$So_T(G_T) = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)Td_{2_T}(u) + T_N^2(v)Td_{2_T}(v))^{\frac{1}{2}},$$
$$So_I(G_T) = \frac{1}{2}\sum_{uv\in E(G)} (I_N^2(u)Td_{2_I}(u) + I_N^2(v)Td_{2_I}(v))^{\frac{1}{2}},$$
$$So_F(G_T) = \frac{1}{2}\sum_{uv\in E(G)} (F_N^2(u)Td_{2_F}(u) + F_N^2(v)Td_{2_F}(v))^{\frac{1}{2}}.$$

$$So(G_T) = \big(So_T(G_T), So_I(G_T), So_F(G_T)\big), \qquad |So(G_T)| = \frac{4 + 2So_T(G_T) - So_I(G_T) - 2So_F(G_T)}{6}.$$

Too, we have:

$$Td_2(u) = \Big(Td_{2_T}(u), Td_{2_I}(u), Td_{2_F}(u)\Big)$$
$$= \left(\sum_{u\neq v\in V} T_M^2(u,v) + T_N^2(u)\,, \sum_{u\neq v\in V} I_M^2(u,v) + I_N^2(u)\,, \sum_{u\neq v\in V} F_M^2(u,v) + F_N^2(u)\right).$$

Definition 11. The Reduced Sombor index of the graph G is defined as:

$$So_{RedT}(G) = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)d_{2Red_T}(u) + T_N^2(v)d_{2Red_T}(v))^{\frac{1}{2}},$$
$$So_{RedI}(G) = \frac{1}{2}\sum_{uv\in E(G)} (I_N^2(u)d_{2Red_I}(u) + I_N^2(v)d_{2Red_I}(v))^{\frac{1}{2}},$$
$$So_{RedF}(G) = \frac{1}{2}\sum_{uv\in E(G)} (F_N^2(u)d_{2Red_F}(u) + F_N^2(v)d_{2Red_F}(v))^{\frac{1}{2}}.$$

$$So_{Red}(G) = \big(So_{RedT}(G), So_{RedI}(G), So_{RedF}(G)\big),$$

$$|So_{Red}(G)| = \frac{4 + 2So_{RedT}(G) - So_{RedI}(G) - 2So_{RedF}(G)}{6}.$$

Too, we have:

$$d_{2Red}(u) = \Big(d_{2Red_T}(u), d_{2Red_I}(u), d_{2Red_F}(u)\Big)$$
$$= \left(\left|\sum_{u\neq v\in V} T_M^2(u,v) - T_N^2(u)\right|, \left|\sum_{u\neq v\in V} I_M^2(u,v) - I_N^2(u)\right|, \left|\sum_{u\neq v\in V} F_M^2(u,v) - F_N^2(u)\right|\right).$$

Definition 12. The Average Sombor index of the graph G is defined as:

$$So_{AvgT}(G) = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)d_{2Avg_T}(u) + T_N^2(v)d_{2Avg_T}(v))^{\frac{1}{2}},$$
$$So_{AvgI}(G) = \frac{1}{2}\sum_{uv\in E(G)} (I_N^2(u)d_{2Avg_I}(u) + I_N^2(v)d_{2Avg_I}(v))^{\frac{1}{2}},$$

$$So_{AvgF}(G) = \frac{1}{2}\sum_{uv\in E(G)} (F_N^2(u)d_{2Avg_F}(u) + F_N^2(v)d_{2Avg_F}(v))^{\frac{1}{2}}.$$

$$So_{Avg}(G) = \left(So_{AvgT}(G), So_{AvgI}(G), So_{AvgF}(G)\right),$$

$$\left|So_{Avg}(G)\right| = \frac{4 + 2So_{AvgT}(G) - So_{AvgI}(G) - 2So_{AvgF}(G)}{6}.$$

Too, we have:

$$\begin{aligned} d_{2Avg}(u) &= \left(d_{2Avg_T}(u), d_{2Avg_I}(u), d_{2Avg_F}(u)\right) \\ &= \left(\left|\sum_{u\neq v\in V} T_M^2(u,v) - \frac{2\sum_{uv\epsilon E(G)} T_M(uv)}{\sum_{u\epsilon V(G)} T_N(u)}\right|, \left|\sum_{u\neq v\in V} I_M^2(u,v)\right. \right. \\ &\quad \left. \left. - \frac{2\sum_{uv\epsilon E(G)} I_M(uv)}{\sum_{u\epsilon V(G)} I_N(u)}\right|, \left|\sum_{u\neq v\in V} F_M^2(u,v) - \frac{2\sum_{uv\epsilon E(G)} F_M(uv)}{\sum_{u\epsilon V(G)} F_N(u)}\right|\right). \end{aligned}$$

Example 1. Let $G = (N, M)$ be the neutrosophic graph with $V = \{a, b, c, d, e\}$ where $(T_N, I_N, F_N)(u_1) = (0.1, 0.2, 0.5)$, $(T_N, I_N, F_N)(u_2) = (0.2, 0.7, 0.3)$, $(T_N, I_N, F_N)(u_3) = (0.4, 0.3, 0.7)$, $(T_N, I_N, F_N)(u_4) = (0.5, 0.2, 0.4)$, and $(T_N, I_N, F_N)(u_5) = (0.4, 0.5, 0.3)$, The edge set contains $(T_M, I_M, F_M)(u_1, u_2) = (0.1, 0.2, 0.5)$, $(T_M, I_M, F_M)(u_3, u_2) = (0.1, 0.2, 0.6)$, $(T_M, I_M, F_M)(u_3, u_4) = (02, 0.3, 0.8)$, and $(T_M, I_M, F_M)(u_5, u_2) = (02, 0.5, 0.3)$. (Figure 1)

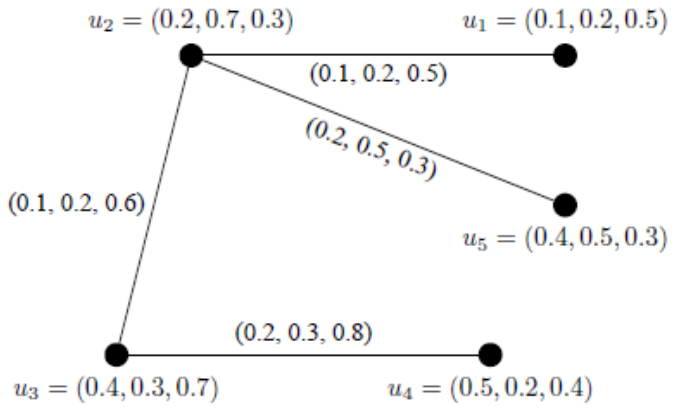

Fig1. $G = (N, M)$ be the neutrosophic graph with $V = \{a, b, c, d, e\}$.

By direct calculations,

$$d(u_1) = (0.1, 0.2, 0.5),$$
$$d(u_2) = (0.1 + 0.2 + 0.1, 0.2 + 0.2 + 0.5, 0.5 + 0.3 + 0.6) = (0.4, 0.9, 1.4),$$
$$d(u_3) = (0.1 + 0.2, 0.2 + 0.3, 0.6 + 0.8) = (0.3, 0.5, 1.4),$$
$$d(u_4) = (0.2, 0.3, 0.8),$$
$$d(u_5) = (0.2, 0.5, 0.3).$$

By definition of $d_m(v)$, we have

$$d_2(v) = \left(\sum_{u\neq v\in V} T_M^2(u,v), \sum_{u\neq v\in V} I_M^2(u,v), \sum_{u\neq v\in V} F_M^2(u,v)\right)$$

Therefore

$$d_2(u_1) = (0.01, 0.04, 0.05), \quad d_2(u_2) = (0.06, 0.33, 0.5), \quad d_2(u_3) = (0.05, 0.13, 1),$$
$$d_2(u_4) = (0.04, 0.09, 0.64), \quad d_2(u_5) = (0.04, 0.25, 0.09).$$

Also, for $u_iu_j \in E(G)$,

$$So_T(G) = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)d_{2_T}(u) + T_N^2(v)d_{2_T}(v))^{\frac{1}{2}} = 0.148,$$
$$So_I(G) = \frac{1}{2}\sum_{uv\in E(G)} (I_N^2(u)d_{2_I}(u) + I_N^2(v)d_{2_I}(v))^{\frac{1}{2}} = 0.771,$$
$$So_F(G) = \frac{1}{2}\sum_{uv\in E(G)} (F_N^2(u)d_{2_F}(u) + F_N^2(v)d_{2_F}(v))^{\frac{1}{2}} = 0.987,$$
$$So(G) = \big(So_T(G), So_I(G), So_F(G)\big) = (0.148, 0.771, 0.987).$$

$$|So(G)| = \frac{4 + 2So_T(G) - So_I(G) - 2So_F(G)}{6} = \frac{4 + 2(0.148) - 0.771 - 2(0.987)}{6} = 0.2585.$$

Theorem 1: let G be the Neutrosophic Graph, and H is the Neutrosophic subgraph of G such that $H = G - u$ then

$$SO(H) < SO(G).$$

Proof: Given that by omitting a vertex of G, a positive value, the sum is lost, so the proof is obvious. □

Theorem 2. Let $G_1 = (N_1, M_1)$ be isomorphic with $G_2 = (N_2, M_2)$. Then all of the following equations are established.

$$So_T(G_1) = So_T(G_2),$$
$$So_I(G_1) = So_I(G_2),$$
$$So_F(G_1) = So_F(G_2),$$

Also, we have $So(G_1) = So(G_2)$.

Proof. Let $G_1 = (N_1, M_1)$ be isomorphic with $G_2 = (N_2, M_2)$, and $f: V_1 \to V_2$ be the bijection from V_1 to V_2 such that

$$T_{N_1}(u) = T_{N_2}\big(f(u)\big), \quad I_{N_1}(u) = I_{N_2}\big(f(u)\big), \quad F_{N_1}(u) = F_{N_2}\big(f(u)\big),$$

For all $u \in V_1$, and

$$T_{M_1}(uv) = T_{M_2}\big(f(u)f(v)\big), \quad I_{M_1}(uv) = I_{M_2}\big(f(u)f(v)\big), \quad F_{M_1}(uv) = F_{M_2}\big(f(u)f(v)\big),$$

For all $uv \in E_1$. Since G_1 isomorphic with G_2, any edge between two vertices, for example, u and v in G_1 are equal to that between $f(u)$ and $f(v)$ in G_2. Hence

$$d_{2T_{G_1}}(u) = d_{2T_{G_2}}\big(f(u)\big), \quad d_{2I_{G_1}}(u) = d_{2I_{G_2}}\big(f(u)\big), \quad d_{2F_{G_1}}(u) = d_{2F_{G_2}}(f(u)),$$

For $u, v \in N_1^*$. Therefore

$$So_T(G_1) = So_T(G_2), \quad So_I(G_1) = So_I(G_2), \quad So_F(G_1) = So_F(G_2),$$

And

$$So(G_1) = \frac{4 + 2So_T(G_1) - 2So_F(G_1) - So_I(G_1)}{6} = \frac{4 + 2So_T(G_2) - 2So_F(G_2) - So_I(G_2)}{6} = So(G_2).$$

□

Theorem 3. Let $G=(N,M)$ be a complete neutrosophic graph whit $V=\{v_1,v_2,\dots,v_n\}$ such that $t_1\le t_2\le\cdots\le t_n$, $i_1\le i_2\le\cdots\le i_n$ and $f_1\ge f_2\ge\cdots\ge f_n$ where $t_j=T_N(v_j)$, $i_j=I_N(v_j)$ and $f_j=F_N(v_j)$ for $j=1,2,\dots,n$. Also, G is the regular neutrosophic graph of the second rank (that is, for each vertex u in $V(G), d_{2G}(v_j)=(d_1,d_2,d_3)$). Then

$$So_T(G)=\frac{\sqrt{d_1}}{2}\sum_{\substack{1\le j<k\\2\le k\le n}}(t_j^2+t_k^2)^{\frac{1}{2}},$$
$$So_I(G)=\frac{\sqrt{d_2}}{2}\sum_{\substack{1\le j<k\\2\le k\le n}}(i_j^2+i_k^2)^{\frac{1}{2}},$$
$$So_F(G)=\frac{\sqrt{d_3}}{2}\sum_{\substack{1\le j<k\\2\le k\le n}}(f_j^2+f_k^2)^{\frac{1}{2}}.$$

Too, we have for u in $V(G)$:

$$d_2(v_j)=\left(d_{2_T}(v_j),d_{2_I}(v_j),d_{2_F}(v_j)\right)=(d_1,d_2,d_3).$$

Proof: Suppose G is a complete neutrosophic graph, we have

$$So_T(G)=\frac{1}{2}\sum_{uv\in E(G)}(T_N^2(u)d_{2_T}(u)+T_N^2(v)d_{2_T}(v))^{\frac{1}{2}}=\frac{1}{2}\sum_{uv\in E(G)}(T_N^2(u)d_1+T_N^2(v)d_1)^{\frac{1}{2}}$$
$$=\frac{1}{2}\sum_{uv\in E(G)}\sqrt{d_1}\left(T_N^2(u)+T_N^2(v)\right)^{\frac{1}{2}}=\frac{\sqrt{d_1}}{2}\sum_{uv\in E(G)}\left(T_N^2(u)+T_N^2(v)\right)^{\frac{1}{2}}$$
$$=\frac{\sqrt{d_1}}{2}\left(\sqrt{t_1^2+t_2^2}+\sqrt{t_1^2+t_3^2}+\cdots+\sqrt{t_1^2+t_n^2}+\sqrt{t_2^2+t_3^2}+\cdots+\sqrt{t_2^2+t_n^2}+\cdots\right.$$
$$\left.+\sqrt{t_{n-2}^2+t_{n-1}^2}+\sqrt{t_{n-2}^2+t_n^2}+\sqrt{t_{n-1}^2+t_n^2}\right)=\frac{\sqrt{d_1}}{2}\sum_{\substack{1\le j<k\\2\le k\le n}}(t_j^2+t_k^2)^{\frac{1}{2}}.$$

Using the same argument, we can prove the other two cases.

□

Theorem 4. Let $G=(N,\ M)$ be a complete neutrosophic graph whit $|V|=n$. Then,

If M is a constant function, that is, $M(uv)=(t_m,\ i_m,\ f_m), for\ uv\ \in E(G)$. Now, we have:

$$So_T(G)=\frac{\sqrt{n-1}}{2}t_m\sum_{uv\in E(G)}(T_N^2(u)+T_N^2(v))^{\frac{1}{2}},$$
$$So_I(G)=\frac{\sqrt{n-1}}{2}i_m\sum_{uv\in E(G)}(I_N^2(u)+I_N^2(v))^{\frac{1}{2}},$$
$$So_F(G)=\frac{\sqrt{n-1}}{2}f_m\sum_{uv\in E(G)}(F_N^2(u)+F_N^2(v))^{\frac{1}{2}}.$$

If $M\ and\ N$ are the constant function, that is, $M(uv)=(t_m,\ i_m,\ f_m), for\ uv\in E(G),\ and\ N(u)=(t_n,i_n,f_n), for\ u\in V(G)$. Now, we have:

$$So_T(G)=\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}}t_m.t_n,$$
$$So_I(G)=\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}}i_m.i_n,$$

$$So_F(G) = \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} f_m . f_n .$$

Also,

$$|So(G)| = \frac{n(n-1)^{\frac{3}{2}}}{12\sqrt{2}} (2t_m . t_n - i_m . i_n - 2f_m . f_n) + \frac{2}{3}.$$

If M *and* N are the constant and the same function, that is, $M(uv) = (t, i, f), for\ uv \in E(G)$, $and\ N(u) = (t, i, f), for\ u \in V(G)$. Now, we have:

$$So_T(G) = \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} t^2,$$

$$So_I(G) = \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} i^2,$$

$$So_F(G) = \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} f^2.$$

Also,

$$|So(G)| = \frac{n(n-1)^{\frac{3}{2}}}{12\sqrt{2}} (2t^2 - i^2 - 2f^2) + \frac{2}{3}.$$

Proof: In the first case, suppose G is a complete neutrosophic graph, with $|V| = n$. Also, $uv \in E(G)$, we have $M(uv) = (t_m,\ i_m,\ f_m)$. Therefore,

$$d_2(u) = \left(d_{2_T}(u), d_{2_I}(u), d_{2_F}(u)\right) = \left(\sum_{u \neq v \in V} T_M^2(u, v), \sum_{u \neq v \in V} I_M^2(u, v), \sum_{u \neq v \in V} F_M^2(u, v)\right)$$
$$= \left((n-1)t_m^2, (n-1)i_m^2, (n-1)f_m^2\right).$$

Now, by placing $d_2(u)$ in relation to the Sombor index we get:

$$So_T(G) = \frac{1}{2} \sum_{uv \in E(G)} (T_N^2(u) d_{2_T}(u) + T_N^2(v) d_{2_T}(v))^{\frac{1}{2}} = \frac{1}{2} \sum_{uv \in E(G)} (T_N^2(u)(n-1)t_m^2 + T_N^2(v)(n-1)t_m^2)^{\frac{1}{2}}$$
$$= \frac{1}{2} \sum_{uv \in E(G)} ((n-1)t_m^2)^{\frac{1}{2}} (T_N^2(u) + T_N^2(v))^{\frac{1}{2}} = \frac{1}{2} ((n-1)t_m^2)^{\frac{1}{2}} \sum_{uv \in E(G)} (T_N^2(u) + T_N^2(v))^{\frac{1}{2}}$$
$$= \frac{\sqrt{n-1}}{2} t_m \sum_{uv \in E(G)} (T_N^2(u) + T_N^2(v))^{\frac{1}{2}}.$$

In the second case, consider $M\ and\ N$ are the constant functions, such that, $M(uv) = (t_m,\ i_m,\ f_m), and\ N(u) = (t_n, i_n, f_n)$. In this state, we have for $u \in V(G)$,

$$d_2(u) = \left(d_{2_T}(u), d_{2_I}(u), d_{2_F}(u)\right) = \left(\sum_{u \neq v \in V} T_M^2(u, v), \sum_{u \neq v \in V} I_M^2(u, v), \sum_{u \neq v \in V} F_M^2(u, v)\right)$$
$$= \left((n-1)t_m^2, (n-1)i_m^2, (n-1)f_m^2\right).$$

by placing $d_2(u)$,

$$So_T(G) = \frac{1}{2} \sum_{uv \in E(G)} (T_N^2(u) d_{2_T}(u) + T_N^2(v) d_{2_T}(v))^{\frac{1}{2}} = \frac{1}{2} \sum_{uv \in E(G)} (T_N^2(u)(n-1)t_m^2 + T_N^2(v)(n-1)t_m^2)^{\frac{1}{2}}$$
$$= \frac{1}{2} \sum_{uv \in E(G)} (t_n^2(n-1)t_m^2 + t_n^2(n-1)t_m^2)^{\frac{1}{2}} = \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} t_m . t_n .$$

The same can be proved for $So_I(G)$ and $So_F(G)$. Also, by direct placing in $|So(G)|$,

$$|So(G)| = \frac{4 + 2So_T(G) - So_I(G) - 2So_F(G)}{6}$$
$$= \frac{4 + 2\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} t_m.t_n - \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} i_m.i_n - 2\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} f_m.f_n}{6}$$
$$= \frac{n(n-1)^{\frac{3}{2}}}{12\sqrt{2}}(2t_m.t_n - i_m.i_n - 2f_m.f_n) + \frac{2}{3}.$$

To prove the third case, it is enough to consider $d_2(u)$ like the previous parts and then replace the value of $M(uv) = (t, i, f), and\ N(u) = (t, i, f)$. The proofs for $So_I(G)$ and $So_F(G)$ are similar to the case for $So_T(G)$.

□

Theorem 5. Let $G = (N, M)$ be a complete neutrosophic graph whit $|V| = n$. Then,

If $M\ and\ N$ are the constant functions, that is, $M(uv) = (t_m,\ i_m,\ f_m), for\ uv\ \in E(G), and\ N(u) = (t_n, i_n, f_n), for\ u \in V(G)$. Now, we have:

$$So_T(G_T) = \frac{n(n-1)}{2\sqrt{2}} t_n((n-1)t_m^2 + t_n^2)^{1/2},$$
$$So_I(G_T) = \frac{n(n-1)}{2\sqrt{2}} i_n((n-1)i_m^2 + i_n^2)^{1/2},$$
$$So_F(G_T) = \frac{n(n-1)}{2\sqrt{2}} f_n((n-1)f_m^2 + f_n^2)^{1/2}.$$

Also,

$$|So(G_T)| = \frac{n(n-1)}{12\sqrt{2}}\left(2t_n((n-1)t_m^2 + t_n^2)^{1/2} - i_n((n-1)i_m^2 + i_n^2)^{1/2} - 2f_n((n-1)f_m^2 + f_n^2)^{1/2}\right) + \frac{2}{3}.$$

If $M\ and\ N$ are the constant and the same functions, that is, $M(uv) = (t, i, f), for\ uv\ \in E(G), and\ N(u) = (t, i, f), for\ u \in V(G)$. Now, we have:

$$So_T(G_T) = \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} t^2,$$
$$So_I(G_T) = \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} i^2,$$
$$So_F(G_T) = \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} f^2.$$

Also,

$$|So(G_T)| = \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}}(2t^2 - i^2 - 2f^2) + \frac{2}{3}.$$

Proof: Suppose G is a complete neutrosophic graph, with $|V| = n$. Also, $uv\ \in E(G)$. The first case, we have $M(uv) = (t_m,\ i_m,\ f_m), for\ uv\ \in E(G), and\ N(u) = (t_n, i_n, f_n), for\ u \in V(G)$. Therefore,

$$Td_2(u) = \left(Td_{2_T}(u), Td_{2_I}(u), Td_{2_F}(u)\right)$$
$$= \left(\sum_{u \neq v \in V} T_M^2(u,v) + T_N^2(u)\,,\ \sum_{u \neq v \in V} I_M^2(u,v) + I_N^2(u)\,,\ \sum_{u \neq v \in V} F_M^2(u,v) + F_N^2(u)\right).$$
$$= \left((n-1)t_m^2 + t_n^2, (n-1)i_m^2 + i_n^2, (n-1)f_m^2 + f_n^2\right).$$

Now, by placing $Td_2(u)$ in relation to the Sombor index we get:

$$So_T(G_T) = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)Td_{2_T}(u) + T_N^2(v)Td_{2_T}(v))^{\frac{1}{2}}$$
$$= \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)((n-1)t_m^2 + t_n^2) + T_N^2(v)((n-1)t_m^2 + t_n^2))^{\frac{1}{2}}$$
$$= \frac{1}{2}\sum_{uv\in E(G)} (((n-1)t_m^2 + t_n^2)(T_N^2(u) + T_N^2(v)))^{\frac{1}{2}}$$
$$= \frac{1}{2}\sum_{uv\in E(G)} (((n-1)t_m^2 + t_n^2)(t_n^2 + t_n^2))^{\frac{1}{2}} = \frac{n(n-1)}{2\sqrt{2}} t_n((n-1)t_m^2 + t_n^2)^{1/2}.$$

It is similarly possible for the other two modes.

$$|So(G_T)| = \frac{4 + 2So_T(G_T) - So_I(G_T) - 2So_F(G_T)}{6}$$
$$= \frac{4 + 2\frac{n(n-1)}{2\sqrt{2}} t_n((n-1)t_m^2 + t_n^2)^{1/2} - \frac{n(n-1)}{2\sqrt{2}} i_n((n-1)i_m^2 + i_n^2)^{1/2} - 2\frac{n(n-1)}{2\sqrt{2}} f_n((n-1)f_m^2 + f_n^2)^{1/2}}{6}$$
$$= \frac{4 + \frac{n(n-1)}{2\sqrt{2}}(2t_n\left((n-1)t_m^2 + t_n^2\right)^{1/2} - i_n\left((n-1)i_m^2 + i_n^2\right)^{1/2} - 2f_n\left((n-1)f_m^2 + f_n^2\right)^{1/2})}{6}$$
$$= \frac{n(n-1)}{12\sqrt{2}}\left(2t_n((n-1)t_m^2 + t_n^2)^{1/2} - i_n((n-1)i_m^2 + i_n^2)^{1/2} - 2f_n((n-1)f_m^2 + f_n^2)^{1/2}\right) + \frac{2}{3}.$$

For case 2: we have $M(uv) = (t, i, f), for\ uv \in E(G), and\ N(u) = (t, i, f), for\ u \in V(G),$ henc:

$$Td_2(u) = \left(Td_{2_T}(u), Td_{2_I}(u), Td_{2_F}(u)\right)$$
$$= \left(\sum_{u\neq v\in V} T_M^2(u,v) + T_N^2(u)\,,\ \sum_{u\neq v\in V} I_M^2(u,v) + I_N^2(u)\,,\ \sum_{u\neq v\in V} F_M^2(u,v) + F_N^2(u)\right).$$
$$= \left((n-1)t^2 + t^2, (n-1)i^2 + i^2, (n-1)f^2 + f^2\right) = (nt^2, ni^2, nf^2).$$

$$So_T(G_T) = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)Td_{2_T}(u) + T_N^2(v)Td_{2_T}(v))^{\frac{1}{2}} = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)(nt^2) + T_N^2(v)(nt^2))^{\frac{1}{2}}$$
$$= \frac{1}{2}\sum_{uv\in E(G)} ((nt^2)(t^2 + t^2))^{\frac{1}{2}} = \frac{1}{2}(nt^2)^{\frac{1}{2}}\sum_{uv\in E(G)} (t^2 + t^2)^{\frac{1}{2}} = \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} t^2.$$

It is similarly proved for the $So_I(G_T)$ and $So_F(G_T)$. Also,

$$|So(G_T)| = \frac{4 + 2So_T(G_T) - So_I(G_T) - 2So_F(G_T)}{6}$$
$$= \frac{4 + 2\frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} t^2 - \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} i^2 - 2\frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} f^2}{6}$$
$$= \frac{4 + \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}}(2t^2 - i^2 - 2f^2)}{6} = \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}}(2t^2 - i^2 - 2f^2) + \frac{2}{3}.$$

The proof is complete.

□

Conclusion

In this paper, for the first time, Sombor indices for neutrosophic graphs are defined. Here, in addition to defining Sombor indices, we have calculated and studied these indices for complete neutrosophic graphs, which are an important category of neutrosophic graphs. In the following, we will generalize these indicators and in our next articles, we will discuss another group of neutrosophic graphs, called neutrosophic regular edge graphs, and then we will discuss its applications.

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Application of the SuperHyperSoft Set for Comprehensive Teaching Quality Analysis in College English Translation Courses for Enhanced Learning Results

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Abstract: This paper applies the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) and the SIWEC for Teaching Quality Assessment in College English Translation Courses. The impacts of normalization on the SIWEC-based TOPSIS approach are examined using SIWEC. It has been discovered that normalization can influence the solution, which in turn influences the contribution of attributes to the distance of each alternative from the ideal solution and the negative ideal solution. The SIWEC method is used to compute the criteria weights and the TOPSIS method is used to rank the alternatives. SuperHyperSoft set is used to deal with different criteria and sub-criteria. Eight criteria and seven alternatives are used in this study. The results show the stability of the ranks under the SuperHyperSoft Set.

Keywords: SuperHyperSoft Set; MCDM Method; Translation Courses; Teaching Quality; College English.

1. Introduction

To ensure that students acquire the linguistic, technical, and critical thinking abilities needed for professional translation, teaching quality evaluation is essential to assessing and enhancing college English translation programs. Universities must constantly improve their curricula, teaching practices, and evaluation techniques to meet industry demands considering the growing globalization and need for qualified translators. A multifaceted study is required for an effective teaching quality evaluation, considering elements including student involvement, faculty expertise, curriculum design, and technology integration[1], [2]. It is difficult to gauge the effectiveness of translation initiatives and make significant changes in the absence of a well-organized assessment mechanism. For the assessment process to offer a thorough grasp of teaching efficacy, both qualitative and quantitative indicators must be considered. Although they offer insights, traditional evaluation techniques including course completion rates, staff self-evaluations, and student feedback might not adequately convey the complexity of translation education. More sophisticated methods, such data-driven analytics, AI-based evaluation tools,

and Multi-Criteria Decision-Making (MCDM) models, can aid in a more accurate analysis of the efficacy of translation training. Institutions can determine the programs' strengths and shortcomings and modify their teaching strategies to better support students' learning outcomes by using these contemporary assessment techniques. The findings of teaching quality evaluation studies frequently point to important areas that require development, including gaps in the actual application of skills, a lack of exposure to industry-standard translation software, and inadequate practical translation training. Some institutions place more of an emphasis on practical experience, such internships and project-based learning, while others prioritize academic understanding[3], [4]. The effectiveness of translation education is greatly impacted by the incorporation of industry collaborations, AI-powered language processing, and technology-driven translation tools. The findings of the assessment emphasize the necessity of a well-rounded strategy that ensures graduates are prepared for the workforce by combining academic knowledge with real-world practice. In the end, a systematic and data-driven approach to teaching quality evaluation facilitates ongoing curriculum improvement and improves students' entire educational experience. The findings of these evaluations help curriculum designers, educators, and legislators improve corporate partnerships, integrate contemporary teaching approaches, and strengthen faculty training. Institutions may improve the quality of English translation instruction by examining assessment results and creating evidence-based plans that guarantee students get the skills necessary to thrive in a fast-paced, technologically advanced translation sector[5], [6]. People have been involved in making decisions pertaining to their everyday lives ever since the beginning of time. The investigation of how humans do this activity has long piqued the curiosity of academics. In this regard, it is essential to model the environment in which we will be moving, that is, in a way that simplifies (represents) the actual system, if it is easily comprehensible and implementable. Therefore, we examine the available options as well as the standards by which those options will be judged. This, which appears straightforward at first glance, is a component of the entire field known as MCDM. MCDM methodology is applied in different issues [7], [8]. When it comes to decision dilemmas, there are several schools of thinking. Multiple objective programming and multiple criteria evaluation are the two types of MCDM challenges. In our instance, the second group is highlighted to concentrate on the evaluation issues. The two primary schools of multicriteria approaches comprise most of these techniques.

There is a subset of MCDM's several compensating techniques that take costs and benefits into account. Among these is the TOPSIS approach, which stands for Technique for Order Performance by Similarity to Ideal Solution[9], [10].

This strategy is used for four primary reasons:

I. The TOPSIS idea enables the search for the best options for each criterion represented in a basic mathematical form.
II. the computing techniques are simple.
III. the logic is logical and intelligible; and

IV. the importance weights are integrated into the comparison procedures[11], [12].

Nevertheless, there are certain disadvantages to the TOPSIS approach. Ranking alternatives is a phenomenon that may be caused by TOPSIS, which is one of its issues. In this phenomenon, adding or removing one option from the choice issue alters the order of preference for the alternatives. This can sometimes result in what is known as total rank alternatives, in which the order of preferences is completely reversed; that is, the best choice becomes the worst when it is added to or removed from the process[13], [14]. In many situations, such an occurrence might not be acceptable.

2. Framework for TOPSIS and SIWEC Evaluation

Numerous options must be assessed and contrasted with MCDM based on several criteria. Aiding the decision-maker when they are choosing between options is the goal of MCDM. In this sense, practical situations are typically defined by numerous competing requirements, and there may be no solution which fulfills all criteria simultaneously. Considering the decision-maker's choices, the answer is thus a compromise. In this regard, TOPSIS is predicated on the idea that the option that is selected ought to be the one that is most far from the Negative Ideal Solution (NIS) and the closest to the Positive Ideal Solution (PIS). The final ranking is calculated by the proximity index[15], [16]. The following steps make up the TOPSIS and SIWEC [17] techniques.

Forming the initial decision matrix

$$Y = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{m1} & \cdots & y_{mn} \end{bmatrix} \quad (1)$$

Normalize the decision matrix

The decision matrix is normalized as:

$$q_{ij} = \frac{y_{ij}}{\max y_{ij}} \quad (2)$$

Compute the standard deviation

Multiply the normalization values by the standard deviation

$$p_{ij} = q_{ij} \times \sigma_j \quad (3)$$

Where σ_j refers to the standard deviation

Compute the sum of each row in p_{ij}

$$r_{ij} = \sum_{j=1}^{n} p_{ij} \quad (4)$$

Compute the criteria weights.

$$W_j = \frac{r_{ij}}{\sum_{j=1}^{n} r_{ij}} \quad (5)$$

The steps of the TOPSIS method are organized as follows:

Normalize the decision matrix

$$e_{ij} = \frac{y_{ij}}{\sqrt{\sum_{j=1}^{m}(y_{ij})^2}} \tag{6}$$

Calculate the weighted decision matrix.

$$v_{ij} = w_j e_{ij} \tag{7}$$

Calculate the positive ideal and negative ideal solutions

$$A^+ = \{v_1^+, \dots, v_n^+\} = \left\{\left(\max_i v_{ij}, j \in J\right)\left(\min_i v_{ij}, j \in B\right)\right\} \tag{8}$$

$$A^- = \{v_1^-, \dots, v_n^-\} = \left\{\left(\min_i v_{ij}, j \in J\right)\left(\max_i v_{ij}, j \in B\right)\right\} \tag{9}$$

Calculate the separation measures

$$d_i^+ = \left\{\sum_{j=1}^{n}\left(v_{ij} - v_j^+\right)^2\right\}^{\frac{1}{2}} \tag{10}$$

$$d_i^- = \left\{\sum_{j=1}^{n}\left(v_{ij} - v_j^-\right)^2\right\}^{\frac{1}{2}} \tag{11}$$

Calculate the relative closeness to ideal solution

$$H_i = \frac{d_i^-}{d_i^- + d_i^+} \tag{12}$$

SuperHyperSoft Set

Let U be a universe of discourse, and let P(U) presents the power set of U. Let $a_1, \dots a_n$ be n distinct attributes and $n \geq 1$. Every attribute a_i has a corresponding set of attributes values A_i, with the $A_i \cap A_j = \emptyset \; for \; all \; i \neq j$ [18], [19].

Let $P(A_i)$ presents the power set of A_i for every $i = 1,2, \dots, n$.

Then the pair $\left(F, P(A_1) \times \; P(A_2) \times P(A_3) \times P(A_4) \times \dots \times P(A_n)\right)$ (13)

$$F: P(A_1) \times \; P(A_2) \times P(A_3) \times P(A_4) \times \dots \times P(A_n) \to P(U) \tag{14}$$

This is called a SuperHyperSoft Set over U.

3. An application

This section shows an application of the proposed approach to show the validation of the proposed approach. We use eight criteria and seven alternatives in this study. The criteria of this study are organized as: Employment and Certification Success Rate (High, Medium), Student Engagement (High, Low), Practical Training (Extensive, Moderate, Limited), Technological Integration (Advanced, Moderate, Basic), Industry Collaboration (Strong, Moderate, Weak),

Curriculum Design (Excellent, Good, Average, Poor), Assessment Methods (Diverse, Standard, Limited), Faculty Expertise (Highly Skilled, Skilled, Basic Knowledge). Alternatives are seven colleges.

We are forming the initial decision matrix using Eq. (1). Four experts rate the criteria and alternatives based on scale between 0.1 to 0.99. They build the decision matrix based on eight criteria and seven alternatives.

Eq. (2) is used to normalize the decision matrix as shown in Fig 1.

We compute the standard deviation.

Eq. (3) is used to multiply the normalization values by the standard deviation as shown in Fig 2.

Eq. (4) is used to compute the sum of each row in p_{ij}.

Eq. (5) is used to compute the criteria weights as shown in Fig 3.

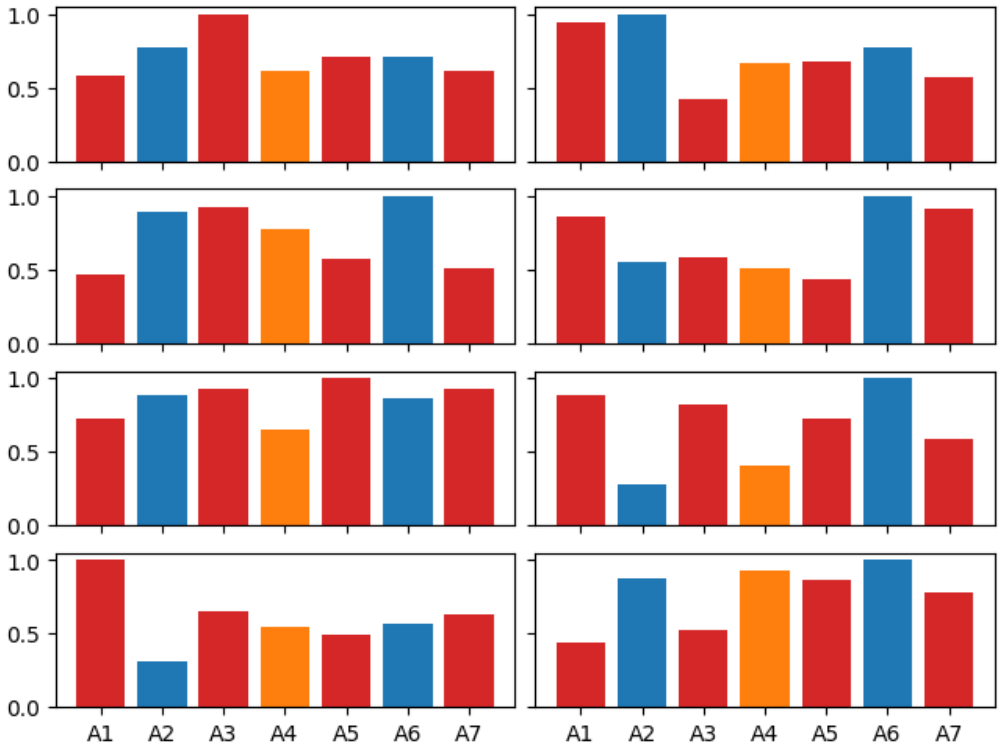

Fig 1. Normalized matrix by the weighting approach.

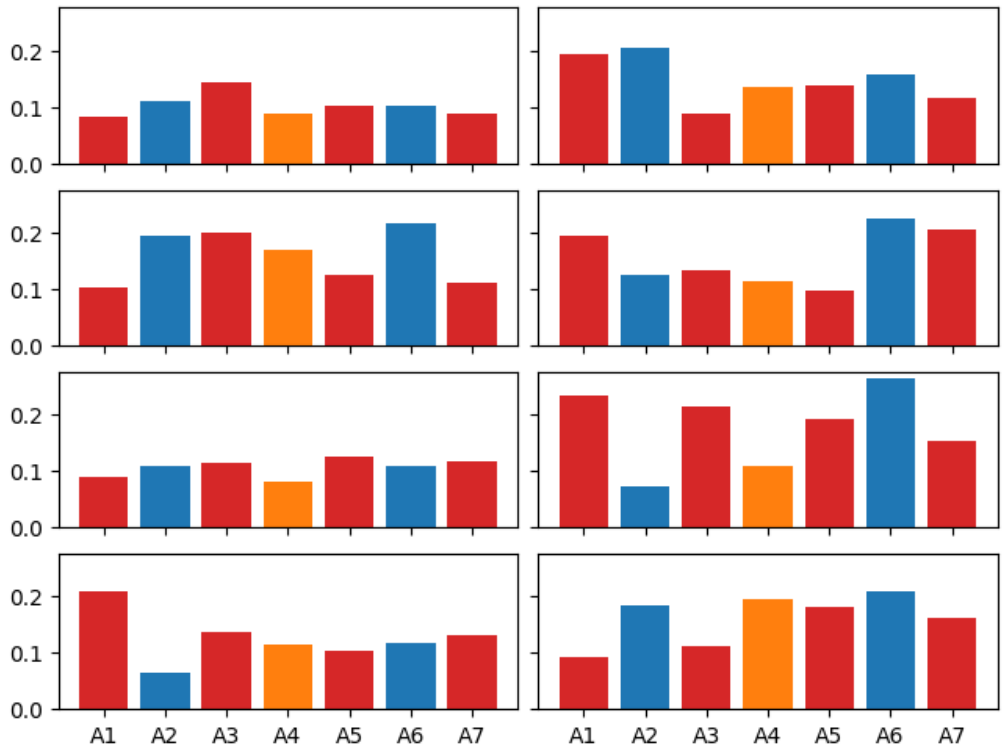

Fig 2. The normalization values by the standard deviation.

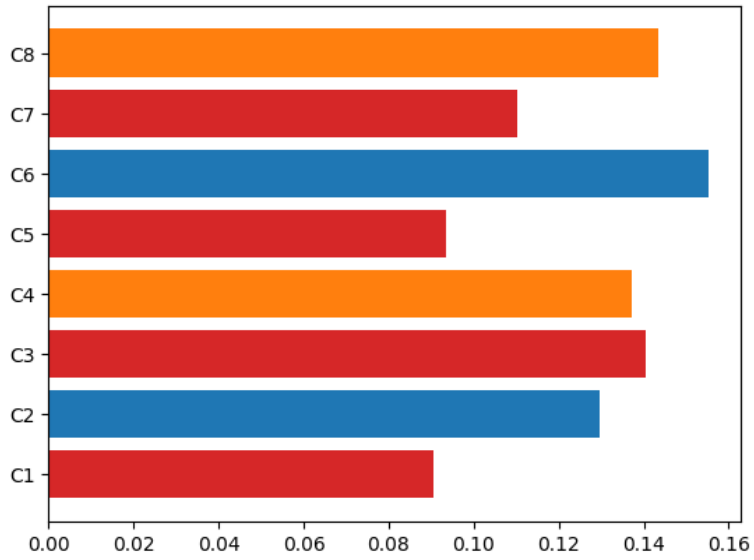

Fig 3. The importance of criteria.

Then we ranked the alternatives based on the TOPSIS method. This study uses the SuperHyperSoft set to deal with the criteria and sub criteria. We select the values of the criteria such as:

{(High, Medium), (High, Low) (Extensive), (Advanced), (Strong), (Good), (Standard), (Skilled)}

So, we propose four cases, each case has a set of values such as:

Case 1: {(High), (High) (Extensive), (Advanced), (Strong), (Good), (Standard), (Skilled)}

Case 2: {(High), (Low) (Extensive), (Advanced), (Strong), (Good), (Standard), (Skilled)}

Case 3: {(Medium), (High) (Extensive), (Advanced), (Strong), (Good), (Standard), (Skilled)}

Case 4: {(Medium), (Low) (Extensive), (Advanced), (Strong), (Good), (Standard), (Skilled)}

Then we ranked the alternatives based on these four cases and finally, we compute the final rank of the alternatives and select the best and the worst.

In Case 1.

Eq. (6) is used to normalize the decision matrix as shown in Fig 4.

Eq. (7) is used to calculate the weighted decision matrix as shown in Fig 5.

Eq. (8 and 9) are used to calculate the positive ideal and negative ideal solutions.

Eq. (10 and 11) are used to calculate the separation measures.

Eq. (12) is used to calculate relative closeness to ideal solution.

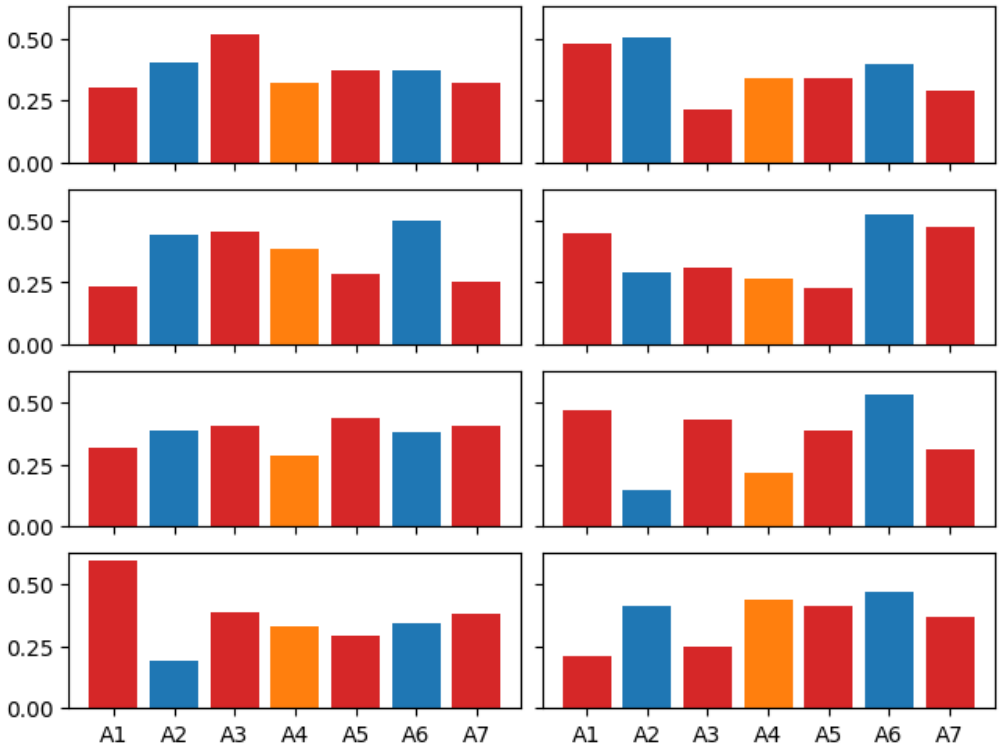

Fig 4. Normalized decision matrix.

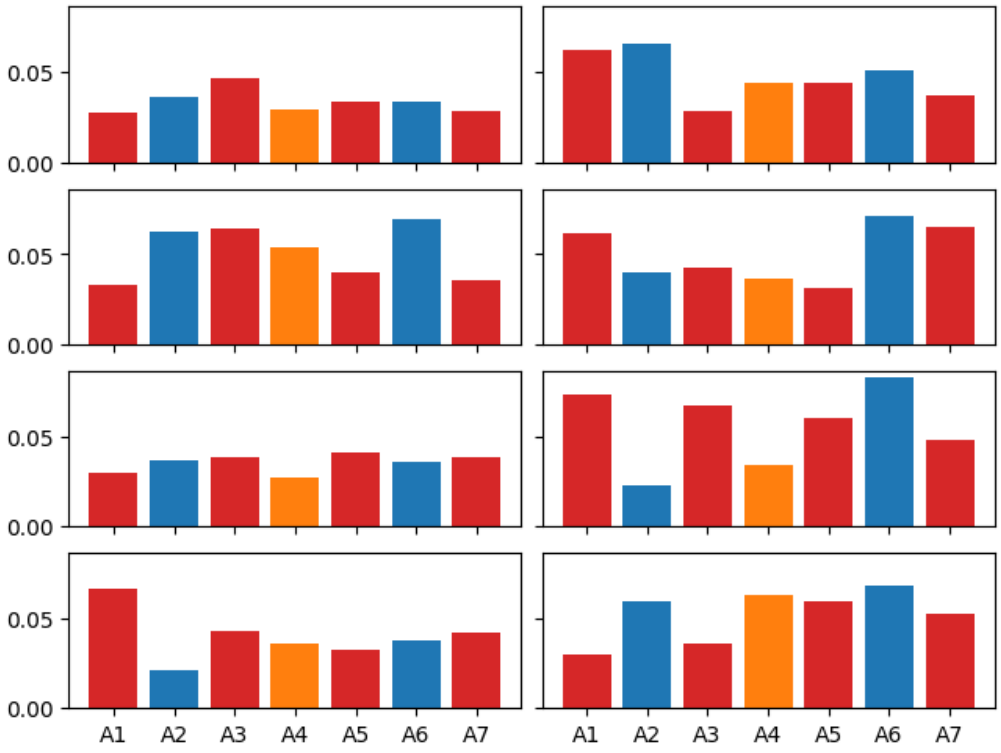

Fig 5. The weighted decision matrix.

In Case 2

We obtain the normalized decision matrix as shown in Fig 6.

The weighted decision matrix as shown in Fig 7.

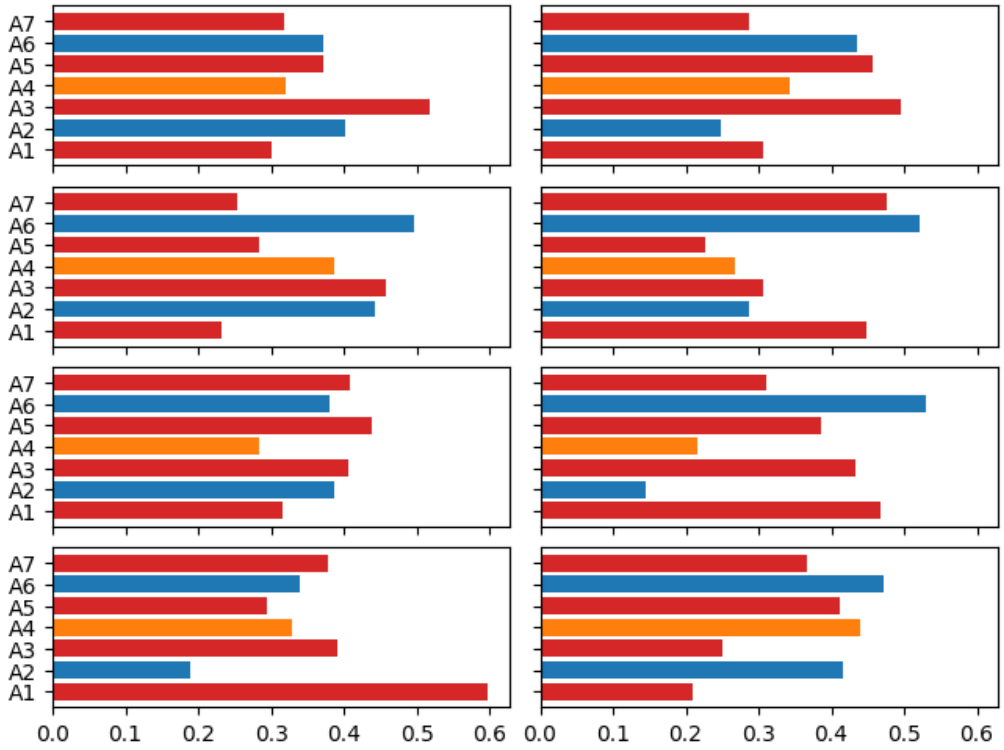

Fig 6. Normalized decision matrix.

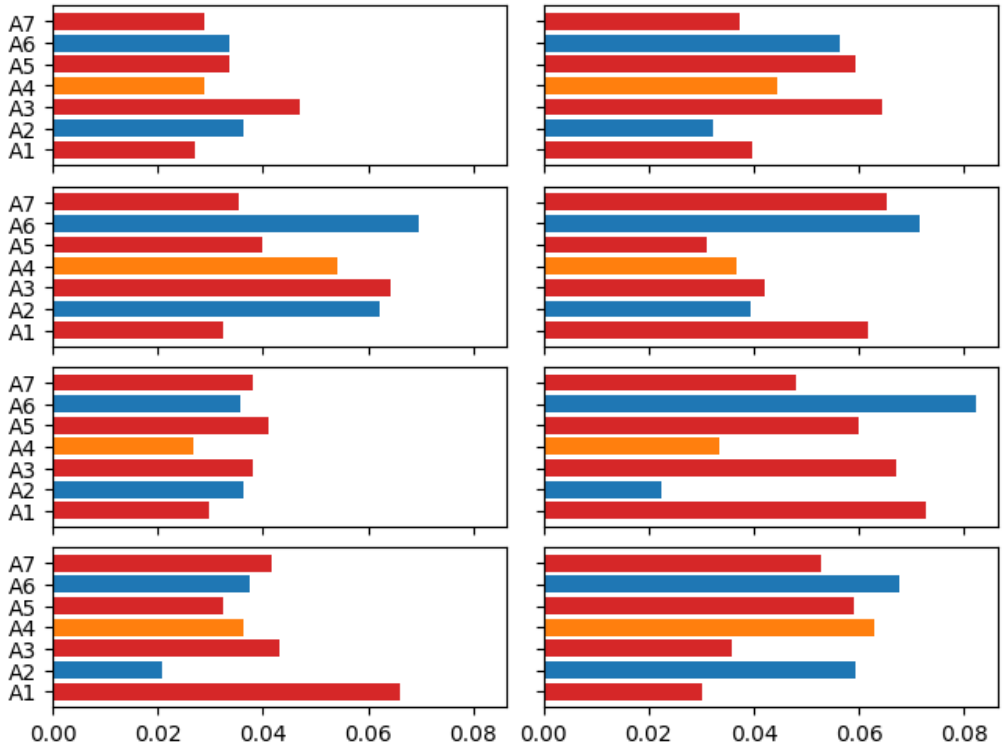

Fig 7. The weighted decision matrix.

In Case 3

We obtain the normalized decision matrix as shown in Fig 8.

The weighted decision matrix as shown in Fig 9.

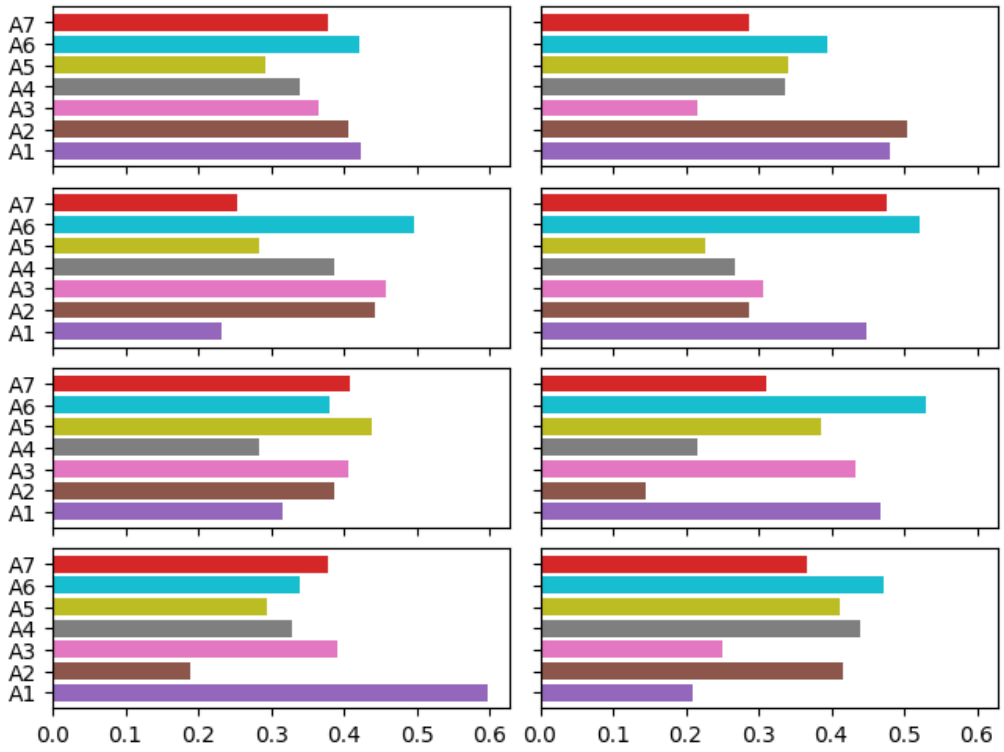

Fig 8. Normalized decision matrix.

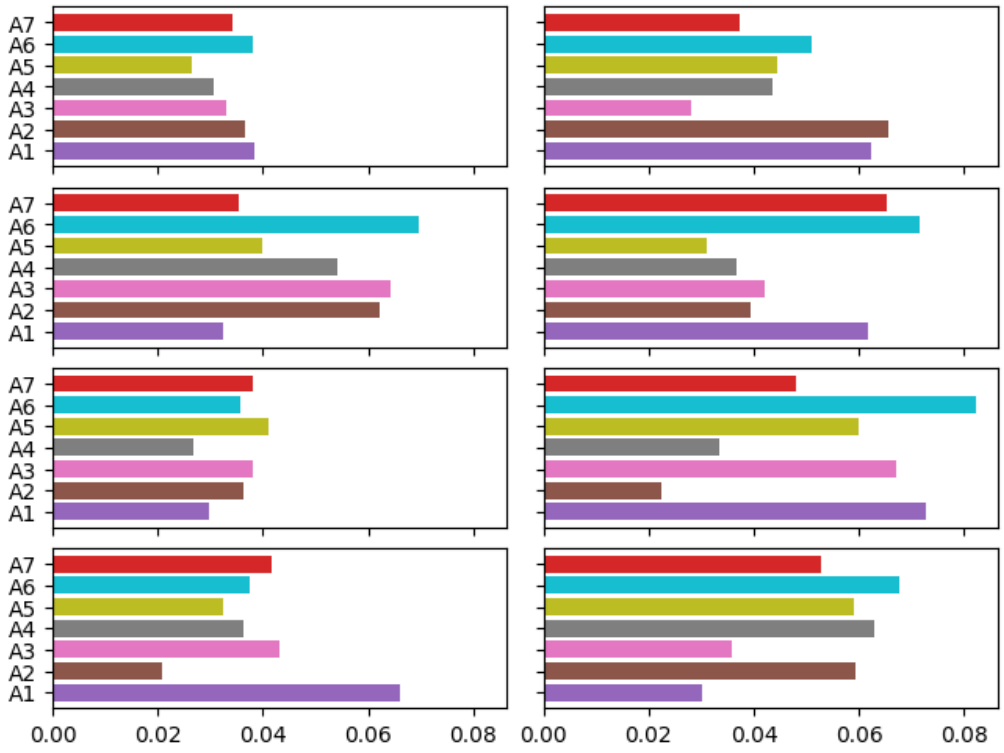

Fig 9. The weighted decision matrix.

In Case 4

We obtain the normalized decision matrix as shown in Fig 10.

The weighted decision matrix as shown in Fig 11.

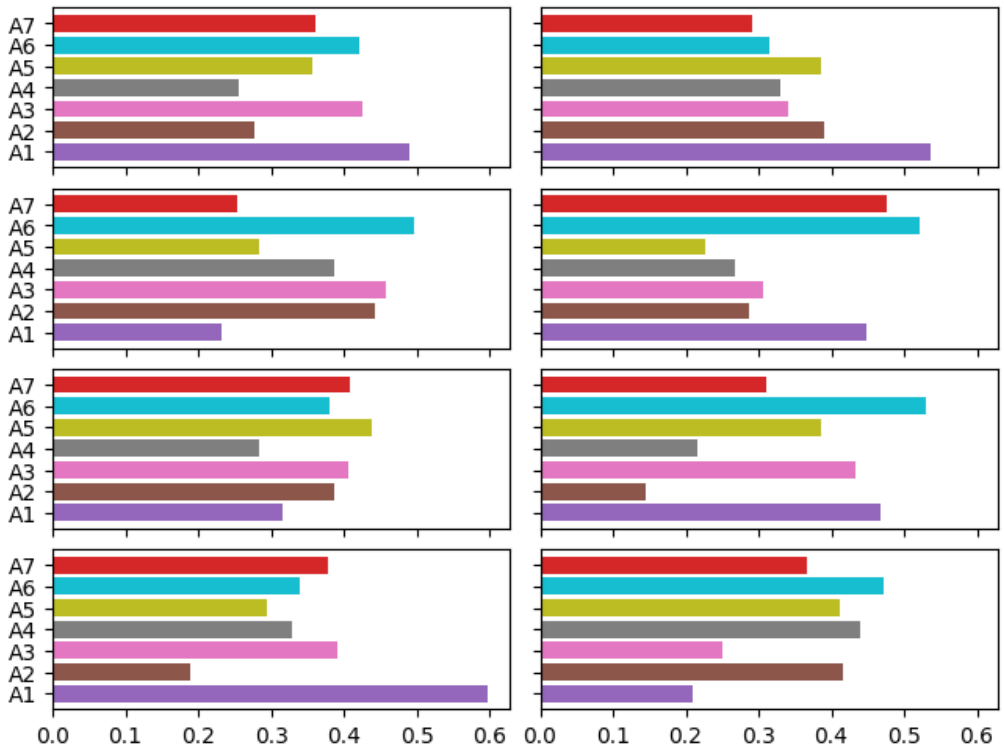

Fig 10. Normalized decision matrix.

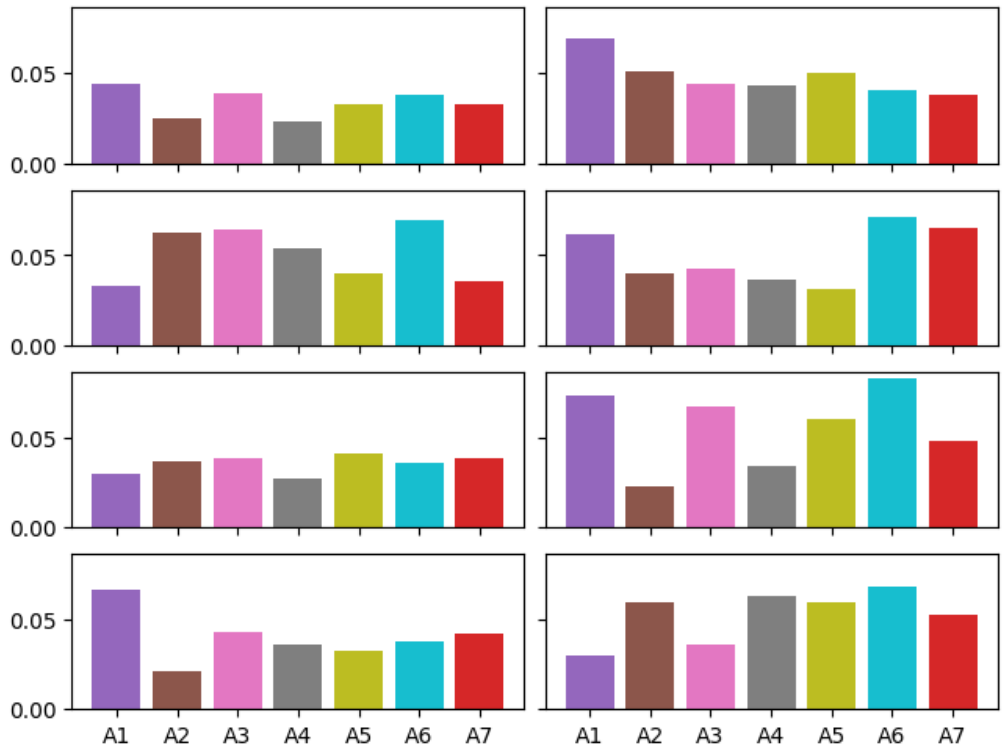

Fig 11. The weighted decision matrix.

Then we compute the rank of the alternatives in each case as shown in Fig 12. We show alternative 6 is the best and alternative 4 is the worst.

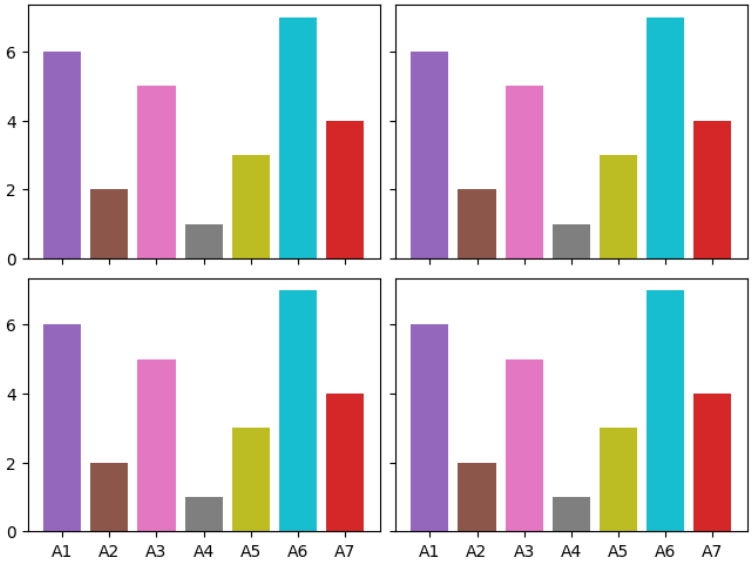

Fig 12. Final ranks of alternatives.

4. Conclusions

The effects of normalizing on the SIWEC-based TOPSIS method are examined in this study, along with the applicability of normalization technique in the TOPSIS method. The findings that have been reached are as follows: (1) Normalization may influence the results, which in turn influences the contribution of attributes to ideal and nonideal solutions. This means that normalization can influence the SIWEC -based TOPSIS method's decision outcome. (2) SuperHyperSoft can affect the decision outcome based on a set of the values in the criteria. (3) SIWEC was used to compute the criteria weights. (4) TOPSIS was used to rank the alternatives. (5) The results show alternative 6 is the best and alternative 4 is the worst.

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m-Polar Quadripartitioned Neutrosophic Graphs with Applications in Decision-Making for Mobile Network Selection

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Abstract. In today's rapidly evolving world, the shift towards multi-polarity is increasingly evident, significantly influencing various scientific and technological domains, particularly in information and data sciences. This study introduces m-polar Quadripartitioned neutrosophic sets and graphs, along with key results and operations, such as the composition and Cartesian product of m-polar quadripartitoned neutrosophic graphs, supported by illustrative examples to clarify the proposed concepts. Additionally, recognizing the complexity of selecting a mobile network shaped by factors like coverage, data speed, pricing, and customer service-we propose a novel decision-making approach using a score function within the framework of m-polar quadripartitioned neutrosophic sets. This methodology enables users to evaluate and compare multiple mobile networks effectively across various criteria.

Keywords: m-Polar; Quadripartitioned Neutrosophic Graphs; Decision-Making; Mobile Network;

1. Introduction

An m-polar neutrosophic model is highly effective, offering greater precision, adaptability, and comparability than classical, fuzzy, or neutrosophic graph models. In neurobiology, multi-polar brain neurons demonstrate significant learning interactions with other neurons, showcasing the relevance of multi-polar systems. Likewise, multi-polar technology finds applications in information technology for managing large-scale systems. However, challenges often arise when dealing with unclear or ambiguous evaluations in the selection of reference sets.

For example, in voting scenarios, individuals may vote in favor, vote against, or abstain [33]. The mathematical properties of m-polar fuzzy graphs are studied in [10], while the concepts of m-polar valued graphs have been extended to various domains of fuzzy set theory [4,11,12]. In electrical engineering, we work with conductors and non-conductors, but some materials act as insulators. Neutrosophic set (NS) theory offers an effective framework for handling such scenarios. In many real-world applications, we encounter multipolarity, along with rankings of truth, falsity, and indeterminacy for various alternatives. To address these uncertainties and hesitations, we introduce the concept of an m-polar NS.

From a practical perspective, the applications of this model span a wide range of fields, including decision-making systems, pattern recognition, image processing, medical diagnosis, and risk management. For example, in decision-making systems, the m-polar quadripartitioned neutrosophic model captures the complex nature of real-world choices, enabling more robust and comprehensive analysis [30, 31]. Similarly, in image processing and pattern recognition, where data may involve ambiguity or uncertainty, this model supports more accurate classification and interpretation [14, 28].

In medical diagnosis, where uncertainty arises from the complexity of biological systems and variability among patients, the m-polar quadripartitioned neutrosophic model offers a systematic framework for integrating diverse information sources, enabling informed decision-making.

Similarly, in risk management, particularly in domains such as finance and insurance, where accurate risk assessment is critical, this model enhances evaluation by addressing various dimensions of risk, including probability, severity, and uncertainty [22]. Consequently, the m-polar quadripartitioned neutrosophic approach not only strengthens the theoretical foundation of neutrosophic set theory but also broadens its practical applicability to a wide array of problems characterized by uncertainty and complexity [31].

While m-polar fuzzy sets (MPFS) define membership values within $[0, 1]^m$ to represent an object's m criteria, they fail to capture the indeterminacy and falsity components of the object. Neutrosophic sets (NS), on the other hand, consider truth, falsity, and ambiguity for a single criterion but lack the capability to integrate information from multiple sources with varying polarities and multiple criteria. To address this limitation, we propose a novel model -the m-polar neutrosophic set (MPNS) which combines the strengths of MPFS and NS. This model accommodates multiple criteria while handling truth, indeterminacy, and falsity for each alternative.

The MPNS builds upon earlier works such as the bipolar neutrosophic set by Deli et al. [9] [18], representing an extension of these concepts. We establish various attributes and operations of MPNS and introduce scoring functions for comparing m-polar neutrosophic

numbers (MPNNs). Throughout the manuscript, we use Δ as an indexing set. In the following paper, [43] studies the concept of a novel neutrosophic multi-criteria decision-making (MCDM) approach to improve the supplier selection process under uncertainty and ambiguity

The concept of single-valued neutrosophic sets was first introduced by H. Wang et al. [27]. By dividing the indeterminacy component of NS into two parts-Unknown' (neither true nor false) and Contradiction' (both true and false)-we derive four components: $\mathcal{A}$ (truth), $\mathcal{C}$ (unknown), $\mathcal{B}$ (contradiction), and $\mathcal{D}$ (falsity). This leads to the quadripartitioned single-valued neutrosophic set, as proposed by Chatterjee et al. [8], rooted in Smarandache's "Four Numerical-Valued Neutrosophic Logic" [24] and Belnap's "Four-Valued Logic" [5]. Neutrosophic graphs were concisely explained in F. Smarandache's book [25], while the degree, order, and size of single-valued neutrosophic graphs (SVNG) were developed by S. Broumi et al. [6]. Recent advancements include bipolar quadripartitioned neutrosophic graphs (BQNG) and quadripartitioned single-valued neutrosophic graphs (QSVNG) with real-world applications [16, 17, 34]. M-polar quadripartitioned neutrosophic sets, graphs, and their topological properties were introduced by S.S. Hussain [29]

Interval neutrosophic graphs (ING), which represent membership, non-membership, and indeterminacy using intervals, offer greater flexibility and ease of use compared to single-valued neutrosophic graphs [19]. Recent studies on m-polar neutrosophic graphs (MPNG) have explored metrics [1, 2] and isomorphism properties [15]. New operations such as join, union, ring sum, and composition, along with six novel products for MPNG, have been introduced in [20].

In this paper, we present the concept of MPQNS and its associated graphs, along with key results. We introduce the composition and Cartesian product of MPQNG and provide examples to illustrate the theoretical concepts. The following highlights the significance and challenges addressed in this work.

Novelty of this paper:

- This paper provides a novel conceptual framework (m-polar quadripartitioned neutrosophic sets) to address uncertainty in decision-making, particularly in scientific and technological domains.
- It introduces innovative operations (e.g., strong product, direct product) on neutrosophic graphs, enhancing analytical capabilities and expanding the toolkit for complex system analysis.
- It explores fundamental concepts like Cartesian product, composition within the context of m-polar neutrosophic graphs, contributing to theoretical understanding and application.

- It gives the properties and theorems of m-polar quadripartitioned neutrosophic graphs, providing a solid theoretical foundation for further research and application.
- Finally, these graphs comprehensively capture decision criteria such as coverage, data speed, pricing, and customer service, enabling a holistic view of mobile network options which is crucial for the dynamic nature of mobile networks.
- The abstract nature of m-polar quadripartitioned neutrosophic sets and associated operations have present challenges in interpretation and understanding, requiring specialized knowledge and expertise for effective utilization.
- In the application viewpoint, using m-polar quadripartitioned neutrosophic sets, utilizing the score function within this framework facilitates quantitative analysis, enabling objective evaluation and comparison of mobile network options, leading to data-driven decision-making.

The flowchart of the proposed QNG model is shown below:1

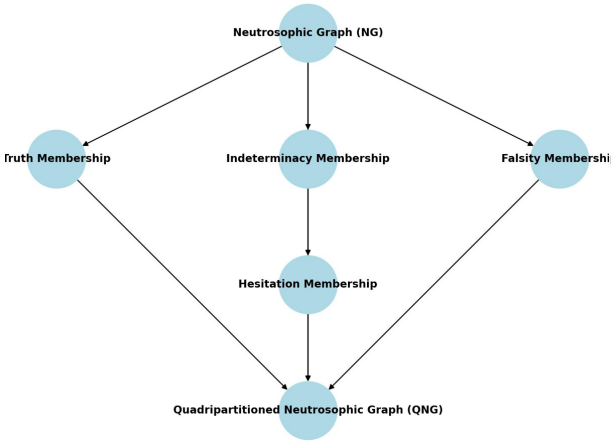

FIGURE 1. Flow Chart

2. Operation on m-polar Quadripartitioned Neutrosophic Graphs

Definition 2.1. A graph $\mathfrak{G} = (\mathbb{V}_{\mathfrak{G}}, \beta, \Gamma)$ of the graph $\mathfrak{G}' = (\mathbb{V}_{\mathfrak{G}}, \mathcal{E}_{\mathfrak{G}})$, where $\beta = (\mathcal{A}_\beta, \mathcal{B}_\beta, \mathcal{C}_\beta, \mathcal{D}_\beta)$, and $\Gamma = (\mathcal{A}_\Gamma, \mathcal{B}_\Gamma, \mathcal{C}_\Gamma, \mathcal{D}_\Gamma)$ is called m-polar QNG if, for each $i = 1, 2, .., m$,

(1) $0 \leq \chi_i(\mathcal{A}_\beta(r)) + \chi_i(\mathcal{B}_\beta(r)) + \chi_i(\mathcal{C}_\beta(r)) + \chi_i(\mathcal{D}_\beta(r)) \leq 4, \forall r \in \mathbb{V}_{\mathfrak{G}}$

(2) $\mathcal{A}_\Gamma : \mathcal{E}_{\mathfrak{G}} \to [0,1]^m, \mathcal{B}_\Gamma : \mathcal{E}_{\mathfrak{G}} \to [0,1]^m, \mathcal{C}_\Gamma : \mathcal{E}_{\mathfrak{G}} \to [0,1]^m, \mathcal{D}_\Gamma : \mathcal{E}_{\mathfrak{G}} \to [0,1]^m$ are the TMF, CMF, IMF and FMF of the edge rs respectively, such that

$\chi_i(\mathcal{A}_\Gamma(rs)) \leq \chi_i(\mathcal{A}_\beta(r)) \wedge \chi_i(\mathcal{A}_\beta(s)), \quad \chi_i(\mathcal{B}_\Gamma(rs)) \leq \chi_i(\mathcal{B}_\beta(r)) \wedge \chi_i(\mathcal{B}_\beta(s)) \ \chi_i(\mathcal{C}_\Gamma(rs)) \leq \chi_i(\mathcal{C}_\beta(r)) \vee \chi_i(\mathcal{C}_\beta(s)), \quad \chi_i(\mathcal{D}_\Gamma(rs)) \leq \chi_i(\mathcal{D}_\beta(r)) \vee \chi_i(\mathcal{D}_\beta(s))$ for all $rs \in \mathcal{E}_{\mathfrak{G}}$

Definition 2.2. The Cartesian products of two m-QNGs $\mathfrak{G} = (\mathbb{V}_{\mathfrak{G}}, \beta_1, \Gamma_1)$ and $\mathbb{H} = (\mathbb{V}_{\mathbb{H}}, \beta_2, \Gamma_2)$ of the graph $\mathfrak{G}' = (\mathbb{V}_{\mathfrak{G}}, \mathcal{E}_{\mathfrak{G}})$ and $\mathbb{H}' = (\mathbb{V}_{\mathbb{H}}, \mathcal{E}_{\mathbb{H}})$ respectively, is denoted by $\mathfrak{G} \times \mathbb{H}$

and is explained as $(\mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}}, \beta_1 \times \beta_2, \Gamma_1 \times \Gamma_2)$ where $\beta_1 \times \beta_2 = (\mathcal{A}_{\beta_1} \times \mathcal{A}_{\beta_2}, \mathcal{B}_{\beta_1} \times \mathcal{B}_{\beta_2}, \mathcal{C}_{\beta_1} \times \mathcal{C}_{\beta_2}, \mathcal{D}_{\beta_1} \times \mathcal{D}_{\beta_2})$ and $\Gamma_1 \times \Gamma_2 = (\mathcal{A}_{\Gamma_1} \times \mathcal{A}_{\Gamma_2}, \mathcal{B}_{\Gamma_1} \times \mathcal{B}_{\Gamma_2}, \mathcal{C}_{\Gamma_1} \times \mathcal{C}_{\Gamma_2}, \mathcal{D}_{\Gamma_1} \times \mathcal{D}_{\Gamma_2})$ s.t for each $i = 1, 2, 3, ..., m$.

$$
\begin{aligned}
(1) \quad \chi_i((\mathcal{A}_{\beta_1} \times \mathcal{A}_{\beta_2})(r,s)) &= \chi_i(\mathcal{A}_{\beta_1}(r)) \wedge \chi_i(\mathcal{A}_{\beta_2}(s)) \\
\chi_i((\mathcal{B}_{\beta_1} \times \mathcal{B}_{\beta_2})(r,s)) &= \chi_i(\mathcal{B}_{\beta_1}(r)) \wedge \chi_i(\mathcal{B}_{\beta_2}(s)) \\
\chi_i((\mathcal{C}_{\beta_1} \times \mathcal{C}_{\beta_2})(r,s)) &= \chi_i(\mathcal{C}_{\beta_1}(r)) \vee \chi_i(\mathcal{C}_{\beta_2}(s)) \\
\chi_i((\mathcal{D}_{\beta_1} \times \mathcal{D}_{\beta_2})(r,s)) &= \chi_i(\mathcal{D}_{\beta_1}(r)) \vee \chi_i(\mathcal{D}_{\beta_2}(s)) \\
\text{for all} \quad (r,s) &\in \mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}},
\end{aligned}
$$

$$
\begin{aligned}
(2) \quad \chi_i((\mathcal{A}_{\Gamma_1} \times \mathcal{A}_{\Gamma_2})(\alpha,r)(\alpha,s)) &= \chi_i(\mathcal{A}_{\beta_1}(\alpha)) \wedge \chi_i(\mathcal{A}_{\Gamma_2}(rs) \\
\chi_i((\mathcal{B}_{\Gamma_1} \times \mathcal{B}_{\Gamma_2})(\alpha,r)(\alpha,s)0 &= \chi_i(\mathcal{B}_{\beta_1}(\alpha)) \wedge \chi_i(\mathcal{B}_{\Gamma_2}(rs)) \\
\chi_i((\mathcal{C}_{\Gamma_1} \times \mathcal{C}_{\Gamma_2})(\alpha,r)(\alpha,s)) &= \chi_i(\mathcal{C}_{\beta_1}(\alpha)) \vee \chi_i(\mathcal{C}_{\Gamma_2}(rs)) \\
\chi_i((\mathcal{D}_{\Gamma_1} \times \mathcal{D}_{\Gamma_2})(\alpha,r)(\alpha,s)) &= \chi_i(\mathcal{D}_{\beta_1}(\alpha)) \vee \chi_i(\mathcal{D}_{\Gamma_2}(rs)) \\
\text{for all} \quad \alpha \in \mathbb{V}_{\mathfrak{G}} &\text{and } \forall rs \in \mathcal{E}_{\mathbb{H}}
\end{aligned}
$$

$$
\begin{aligned}
(iii) \quad \chi_i((\mathcal{A}_{\Gamma_1} \times \mathcal{A}_{\Gamma_2})(r,\alpha)(s,\alpha)) &= \chi_i(\mathcal{A}_{\Gamma_1}(rs)) \wedge \chi_i(\mathcal{A}_{\beta_2}(\alpha)) \\
\chi_i((\mathcal{B}_{\Gamma_1} \times \mathcal{B}_{\Gamma_2})(r,\alpha)(s,\alpha)) &= \chi_i(\mathcal{B}_{\Gamma_1}(rs)) \wedge \chi_i(\mathcal{B}_{\beta_2}(\alpha)) \\
\chi_i((\mathcal{C}_{\Gamma_1} \times \mathcal{C}_{\Gamma_2})(r,\alpha)(s,\alpha)) &= \chi_i(\mathcal{C}_{\Gamma_1}(rs)) \vee \chi_i(\mathcal{C}_{\beta_2}(\alpha)) \\
\chi_i((\mathcal{D}_{\Gamma_1} \times \mathcal{D}_{\Gamma_2})(r,\alpha)(s,\alpha)) &= \chi_i(\mathcal{D}_{\Gamma_1}(rs)) \vee \chi_i(\mathcal{D}_{\beta_2}(\alpha)) \\
\text{for all} \quad \alpha \in \mathbb{V}_{\mathbb{H}} &\text{and } \forall rs \in \mathcal{E}_{\mathfrak{G}}
\end{aligned}
$$

$$
\begin{aligned}
(iv)\chi_i((\mathcal{A}_{\Gamma_1} \times \mathcal{A}_{\Gamma_2})(r,s)(\alpha,\delta)) &= 0 \\
\chi_i((\mathcal{B}_{\Gamma_1} \times \mathcal{B}_{\Gamma_2})(r,s)(\alpha,\delta)) &= 0 \\
\chi_i((\mathcal{C}_{\Gamma_1} \times \mathcal{C}_{\Gamma_2})(r,s)(\alpha,\delta)) &= 0 \\
\chi_i((\mathcal{D}_{\Gamma_1} \times \mathcal{D}_{\Gamma_2})(r,s)(\alpha,\delta)) &= 0 \\
(r,s)(\alpha,\delta) &\in (\mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}})^2 - \mathcal{E}.
\end{aligned}
$$

Here $\mathcal{E} = \{(r,\alpha)(s,\alpha) : rs \in \mathcal{E}_{\mathfrak{G}}, \alpha \in \mathbb{V}_{\mathbb{H}}\} \cup \{(r,\alpha)(r,\delta) : r \in \mathbb{V}_{\mathfrak{G}}, \alpha\delta \in \mathcal{E}_{\mathbb{H}}\}$.

Theorem 2.3. *The Cartesian Product* $(\mathfrak{G} \times \mathbb{H}) = (\mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}}, \beta_{\mathfrak{G}} \times \beta_{\mathbb{H}}, \Gamma_{\mathfrak{G}} \times \Gamma_{\mathbb{H}})$ *where* $\beta_{\mathfrak{G}} \times \beta_{\mathbb{H}} = (\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}}, \mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}}, \mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}}, \mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})$ *and* $\Gamma_{\mathfrak{G}} \times \Gamma_{\mathbb{H}} = (\mathcal{A}_{\Gamma_{\mathfrak{G}}} \times \mathcal{A}_{\Gamma_{\mathbb{H}}}, \mathcal{B}_{\Gamma_{\mathfrak{G}}} \times \mathcal{B}_{\Gamma_{\mathbb{H}}}, \mathcal{C}_{\Gamma_{\mathfrak{G}}} \times \mathcal{C}_{\Gamma_{\mathbb{H}}}, \mathcal{D}_{\Gamma_{\mathfrak{G}}} \times \mathcal{D}_{\Gamma_{\mathbb{H}}})$ *of two m-polar QNG* $\mathfrak{G} = (\mathbb{V}_{\mathfrak{G}}, \beta_{\mathfrak{G}}, \Gamma_{\mathfrak{G}})$ *and* $\mathbb{H} = (\mathbb{V}_{\mathbb{H}}, \beta_{\mathbb{H}}, \Gamma_{\mathbb{H}})$ *of the graphs* $\mathfrak{G}' = (\mathbb{V}_{\mathfrak{G}} \ \mathcal{E}_{\mathfrak{G}})$ *and* $\mathbb{H}' = (\mathbb{V}_{\mathbb{H}} \ \mathcal{E}_{\mathbb{H}})$ *is an* $\mathfrak{m}$*-polar quadripartitioned neutrosophic graph of* $\mathfrak{G}' \times \mathbb{H}'$

Proof. Let $\alpha \in \mathbb{V}_{\mathfrak{G}}$ and $rs \in \mathcal{E}_{\mathbb{H}}$ then for each $i = 1, 2, 3, .., m$.

$$\begin{aligned}
\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \times \mathcal{A}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &= \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{A}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\leq \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \{\chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(s))\} \\
&= \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(r))\} \wedge \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(s))\} \\
&= \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})((\alpha, r)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s))) \\
\therefore \quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \times \mathcal{A}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &\leq \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})((\alpha, r)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s)))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \times \mathcal{B}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &= \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\leq \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(s))\} \\
&= \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(r))\} \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(s))\} \\
&= \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})((\alpha, r)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s))) \\
\therefore \quad \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \times \mathcal{B}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &\leq \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})((\alpha, r)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s)))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \times \mathcal{C}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &= \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\leq \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \{\chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(r)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(s))\} \\
&= \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(r))\} \vee \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(s))\} \\
&= \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})((\alpha, r)) \vee \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s))) \\
\therefore \quad \chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \times \mathcal{C}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &\leq \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})((\alpha, r)) \vee \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s)))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \times \mathcal{D}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &= \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\leq \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \{\chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(r)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(s))\} \\
&= \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(r))\} \vee \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(s))\} \\
&= \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})((\alpha, r)) \vee \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s))) \\
\therefore \quad \chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \times \mathcal{D}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &\leq \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})((\alpha, r)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})(\alpha, s))
\end{aligned}$$

Let $\alpha \in \mathbb{V}_{\mathbb{H}}$ and $rs \in \mathcal{E}_{\mathfrak{G}}$ then for each $i = 1, 2, 3, .., m$.

$$\begin{aligned}\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \times \mathcal{A}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{A}_{\Gamma_{\mathfrak{G}}}(rs)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha)) \\ &\leq \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(s))\} \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha))\} \wedge \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(s,\alpha))) \\ \therefore \quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \times \mathcal{A}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &\leq \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(s,\alpha)))\end{aligned}$$

$$\begin{aligned}\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \times \mathcal{B}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{B}_{\Gamma_{\mathfrak{G}}}(rs)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha)) \\ &\leq \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(r)) \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(r))\} \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})((\alpha,r)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})(s,\alpha))) \\ \therefore \quad \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \times \mathcal{B}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &\leq \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})(s,\alpha))\end{aligned}$$

$$\begin{aligned}\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \times \mathcal{C}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{C}_{\Gamma_{\mathfrak{G}}}(rs)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha)) \\ &\leq \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(r)) \vee \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha))\} \vee \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})((r,\alpha)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})(s,\alpha))) \\ \therefore \quad \chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \times \mathcal{C}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &\leq \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})((r,\alpha)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})(s,\alpha))\end{aligned}$$

$$\begin{aligned}\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \times \mathcal{D}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{D}_{\Gamma_{\mathfrak{G}}}(rs)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha)) \\ &\leq \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(r)) \vee \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha))\} \vee \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})((r,\alpha)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})(s,\alpha))) \\ \therefore \quad \chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \times \mathcal{D}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &\leq \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})((r,\alpha)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})(s,\alpha))\end{aligned}$$

Let $(r,s)(\alpha,\delta) \in (\mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}})^2 - \mathcal{E}$. Then,

$$\begin{aligned}\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \times \mathcal{A}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) &= 0 \leq \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(r,s)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha,\delta)) \\ \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \times \mathcal{B}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) &= 0 \leq \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})(r,s)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})(\alpha,\delta)) \\ \chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \times \mathcal{C}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) &= 0 \leq \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})(r,s)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})(\alpha,\delta)) \\ \chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \times \mathcal{D}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) &= 0 \leq \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})(r,s)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})(\alpha,\delta))\end{aligned}$$

for each $i = 1, 2, 3, ..., m$. □

Definition 2.4. The composition $\mathfrak{G} \circ \mathbb{H} = (\mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}}, \beta_{\mathfrak{G}} \times \beta_{\mathbb{H}}, \Gamma_{\mathfrak{G}} \times \Gamma_{\mathbb{H}})$ where $(\beta_{\mathfrak{G}} \circ \beta_{\mathbb{H}}) = (\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}}, \mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}}, \mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}}, \mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})$ and $(\Gamma_{\mathfrak{G}} \circ \Gamma_{\mathbb{H}}) = (\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}}, \mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}}, \mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}}, \mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})$ of two m-polar QNG $\mathfrak{G} = (\mathbb{V}_{\mathfrak{G}}, \beta_{\mathfrak{G}}, \Gamma_{\mathfrak{G}})$ and $\mathbb{H} = (\mathbb{V}_{\mathbb{H}}, \beta_{\mathbb{H}}, \Gamma_{\mathbb{H}})$, where $\beta_{\mathfrak{G}} = (\mathcal{A}_{\beta_{\mathfrak{G}}}, \mathcal{B}_{\beta_{\mathfrak{G}}}, \mathcal{C}_{\beta_{\mathfrak{G}}}, \mathcal{D}_{\beta_{\mathfrak{G}}})$, $\beta_{\mathbb{H}} = (\mathcal{A}_{\beta_{\mathbb{H}}}, \mathcal{B}_{\beta_{\mathbb{H}}}, \mathcal{C}_{\beta_{\mathbb{H}}}, \mathcal{D}_{\beta_{\mathbb{H}}})$, $\Gamma_{\mathfrak{G}} = (\mathcal{A}_{\Gamma_{\mathfrak{G}}}, \mathcal{B}_{\Gamma_{\mathfrak{G}}}, \mathcal{C}_{\Gamma_{\mathfrak{G}}}, \mathcal{D}_{\Gamma_{\mathfrak{G}}})$, $\Gamma_{\mathbb{H}} = (\mathcal{A}_{\Gamma_{\mathbb{H}}}, \mathcal{B}_{\Gamma_{\mathbb{H}}}, \mathcal{C}_{\Gamma_{\mathbb{H}}}, \mathcal{D}_{\Gamma_{\mathbb{H}}})$, of the graphs $\mathfrak{G}' = (\mathbb{V}_{\mathfrak{G}}, \mathcal{E}_{\mathfrak{G}})$ and $\mathbb{H}' = (\mathbb{V}_{\mathbb{H}}, \mathcal{E}_{\mathbb{H}})$ respectively, is explained as follows: for each $i = 1, 2, .., m$,

$$
\begin{aligned}
(i) \quad \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(rs)) &= \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(s)) \\
\chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(rs)) &= \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(s)) \\
\chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(rs)) &= \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(s)) \\
\chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(rs)) &= \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(s)) \\
&\text{for} \quad \text{all} \quad rs \in \mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}}
\end{aligned}
$$

$$
\begin{aligned}
(ii) \quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})(\alpha, r)(\alpha, s)) &= \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{A}_{\Gamma_{\mathbb{H}}}(rs)) \\
\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})(\alpha, r)(\alpha, s)) &= \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\Gamma_{\mathbb{H}}}(rs)) \\
\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})(\alpha, r)(\alpha, s)) &= \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\Gamma_{\mathbb{H}}}(rs)) \\
\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})(\alpha, r)(\alpha, s)) &= \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\text{for} \quad \text{all} \quad \alpha \in \mathbb{V}_{\mathfrak{G}} \text{and} \quad (rs) \in \mathcal{E}_H.
\end{aligned}
$$

$$
\begin{aligned}
(iii) \quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})(r, \alpha)(s, \alpha)) &= \chi_i(\mathcal{A}_{\Gamma_{\mathfrak{G}}}(rs)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha)) \\
\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})(r, \alpha)(s, \alpha)) &= \chi_i(\mathcal{B}_{\Gamma_{\mathfrak{G}}}(rs)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha)) \\
\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})(r, \alpha)(s, \alpha)) &= \chi_i(\mathcal{C}_{\Gamma_{\mathfrak{G}}}(rs)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha)) \\
\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})(r, \alpha)(s, \alpha)) &= \chi_i(\mathcal{D}_{\Gamma_{\mathfrak{G}}}(rs)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha)) \\
&\text{for} \quad \text{all} \quad \alpha \in \mathbb{V}_{\mathbb{H}} \text{and} \quad (rs) \in \mathcal{E}_G.
\end{aligned}
$$

$$(iv)\quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{A}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta))=\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}}(r\alpha)))\wedge\chi_i((\mathcal{A}_{\beta_{\mathbb{H}}}(s))\wedge\chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\delta)))$$
$$\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{B}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta))=\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}}(r\alpha)))\wedge\chi_i((\mathcal{B}_{\beta_{\mathbb{H}}}(s))\wedge\chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\delta)))$$
$$\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{C}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta))=\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}}(r\alpha)))\vee\chi_i((\mathcal{C}_{\beta_{\mathbb{H}}}(s))\vee\chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\delta)))$$
$$\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{D}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta))=\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}}(r\alpha)))\vee\chi_i((\mathcal{D}_{\beta_{\mathbb{H}}}(s))\vee\chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\delta)))$$
$$\text{for all}\quad (rs)(\alpha\delta)\in\mathcal{E}_0-\mathcal{E}$$

$$(v)\quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{A}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta))=0$$
$$\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{B}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta))=0$$
$$\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{C}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta))=0$$
$$\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{D}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta))=0$$
$$\text{for}\quad \text{all}\quad (rs)(\alpha\delta)\in(\mathbb{V}_{\mathfrak{G}}\times\mathbb{V}_{\mathbb{H}})^2-\mathcal{E}_0$$

Here $\mathcal{E}=\{((r,\alpha)(s\alpha)):rs\in\mathcal{E}_{\mathfrak{G}},\alpha\in\mathbb{V}_{\mathbb{H}}\}\cup((r,\alpha)(r\delta)):r\in\mathbb{V}_{\mathfrak{G}},\alpha\delta\in\mathcal{E}_{\mathbb{H}}$ and $\mathcal{E}^0=\mathcal{E}\cup\{(r,\alpha)(s,\delta):r\alpha\in\mathcal{E}_{\mathfrak{G}},\alpha\neq\delta\in\mathbb{V}_{\mathbb{H}}\}$

Theorem 2.5. *The composition* $(\mathfrak{G}\circ\mathbb{H})=(\mathbb{V}_{\mathfrak{G}}\circ\mathbb{V}_{\mathbb{H}},\beta_{\mathfrak{G}}\circ\beta_{\mathbb{H}},\Gamma_{\mathfrak{G}}\circ\Gamma_{\mathbb{H}})$ *where* $\beta_{\mathfrak{G}}\circ\beta_{\mathbb{H}}=(\mathcal{A}_{\beta_{\mathfrak{G}}}\circ\mathcal{A}_{\beta_{\mathbb{H}}},\mathcal{B}_{\beta_{\mathfrak{G}}}\circ\mathcal{B}_{\beta_{\mathbb{H}}},\mathcal{C}_{\beta_{\mathfrak{G}}}\circ\mathcal{C}_{\beta_{\mathbb{H}}},\mathcal{D}_{\beta_{\mathfrak{G}}}\circ\mathcal{D}_{\beta_{\mathbb{H}}})$ *and* $\Gamma_{\mathfrak{G}}\circ\Gamma_{\mathbb{H}}=(\mathcal{A}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{A}_{\Gamma_{\mathbb{H}}},\mathcal{B}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{B}_{\Gamma_{\mathbb{H}}},\mathcal{C}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{C}_{\Gamma_{\mathbb{H}}},\mathcal{D}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{D}_{\Gamma_{\mathbb{H}}})$ *of two* $\mathfrak{m}$*-polar QNG* $\mathfrak{G}=(\mathbb{V}_{\mathfrak{G}},\beta_{\mathfrak{G}},\Gamma_{\mathfrak{G}})$ *and* $\mathbb{H}=(\mathbb{V}_{\mathbb{H}},\beta_{\mathbb{H}},\Gamma_{\mathbb{H}})$ *of the graphs* $\mathfrak{G}'=(\mathbb{V}_{\mathfrak{G}},\mathcal{E}_{\mathfrak{G}})$ *and* $\mathbb{H}'=(\mathbb{V}_{\mathbb{H}},\mathcal{E}_{\mathbb{H}})$ *is an* $\mathfrak{m}$*-QNG of* $\mathfrak{G}'\circ\mathbb{H}'$

Proof. Let $\alpha\in\mathbb{V}_{\mathfrak{G}}$ and $rs\in\mathcal{E}_{\mathbb{H}}$ then for each $i=1,2,3,..,m$.

$$\begin{aligned}\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{A}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s)))&=\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha))\wedge\chi_i(\mathcal{A}_{\Gamma_{\mathbb{H}}}(rs))\\&\leq\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha))\wedge\{\chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(r))\wedge\chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(s))\}\\&=\{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha))\wedge\chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(r))\}\wedge\{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha))\wedge\chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(s))\}\\&=\chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}}\circ\mathcal{A}_{\beta_{\mathbb{H}}})((\alpha,r))\wedge\chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}}\circ\mathcal{A}_{\beta_{\mathbb{H}}})(\alpha,s)))\end{aligned}$$
$$\therefore\quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}}\circ\mathcal{A}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s)))\leq\chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}}\circ\mathcal{A}_{\beta_{\mathbb{H}}})((\alpha,r))\wedge\chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}}\circ\mathcal{A}_{\beta_{\mathbb{H}}})(\alpha,s))$$

$$\begin{aligned}\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) &= \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\Gamma_{\mathbb{H}}}(rs)) \\ &\leq \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(s))\} \\ &= \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(r))\} \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(s))\} \\ &= \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})((\alpha,r)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha,s)))\end{aligned}$$

$$\therefore \quad \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) \leq \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})((\alpha,r)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(\alpha,s))$$

$$\begin{aligned}\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) &= \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\Gamma_{\mathbb{H}}}(rs)) \\ &\leq \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \{\chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(r)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(s))\} \\ &= \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(r))\} \vee \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(s))\} \\ &= \chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})((\alpha,r)) \vee \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})(\alpha,s)))\end{aligned}$$

$$\therefore \quad \chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) \leq \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})((\alpha,r)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(\alpha,s))$$

$$\begin{aligned}\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) &= \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\Gamma_{\mathbb{H}}}(rs)) \\ &\leq \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \{\chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(r)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(s))\} \\ &= \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(r))\} \vee \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(s))\} \\ &= \chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})((\alpha,r)) \vee \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})(\alpha,s)))\end{aligned}$$

$$\therefore \quad \chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) \leq \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})((\alpha,r)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(\alpha,s))$$

Let $\alpha \in \mathbb{V}_{\mathbb{H}}$ and $rs \in \mathcal{E}_{\mathfrak{G}}$ then for each $i = 1, 2, 3, .., m$.

$$\begin{aligned}\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{A}_{\Gamma_{\mathfrak{G}}}(rs)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha)) \\ &\leq \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(r)) \wedge \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha))\} \wedge \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(s,\alpha)))\end{aligned}$$

$$\therefore \quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) \leq \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(s,\alpha))$$

$$\begin{aligned}\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{B}_{\Gamma_{\mathfrak{G}}}(rs)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha)) \\ &\leq \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(r)) \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha))\} \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(s,\alpha)))\end{aligned}$$

$$\therefore \quad \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) \leq \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(s,\alpha))$$

$$\begin{aligned}\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{C}_{\Gamma_{\mathfrak{G}}}(rs)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha)) \\ &\leq \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(r)) \vee \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha))\} \vee \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})((\alpha,r)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(s,\alpha)))\end{aligned}$$

$$\therefore \quad \chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) \leq \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})((r,\alpha)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(s,\alpha))$$

$$\begin{aligned}\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{D}_{\Gamma_{\mathfrak{G}}}(rs)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha)) \\ &\leq \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(r)) \vee \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha))\} \vee \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha))\} \\ &= \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})((\alpha,r)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(s,\alpha)))\end{aligned}$$

$$\therefore \quad \chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) \leq \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})((r,\alpha)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(s,\alpha))$$

Let $(r,s)(\alpha,\delta) \in (\mathbb{V}_{\mathfrak{G}} \circ \mathbb{V}_{\mathbb{H}})^2 - \mathcal{E}$. Then for each $i = 1,2,3,...,m$.

$$\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) = 0 \leq \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(r,s)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha,\delta))$$

$$\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) = 0 \leq \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(r,s)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(\alpha,\delta))$$

$$\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) = 0 \leq \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(r,s)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(\alpha,\delta))$$

$$\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) = 0 \leq \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(r,s)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(\alpha,\delta))$$

Let $(r,s)(\alpha,\delta) \in \mathcal{E}_0 - \mathcal{E}$. Then for each $i = 1,2,3,...,m$.

$$\begin{aligned}
\chi_i((\mathcal{A}_{\Gamma_\mathfrak{G}} \circ \mathcal{A}_{\Gamma_\mathbb{H}})(r,s)(\alpha,\delta)) &= \chi_i((\mathcal{A}_{\Gamma_\mathfrak{G}}(r\alpha))) \wedge \chi_i((\mathcal{A}_{\beta_\mathbb{H}}(s) \wedge \mathcal{A}_{\beta_\mathbb{H}}(\delta))) \\
&\leq \{(\chi_i(\mathcal{A}_{\beta_\mathfrak{G}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_\mathfrak{G}}(\alpha)))\} \wedge \{\chi_i((\mathcal{A}_{\beta_\mathbb{H}}(s)) \wedge \chi_i(\mathcal{A}_{\beta_\mathbb{H}}(\delta)))\} \\
&= \{\chi_i((\mathcal{A}_{\beta_\mathfrak{G}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_\mathbb{H}}(s)))\} \wedge \{\chi_i((\mathcal{A}_{\beta_\mathfrak{G}}(\alpha)) \wedge \chi_i(\mathcal{A}_{\beta_\mathbb{H}}(\delta)))\} \\
&= \chi_i((\mathcal{A}_{\beta_\mathfrak{G}} \circ \mathcal{A}_{\beta_\mathbb{H}})(r,s)) \wedge \chi_i((\mathcal{A}_{\beta_\mathfrak{G}} \circ \mathcal{A}_{\beta_\mathbb{H}})(\alpha,\delta)) \\
\therefore \quad \chi_i((\mathcal{A}_{\Gamma_\mathfrak{G}} \circ \mathcal{A}_{\Gamma_\mathbb{H}})(r,s)(\alpha,\delta)) &\leq \chi_i((\mathcal{A}_{\beta_\mathfrak{G}} \circ \mathcal{A}_{\beta_\mathbb{H}})(r,s)) \wedge \chi_i((\mathcal{A}_{\beta_\mathfrak{G}} \circ \mathcal{A}_{\beta_\mathbb{H}})(\alpha,\delta))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{B}_{\Gamma_\mathfrak{G}} \circ \mathcal{B}_{\Gamma_\mathbb{H}})(r,s)(\alpha,\delta)) &= \chi_i((\mathcal{B}_{\Gamma_\mathfrak{G}}(r\alpha))) \wedge \chi_i((\mathcal{B}_{\beta_\mathbb{H}}(s)) \wedge \chi_i(\mathcal{B}_{\beta_\mathbb{H}}(\delta))) \\
&\leq \{\chi_i((\mathcal{B}_{\beta_\mathfrak{G}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_\mathfrak{G}}(\alpha)))\} \wedge \{\chi_i((\mathcal{B}_{\beta_\mathbb{H}}(s)) \wedge \chi_i(\mathcal{B}_{\beta_\mathbb{H}}(\delta)))\} \\
&= \{\chi_i((\mathcal{B}_{\beta_\mathfrak{G}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_\mathbb{H}}(s)))\} \wedge \{\chi_i((\mathcal{B}_{\beta_\mathfrak{G}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\beta_\mathbb{H}}(\delta)))\} \\
&= \chi_i((\mathcal{B}_{\beta_\mathfrak{G}} \circ \mathcal{B}_{\beta_\mathbb{H}})(r,s)) \wedge \chi_i((\mathcal{B}_{\beta_\mathfrak{G}} \circ \mathcal{B}_{\beta_\mathbb{H}})(\alpha,\delta)) \\
\therefore \quad \chi_i((\mathcal{B}_{\Gamma_\mathfrak{G}} \circ \mathcal{B}_{\Gamma_\mathbb{H}})(r,s)(\alpha,\delta)) &\leq \chi_i((\mathcal{B}_{\beta_\mathfrak{G}} \circ \mathcal{B}_{\beta_\mathbb{H}})(r,s)) \wedge \chi_i((\mathcal{B}_{\beta_\mathfrak{G}} \circ \mathcal{B}_{\beta_\mathbb{H}})(\alpha,\delta))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{C}_{\Gamma_\mathfrak{G}} \circ \mathcal{C}_{\Gamma_\mathbb{H}})(r,s)(\alpha,\delta)) &= \chi_i((\mathcal{C}_{\Gamma_\mathfrak{G}}(r\alpha))) \vee \chi_i((\mathcal{C}_{\beta_\mathbb{H}}(s)) \vee \chi_i(\mathcal{C}_{\beta_\mathbb{H}}(\delta))) \\
&\leq \{\chi_i((\mathcal{C}_{\beta_\mathfrak{G}}(r)) \vee \chi_i(\mathcal{C}_{\beta_\mathfrak{G}}(\alpha)))\} \vee \{\chi_i((\mathcal{C}_{\beta_\mathbb{H}}(s)) \vee \chi_i(\mathcal{C}_{\beta_\mathbb{H}}(\delta)))\} \\
&= \{\chi_i((\mathcal{C}_{\beta_\mathfrak{G}}(r)) \vee \chi_i(\mathcal{C}_{\beta_\mathbb{H}}(s)))\} \vee \{\chi_i((\mathcal{C}_{\beta_\mathfrak{G}}(\alpha)) \vee \chi_i(\mathcal{C}_{\beta_\mathbb{H}}(\delta)))\} \\
&= \chi_i((\mathcal{C}_{\beta_\mathfrak{G}} \circ \mathcal{C}_{\beta_\mathbb{H}})(r,s)) \vee \chi_i((\mathcal{C}_{\beta_\mathfrak{G}} \circ \mathcal{C}_{\beta_\mathbb{H}})(\alpha,\delta)) \\
\therefore \quad \chi_i((\mathcal{B}_{\Gamma_\mathfrak{G}} \circ \mathcal{C}_{\Gamma_\mathbb{H}})(r,s)(\alpha,\delta)) &\leq \chi_i((\mathcal{C}_{\beta_\mathfrak{G}} \circ \mathcal{C}_{\beta_\mathbb{H}})(r,s)) \vee \chi_i((\mathcal{C}_{\beta_\mathfrak{G}} \circ \mathcal{C}_{\beta_\mathbb{H}})(\alpha,\delta))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{D}_{\Gamma_\mathfrak{G}} \circ \mathcal{D}_{\Gamma_\mathbb{H}})(r,s)(\alpha,\delta)) &= \chi_i((\mathcal{D}_{\Gamma_\mathfrak{G}}(r\alpha))) \vee \chi_i((\mathcal{D}_{\beta_\mathbb{H}}(s)) \vee \chi_i(\mathcal{D}_{\beta_\mathbb{H}}(\delta))) \\
&\leq \{\chi_i((\mathcal{D}_{\beta_\mathfrak{G}}(r)) \vee \chi_i(\mathcal{D}_{\beta_\mathfrak{G}}(\alpha)))\} \vee \{\chi_i((\mathcal{D}_{\beta_\mathbb{H}}(s)) \vee \chi_i(\mathcal{D}_{\beta_\mathbb{H}}(\delta)))\} \\
&= \{\chi_i((\mathcal{D}_{\beta_\mathfrak{G}}(r)) \vee \chi_i(\mathcal{D}_{\beta_\mathbb{H}}(s)))\} \vee \{\chi_i((\mathcal{D}_{\beta_\mathfrak{G}}(\alpha)) \vee \chi_i(\mathcal{D}_{\beta_\mathbb{H}}(\delta)))\} \\
&= \chi_i((\mathcal{D}_{\beta_\mathfrak{G}} \circ \mathcal{D}_{\beta_\mathbb{H}})(r,s)) \vee \chi_i((\mathcal{D}_{\beta_\mathfrak{G}} \circ \mathcal{D}_{\beta_\mathbb{H}})(\alpha,\delta)) \\
\therefore \quad \chi_i((\mathcal{B}_{\Gamma_\mathfrak{G}} \circ \mathcal{D}_{\Gamma_\mathbb{H}})(r,s)(\alpha,\delta)) &\leq \chi_i((\mathcal{D}_{\beta_\mathfrak{G}} \circ \mathcal{D}_{\beta_\mathbb{H}})(r,s)) \vee \chi_i((\mathcal{D}_{\beta_\mathfrak{G}} \circ \mathcal{D}_{\beta_\mathbb{H}})(\alpha,\delta))
\end{aligned}$$

□

3. **Application: Network Selection for Mobile Service**

In this section, m-polar quadripartitioned neutrosophic decision making model for "Network Selection for Mobile Service". The criteria are:

1. Coverage and Signal Strength
2. Data Speed and Performance
3. Service Plans and Pricing
4. Customer Service and Support

Let $\mathfrak{N} = \{\mathfrak{N}_1, \mathfrak{N}_2, \ldots, \mathfrak{N}_m\}$ be the set of alternatives and $\mathfrak{C} = \{\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_n\}$ be the set of criteria. We consider that ω_j be the weight corresponding to the criteria $\mathfrak{C}_j$, $(j = 1, 2, \ldots, n)$. In the decision making problem, the evaluation information of the alternative $\mathfrak{N}_i$, $(i = 1, 2, \ldots, m)$ on the criteria is represented by the form of a quadripartitioned neutrosophic number. Coverage and signal strength are fundamental criteria for mobile network selection, ensuring reliable connectivity across different geographical areas. Consumers prioritize networks with extensive coverage, minimal dead zones, and strong signal strength, enabling uninterrupted communication and data access. Data speed and performance significantly impact the user experience, especially for tasks such as streaming, downloading, and online gaming. Buyers seek networks with high-speed data connections, low latency, and consistent performance, allowing for fast and seamless browsing, streaming, and data transfer. Service plans and pricing play a crucial role in network selection, with consumers considering factors such as monthly fees, data allowances, contract terms, and additional charges. Customers look for cost-effective plans that offer value for money, competitive pricing, and flexibility to meet their usage requirements. Customer service and support are essential considerations, as users may require assistance with billing inquiries, technical support, or network-related issues. Buyers prefer networks with responsive customer service, accessible support channels, and knowledgeable representatives who can address their concerns promptly and effectively.

3.1. *Algorithm:*

`Step-1:` Define Criteria and Weights

`Step-2:` Collect Neutrosophic Evaluations and construct single-valued quadripartitioned neutrosophic decision matrix $\mathcal{D}$:

$$\mathcal{D} = (a_{i,j})_{m,n} = \begin{pmatrix} (T_{1,1}, C_{1,1}, U_{1,1}, F_{1,1}) & (T_{1,2}, C_{1,2}, U_{1,2}, F_{1,2}) & \ldots & (T_{1,n}, C_{1,n}, U_{1,n}, F_{1,n}) \\ (T_{2,1}, C_{2,1}, U_{2,1}, F_{2,1}) & (T_{2,2}, C_{2,2}, U_{2,2}, F_{2,2}) & \ldots & (T_{2,n}, C_{2,n}, U_{2,n}, F_{2,n}) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{m,1}, C_{m,1}, U_{m,1}, F_{m,1}) & (T_{m,2}, C_{m,2}, U_{m,2}, F_{m,2}) & \ldots & (T_{m,n}, C_{m,n}, U_{m,n}, F_{m,n}) \end{pmatrix}$$

`Step-3:` Aggregate Evaluations- Aggregate the evaluations using a neutrosophic arithmetic average operator

`Step-4:` Calculate Scores and Rank Networks - Compute the score and accuracy value for each networks based on the aggregated evaluations. We determine some score function for

the ranking of mQNs, in order to solve the problems with multi-criteria decision-making with m-Polar quadripartitioned neutrosophic numbers (mQNNs).

The mQNS score function defined as

$$\mathfrak{S}(\mathfrak{Q}) = \frac{m}{2} + \sum_{i=0}^{m} \frac{\omega_T \cdot T_i + \omega_C \cdot C_i + \omega_U \cdot U_i + \omega_F \cdot F_i}{\omega_T + \omega_C + \omega_U + \omega_F}$$

$$\mathfrak{A}(\mathfrak{Q}) = \sum_{i=0}^{m} \sqrt{\frac{(\omega_T \cdot T_i)^2 + (\omega_C \cdot C_i)^2 + (\omega_U \cdot U_i)^2 + (\omega_F \cdot F_i)^2}{2}}$$

`Step-5:` Make Decision - Rank the Networks based on their scores and select the one with the highest rank as the best option.

For our calculation, we are considering the data for 4-polar QNs are given as follows:

Network $\mathfrak{N}_1$:

1. Coverage and Signal Strength - $(0.4, 0.3, 0.2, 0.1)$
2. Data Speed and Performance - $(0.4, 0.6, 0.2, 0.1)$
3. Service Plans and Pricing - $(0.2, 0.5, 0.1, 0.2)$
4. Customer Service and Support - $(0.1, 0.2, 0.5, 0.2)$

Network $\mathfrak{N}_2$:

1. Coverage and Signal Strength - $(0.8, 0.4, 0.4, 0.1)$
2. Data Speed and Performance - $(0.6, 0.2, 0.7, 0.1)$
3. Service Plans and Pricing - $(0.7, 0.3, 0.8, 0.1)$
4. Customer Service and Support - $(0.1, 0.4, 0.7, 0.1)$

Network $\mathfrak{N}_3$:

1. Coverage and Signal Strength - $(0.9, 0.1, 0.1, 0)$
2. Data Speed and Performance - $(0.6, 0.3, 0.2, 0.1)$
3. Service Plans and Pricing - $(0.8, 0.8, 0.2, 0.2)$
4. Customer Service and Support - $(0.4, 0.7, 0.1, 0.1)$

Network $\mathfrak{N}_4$:

1. Coverage and Signal Strength - $(0.7, 0.4, 0.1, 0)$
2. Data Speed and Performance - $(0.5, 0.3, 0.7, 0)$
3. Service Plans and Pricing - $(0.8, 0.1, 0.3, 0)$
4. Customer Service and Support - $(0.8, 0.4, 0.2, 0.1)$.

Now, this Figure 2 generate the given data with each group representing a criterion and each bar within a group representing a network.

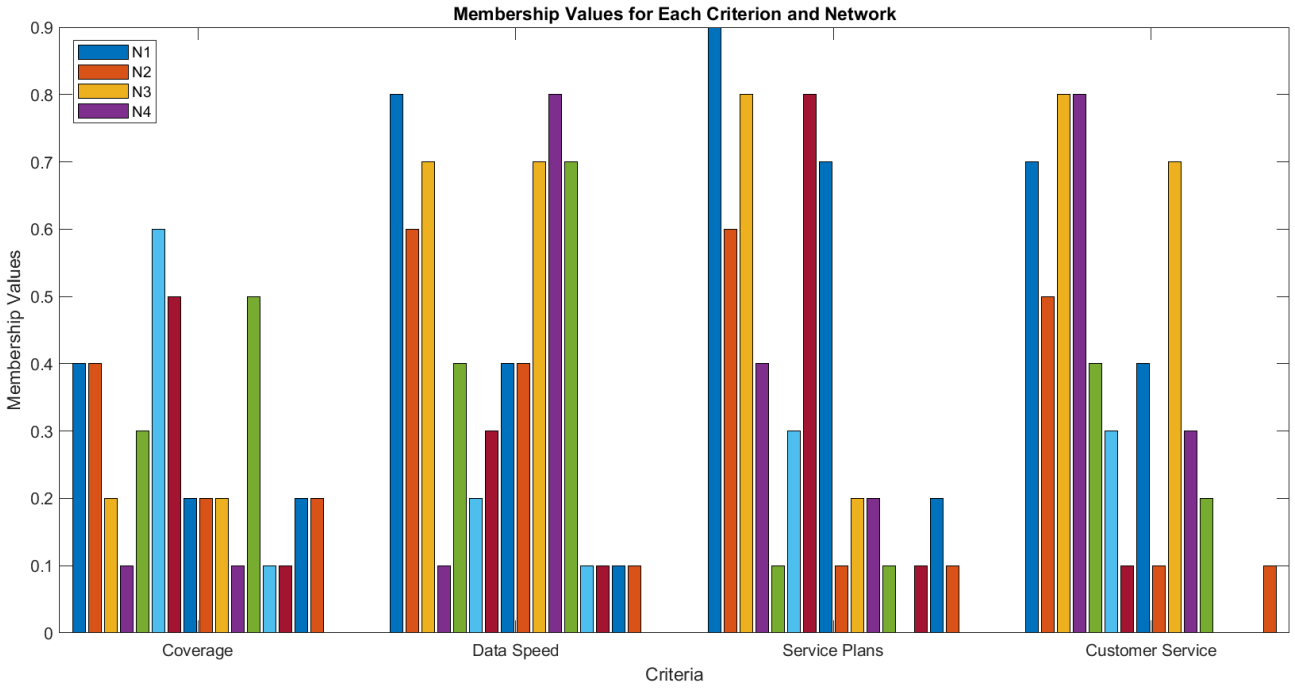

FIGURE 2. Membership values for each Criterion and Network

The weights are given as $\omega = (0.4, 0.5, 0.2, 0.1)$. Here in $\mathcal{D}$ each row represents a networks, and each column represents a criterion.

$$\mathcal{D} = \begin{pmatrix} (0.4,0.3,0.2,0.1) & (0.4,0.6,0.2,0.1) & (0.2,0.5,0.1,0.2) & (0.1,0.2,0.5,0.2) \\ (0.8,0.4,0.4,0.1) & (0.6,0.2,0.7,0.1) & (0.7,0.3,0.8,0.1) & (0.1,0.4,0.7,0.1) \\ (0.9,0.1,0.1,0) & (0.6,0.3,0.2,0.1) & (0.8,0.8,0.2,0.2) & (0.4,0.7,0.1,0.1) \\ (0.7,0.4,0.1,0) & (0.5,0.3,0.7,0) & (0.8,0.1,0.3,0) & (0.8,0.4,0.2,0.1) \end{pmatrix}$$

To aggregate the evaluations, we shall compute the quadripartitioned neutrosophic arithmetic average for each criterion across all networks. This will give a single quadripartitioned neutrosophic value for each criterion.

Let's denote the aggregated quadripartioned neutrosophic values as $\mathfrak{Q}_i$ for each criterion i. $\mathfrak{Q}_1 = \left(\frac{0.4+0.4+0.2+0.1}{4}, \frac{0.3+0.6+0.5+0.2}{4}, \frac{0.2+0.2+0.1+0.5}{4}, \frac{0.1+0.1+0.2+0.2}{4}\right) = (0.275, 0.4, 0.25, 0.15)$.

In the similar fashion, we can also compute:

$\mathfrak{Q}_2 = (0.55, 0.325, 0.65, 0.025)$,

$\mathfrak{Q}_3 = (0.675, 0.475, 0.15, 0.1)$,

$\mathfrak{Q}_4 = (0.7, 0.3, 0.325, 0.025)$.

Now, let's calculate the scores and accuracy values for each network based on these aggregated values.

To calculate the scores and accuracy values for each networks, we shall use the aggregated quadripartitioned neutrosophic values obtained in the previous step. The score function

$\mathfrak{S}(\mathfrak{Q}_1) = 0.729$

$\mathfrak{S}(\mathfrak{Q}_2) = 0.8458$

$\mathfrak{S}(\mathfrak{Q}_3) = 0.8745$

$\mathfrak{S}(\mathfrak{Q}_4) = 0.8313.$

The accuracy function $\mathfrak{A}(\mathfrak{Q}_1) = 0.166$

$\mathfrak{A}(\mathfrak{Q}_2) = 0.219$

$\mathfrak{A}(\mathfrak{Q}_3) = 0.257$

$\mathfrak{A}(\mathfrak{Q}_4) = 0.221$

Now, let's rank the networks based on their scores and select the one with the highest rank as the best network (in Fig 3). Based on the calculated scores and accuracy values, we can rank the networks as follows: $\mathfrak{N}_3 > \mathfrak{N}_2 > \mathfrak{N}_4 > \mathfrak{N}_1$ Hence the best network is $\mathfrak{N}_3$.

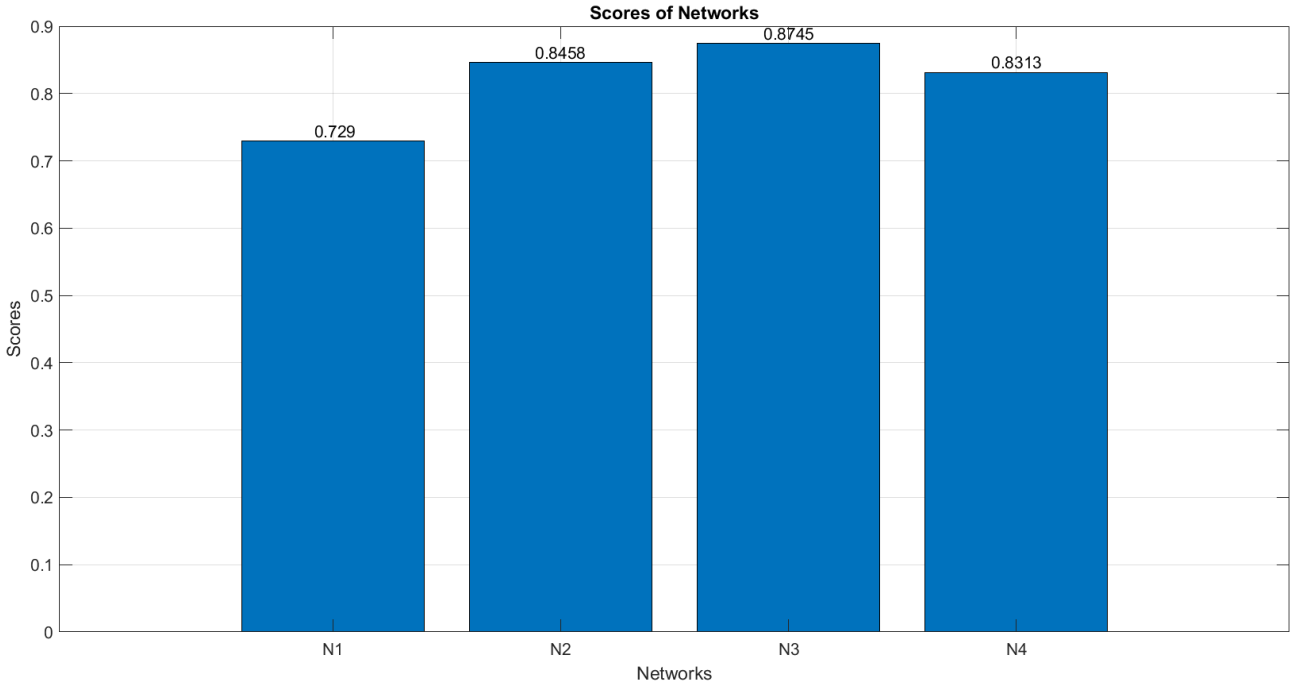

FIGURE 3. Scores of the networks

Limitations

Quadripartitioned neutrosophic graphs (QNGs) are an advanced mathematical tool with potential applications in decision-making and uncertainty modeling. However, like any emerging concept, QNGs have certain limitations, including:

Complexity in Computation:

Handling and analyzing QNGs requires significant computational effort, especially for large-scale graphs, as they involve multiple degrees of membership (truth, indeterminacy, falsity,

and hesitation).

Interpretability Challenges:
The interpretation of results derived from QNGs can be complex for non-specialists, limiting their applicability in real-world scenarios where stakeholders may lack technical expertise.

Data Dependency:
QNGs rely on accurate and detailed input data for constructing membership functions. Any bias or error in the data can significantly affect the reliability of the results.

Limited Theoretical Development:
As a relatively new field, QNGs lack comprehensive theoretical frameworks, algorithms, and tools compared to traditional graph theories, which limits their widespread application.

Scalability Issues:
The practical implementation of QNGs on large datasets, such as social networks or supply chain systems, faces scalability problems due to their mathematical and computational complexity.

Integration with Existing Models:
Integrating QNGs with other models or frameworks, such as machine learning or optimization techniques, is still underexplored, which restricts their versatility.

Conclusion

In this work, we introduced and established the concept of m-polar neutrosophic structures (m-PNS) and their corresponding graphs, accompanied by several foundational results. Operations such as the Cartesian product and composition of m-polar neutrosophic graphs (m-PNGs) have been systematically defined and explored, demonstrating their applicability through illustrative examples. These developments provide a robust framework for modeling and analyzing complex systems involving uncertainty and multi-polarity.

Looking ahead, we aim to extend this study to investigate the algebraic properties of m-polar quadripartitioned neutrosophic graphs, focusing on their structural behavior and potential applications in real-world problems. The incorporation of score functions further enhances decision-making processes by providing a comprehensive assessment that balances the relative importance of multiple criteria. This ensures that decisions are not only mathematically sound but also aligned with users' preferences and priorities.

Through our proposed framework, practical applications such as mobile network selection can be significantly improved, enabling users to make informed decisions that optimize performance, affordability, and overall satisfaction. Additionally, we foresee the application of these concepts in diverse fields, such as supply chain management, social network analysis, and computational biology, paving the way for further advancements in the study of neutrosophic graphs

Compliance with ethical standards

Data availability statements Not applicable.

Conflict of interest: The authors declare that they have no conflict of interest.

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Decision Support Model with TreeSoft Set for Assessment of Coupled and Coordinated Development of Digital Transportation Technology and Industry

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Abstract: The integration of digital transportation technology and the transportation industry has become a crucial factor in shaping modern mobility systems, enhancing operational efficiency, and promoting sustainability. This study evaluates the coupled and coordinated development of digital transportation innovations. Using multi-criteria decision-making (MCDM) methodologies, this research assesses key factors using BWM to compute the criteria weights. The MOOSRA method is used to rank the alternatives. This study uses the TreeSoft set to divide the criteria and sub criteria as set of levels and in each level, we compute the criteria weights. A case study with five criteria and different sub criteria with seven alternatives is conducted in this study. We conducted sensitivity analysis by using different criteria weights. This evaluation provides a systematic framework for decision-makers to optimize digital transportation strategies, ensuring a harmonized, resilient, and future-ready transportation ecosystem.

Keywords: TreeSoft Set; Decision Support Model; Coupled and Coordinated; Digital Transportation Technology and Industry.

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1. Introduction

The rapid advancement of digital transportation technology is transforming the global transportation industry, leading to smarter, more efficient, and sustainable mobility solutions. The integration of artificial intelligence (AI), the Internet of Things (IoT), big data analytics, blockchain, and cloud computing has significantly improved traffic management, logistics optimization, and passenger convenience. However, the successful adoption of these digital innovations depends on the coordinated development between technology and the transportation industry, ensuring that infrastructure, policies, and operational frameworks align with technological progress. As digitalization reshapes transportation networks, evaluating the

level of integration and coordination between these two sectors is crucial for enhancing system efficiency, reducing costs, and maximizing societal benefits[1], [2].

The coupling and coordination of digital technology and the transportation industry require a balanced approach that fosters innovation while addressing industry-specific challenges. While advanced technologies such as autonomous vehicles, intelligent traffic systems, and Mobility-as-a-Service (MaaS) platforms offer promising solutions, their effectiveness depends on industry readiness, government regulations, and financial investments. Some regions and sectors have rapidly adopted digital transportation solutions, while others face barriers such as infrastructure limitations, cybersecurity risks, and resistance to change. A well-structured evaluation framework can provide insights into how well these technologies are being integrated into different transportation models and help identify the gaps that hinder seamless collaboration between technology providers and transportation service operators[3], [3].

To assess the effectiveness of digital transportation technology in the industry, multi-criteria decision-making (MCDM) methodologies can be employed. These methods allow researchers and policymakers to evaluate key performance indicators such as technological integration, industry adaptability, investment in digital infrastructure, operational efficiency, and sustainability. By applying MCDM techniques, decision-makers can rank transportation models based on their digital readiness, economic feasibility, and societal impact. Such an approach ensures that digital transformation efforts align with broader industry goals, improving the overall quality and reliability of transportation systems while fostering environmental sustainability[4], [5].

Despite its advantages, the integration of digital technology into transportation is not without challenges. The unequal distribution of digital infrastructure, cybersecurity concerns, high initial investment costs, and workforce training remain critical obstacles. Additionally, the rate of technological change often outpaces industry adaptation, leading to gaps in policymaking and regulation. Therefore, evaluating the coupled and coordinated development of digital transportation technology and industry is not just about measuring technological progress but also about understanding the institutional, economic, and societal factors that influence adoption and scalability. Addressing these challenges requires collaboration between governments, private enterprises, and research institutions to develop scalable, adaptable, and future-proof digital transportation solutions.

A systematic evaluation of the relationship between digital transportation technology and industry will provide valuable insights for stakeholders looking to enhance efficiency, promote sustainable mobility, and future-proof transportation systems. As the world moves toward smart cities and intelligent transportation networks, it is essential to ensure that technological advancements are effectively integrated into existing infrastructure, regulations, and industry practices. By assessing the level of digital coordination and its impact on operational

performance, policymakers and industry leaders can make informed decisions that lead to a more connected, resilient, and sustainable transportation ecosystem[6], [7].

This study proposed a decision support model for Evaluation of Coupled and Coordinated Development of Digital Transportation Technology and Industry using MCDM methods. The main contributions of this study are organized as follows:

Two MCDM methods are used in this study such as BWM to compute the criteria weights and the MOOSRA method to rank the alternatives.

TreeSoft Set is used to divide the criteria and sub criteria as levels and each level includes different nodes.

The case Study is provided to show the validation of the proposed decision support system by including five main criteria and seven alternatives.

Sensitivity analysis is conducted to show the stability of the ranks of the alternatives under different criteria weights.

2. Decision Support Model

The experiment is conducted for Evaluation of Coupled and Coordinated Development of Digital Transportation Technology and Industry. This study uses two MCDM methods for this evaluation such as BWM and MOOSRA. BWM is used to compute the criteria weights and the MOOSRA method is used to rank alternatives[8], [9], [10]. We use the TreeSoft Set to deal with criteria and sub criteria. The steps of the decision support model are organized as follows:

Step 1. Build a Tree and compute its nodes.

- Level 1: This level contains the main criteria; these criteria are represented as nodes. This study uses five main criteria, so we have five nodes in the first level.
- Level 2: This level is divided into different branches; each branch shows the sub criteria. Each node has different sub criteria, these sub criteria are presented as sub nodes[11], [12].

Let U be a universal set and H be a non-empty subset of U, the power set of H is $P(H)$. The set of criteria are $C_1, \dots, C_n; n \geq 2$. $C_1 \cap C_2 \cap \dots \cap C_n = \emptyset; F: P(C) \to P(H)$ is a MultiSoft set.

The TreeSoft Set can be defined as:

$$\begin{aligned} C_1 &= \{C_{1,1}, C_{1,2}, \dots\} \\ C_2 &= \{C_{2,1}, C_{2,2}, \dots\} \\ C_3 &= \{C_{3,1}, C_{3,2}, \dots\} \\ &\vdots \\ C_n &= \{C_{n,1}, C_{n,2}, \dots\} \end{aligned} \quad (1)$$

The TreeSoft set can be defined as a set of levels with the main level is root node[13], [14].

$$Tree(C) = \{C_{i_1}, i_1 = 1,2, \dots\} \cup \{C_{i_1,i_2}, i_1, i_2 = 1,2, \dots\} \cup \dots \cup \{C_{i_1,i_2,\dots i_m}, i_1, i_2, \dots . i_m = 1,2, \dots\} \quad (2)$$

Step 2. Computing the criteria weights.

In this step, we compute the criteria weights by the BWM. The steps of this method are organized as follows:

- Identification of the decision criteria. We define a set of criteria and sub criteria by the opinions of experts and literature reviews such as $C_1, \dots, C_n$.
- Define the best and worst criterion. Experts can define the best and worst criterion. We can present the best criterion as C_B and the worst criterion as C_W.
- Conduct comparison between the best criterion and other criteria. Experts can perform this step to show the best criterion over other criteria such as $C_B = (C_{B1}, \dots, C_{Bn})$.
- Conduct comparison between the worst criterion and other criteria. Experts can perform this step to show the worst criterion over other criteria such as $C_W = (C_{W1}, \dots, C_{Wn})^T$.
- Compute the optimal weights. The criteria weights can be computed as:
- $\min \max \left\{ \left| w_B - C_{B_j} w_j \right|, \left| w_B - C_{W_j} w_W \right| \right\}$ subject to (3)
- $\sum_j w_j = 1$ (4)
- $w_j > 0;\ \forall_j$ (5)

Step 3. Ranking the alternatives.

In this step, we can obtain the rank of the alternatives by using the MOOSR method. The steps of this method are organized as follows:

- Build the decision matrix.
- $X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}$ (6)
- Compute the normalized decision matrix.
- $h_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}$ (7)
- Compute the weighted decision matrix.

- $u_{ij} = w_j h_{ij}$ (8)
- Compute the MOOSRA value as:
- $Q_{ij} = \frac{\sum_{j=1}^{g} u_{ij}}{\sum_{g+1}^{n} u_{ij}}$ (9)
- Rank the alternatives.

3. Results and Discussion

This section shows the results of the decision support model by providing the criteria weights and ranking of the alternatives. This study uses five criteria and different sub criteria with seven alternatives.

Step 1. We build a tree with two levels.

- Level 1: This level shows the five main criteria.
- Level 2: This level shows the sub criteria values. Fig 1 shows the Tree nodes of this study.

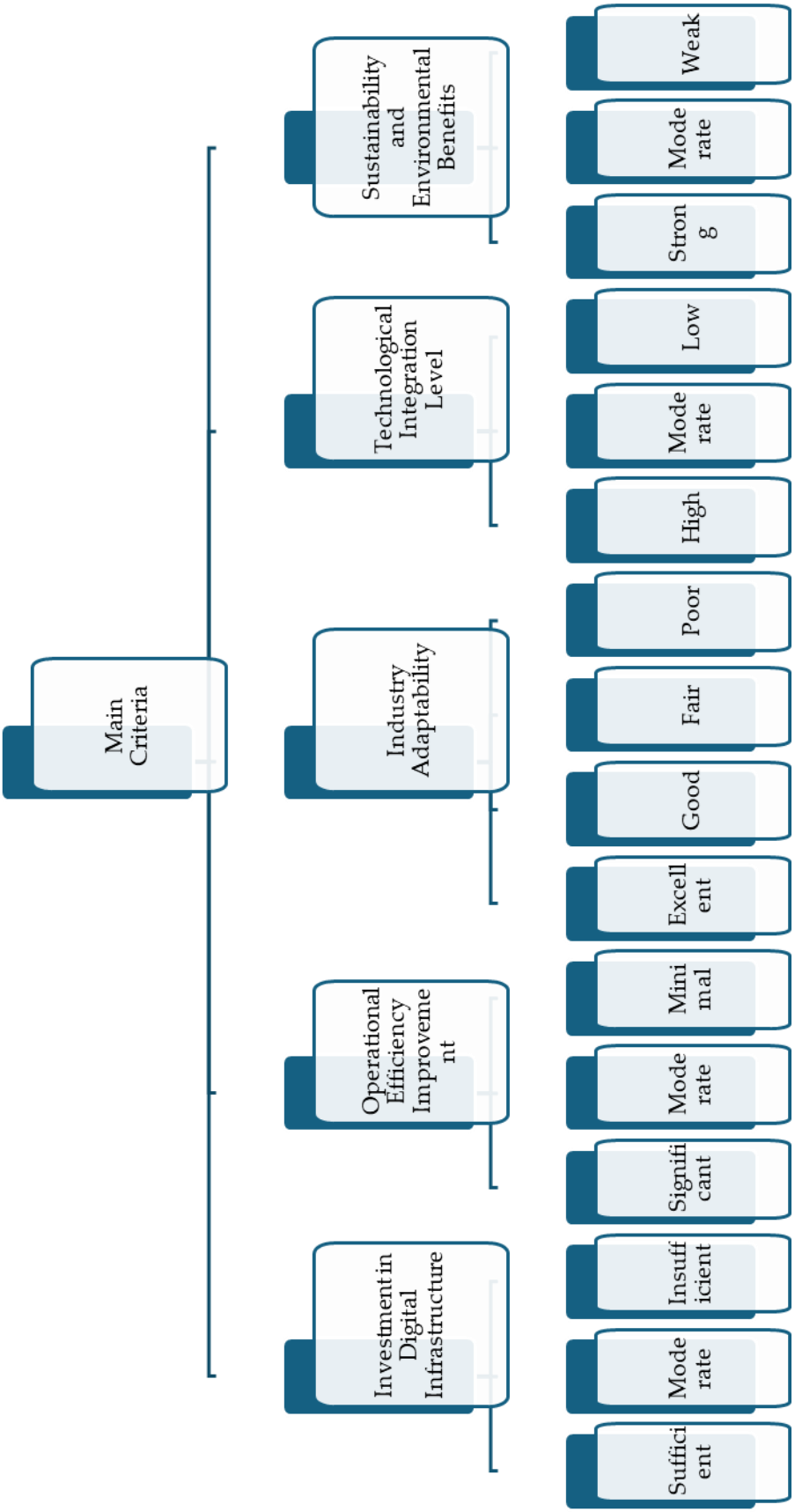
Main Criteria
Investment in Digital Infrastructure
Sufficient
Moderate
Insufficient
Operational Efficiency Improvement
Significant
Moderate
Minimal
Industry Adaptability
Excellent
Good
Fair
Poor
Technological Integration Level
High
Moderate
Low
Sustainability and Environmental Benefits
Strong
Moderate
Weak

Fig 1. Tree nodes of this study.

Step 2. We compute the criteria weights in each level. We compute the criteria weights in the main criteria and the sub criteria.

The results of the BWM are shown as:

- We define five main criteria and sub criteria as shown in Fig 1.
- Experts and decision makers are defined the best and worst criterion.
- We conducted the comparison between the best criterion over other criteria.
- We conducted the comparison between the worst criterion over other criteria.
- Then we obtain the criteria weights of the main criteria as shown in Fig 2.
- Then we apply the previous steps to compute the weights of the sub criteria as shown in Fig 3.

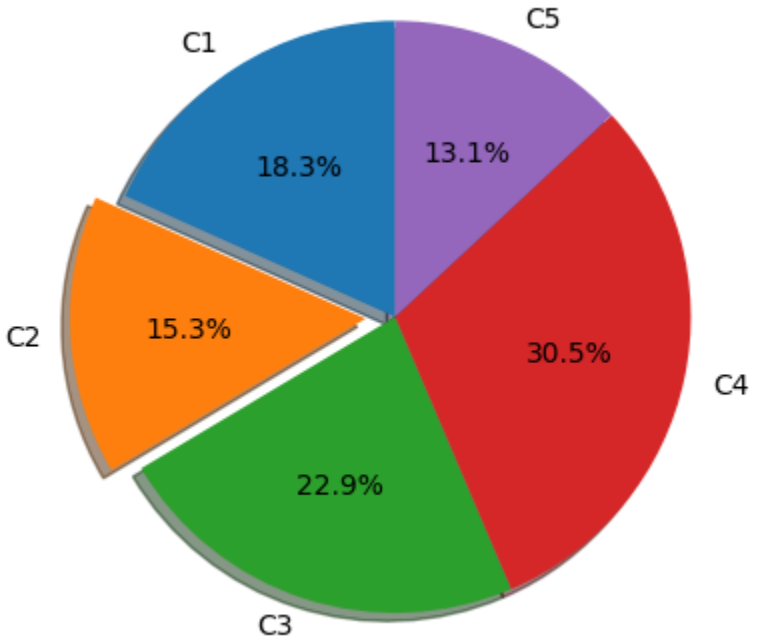

Fig 2. The weights of main criteria.

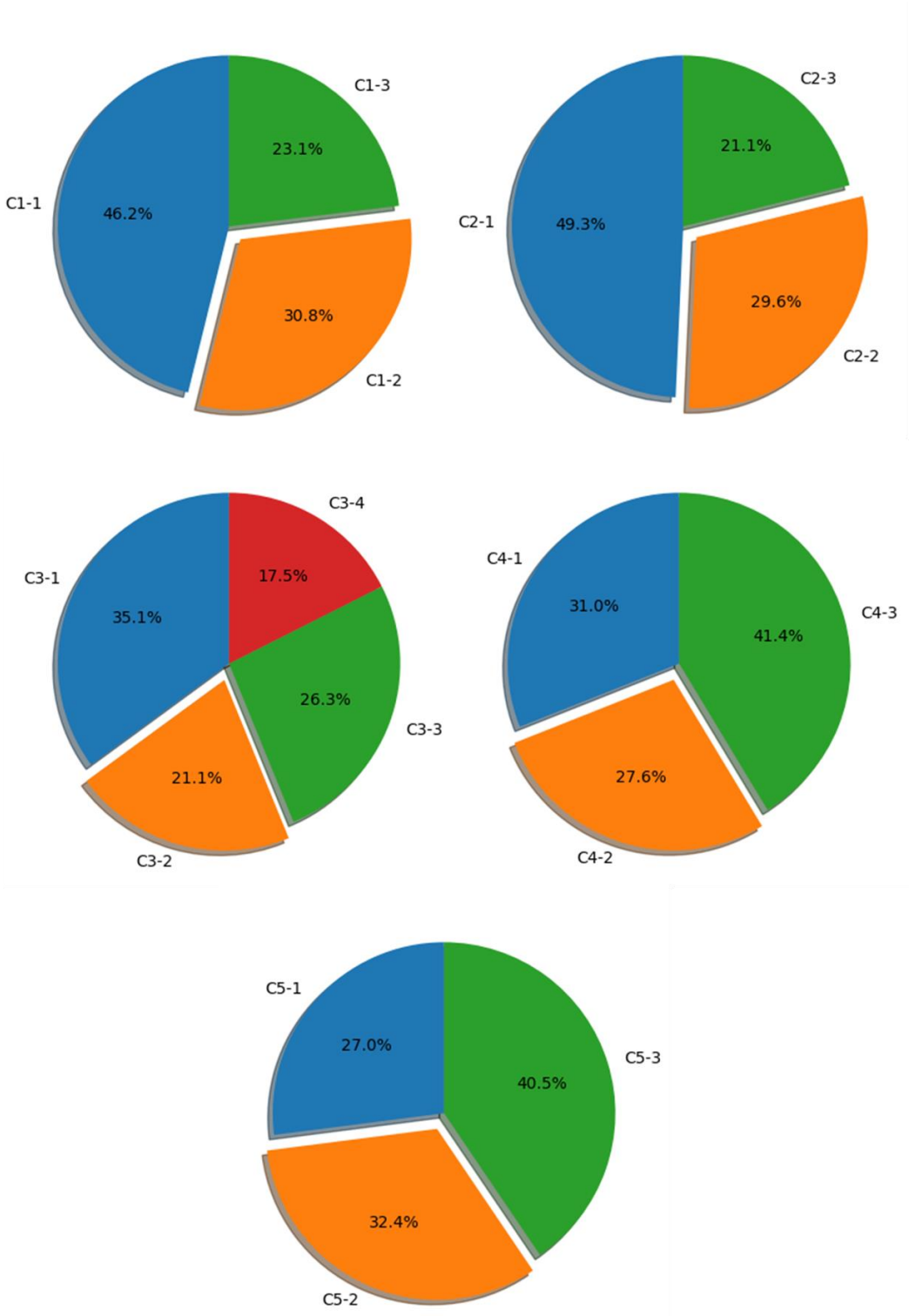

Fig 3. The weights of sub criteria.

Step 3. Ranking the alternatives.

We apply the steps of the MOOSRA method to rank the alternatives.

- We build the decision matrix using Eq. (1). Experts are used scale between 0.9 to 0.1 to evaluate the criteria and alternatives.
- We compute the normalized decision matrix using Eq. (7) as shown in Fig 4.
- Then we compute the weighted decision matrix using Eq. (8) as shown in Fig 5.
- Then we compute the MOOSRA value using Eq. (9) as shown in Fig 6.
- Then we rank the alternatives as shown in Fig 7.

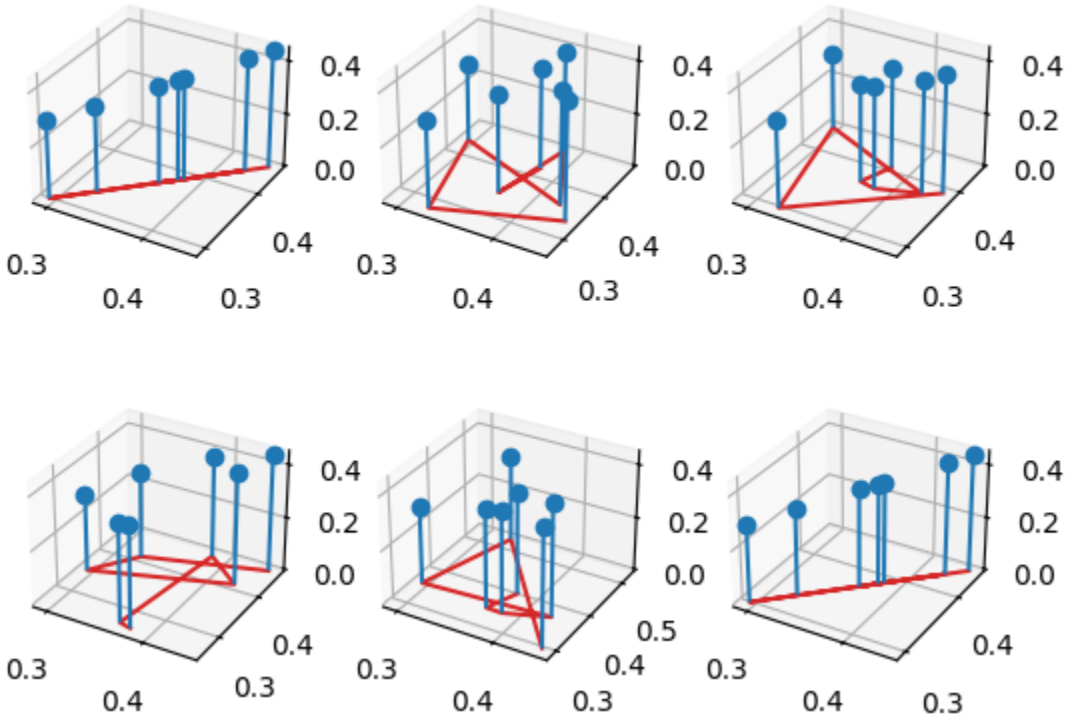

Fig 4. The normalized decision matrix.

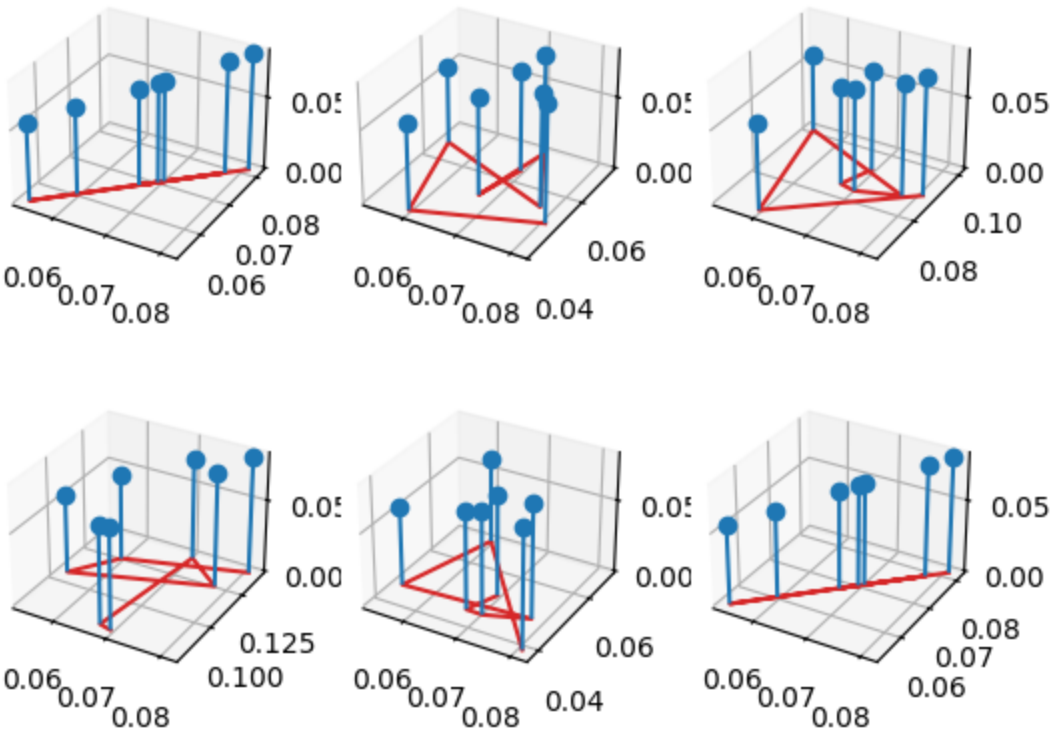

Fig 5. The weighted decision matrix.

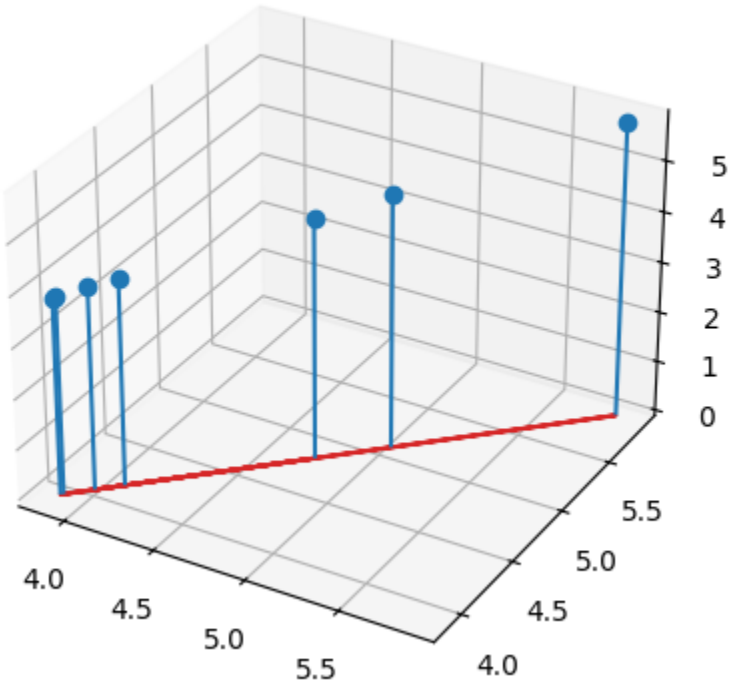

Fig 6. The MOOSRA value of each alternative.

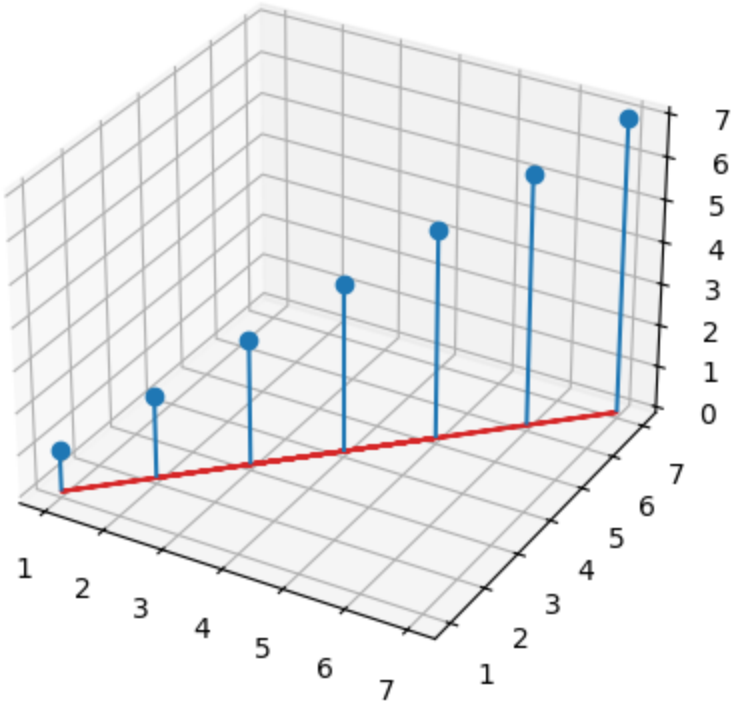

Fig 7. The rank of alternatives.

4. Analysis

This section conducted sensitivity analysis to show the rank of alternatives under different cases. This study proposes six cases in the criteria weights. We change the criteria weight, then we rank the alternatives. Fig 8 shows the different values of each criterion. In the first case, the criteria have the same weights. The second scale the first criterion weight is increased by 30% and other criteria have the same weights. The third scale the second criterion weight is increased by 30% and other criteria have the same weights. On the fourth scale the third criterion weight is increased by 30% and other criteria have the same weights.

The ranks of the alternatives are shown in Fig 9. We show the rank of alternatives in each case. The results show alternative 3 is the best and alternative 1 is the worst. We show the ranks of alternatives are stable under different cases. Fig 10. Shows the values of the MOOSRA score.

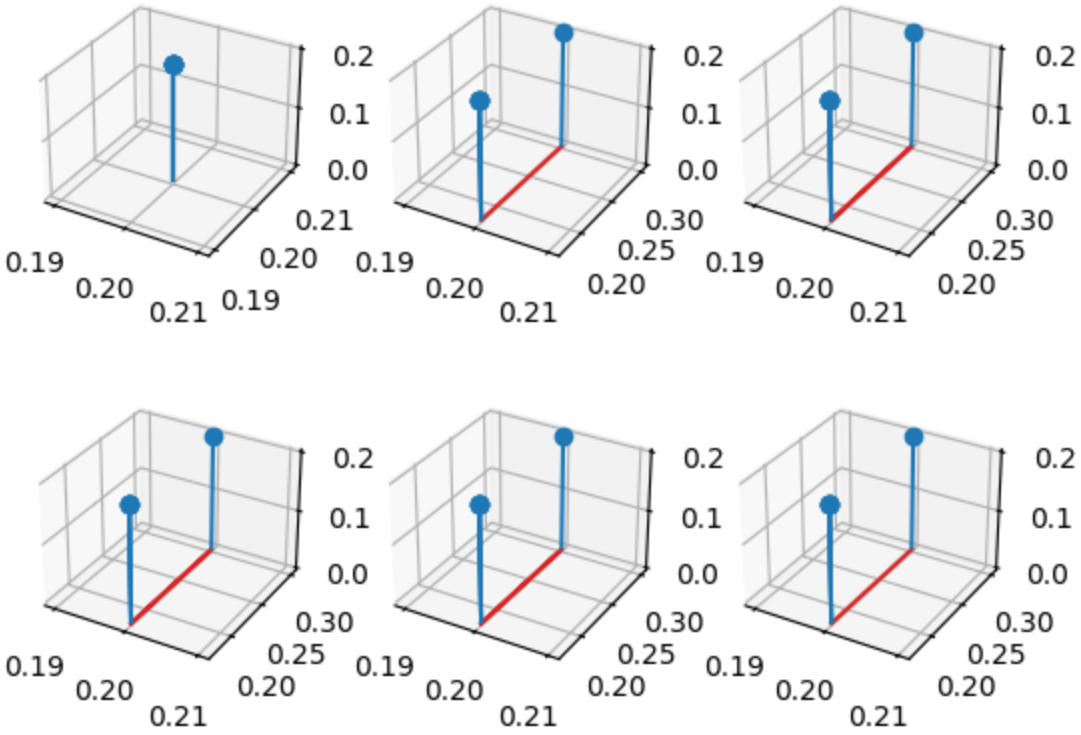

Fig 8. The different criteria weights.

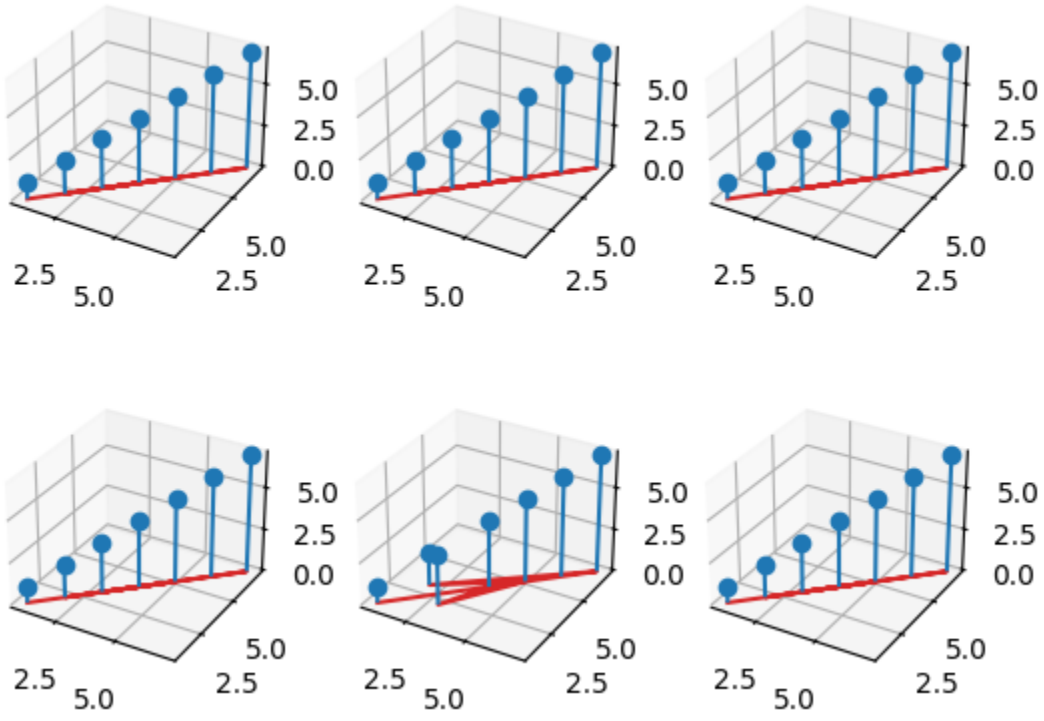

Fig 9. The different ranks of alternatives.

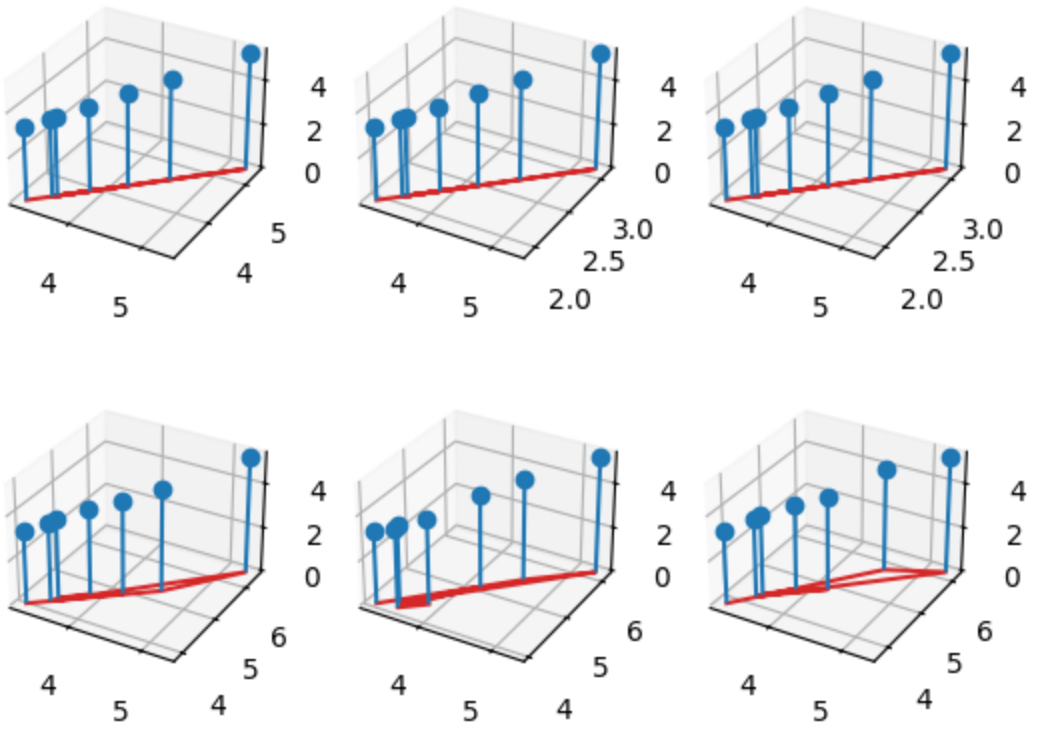

Fig 10. The different values of MOOSRA score.

5. Conclusions

The findings of this study underscore the importance of coupling and coordinating digital transportation technology with industry practices to enhance efficiency, reduce operational costs, and improve sustainability. While digital advancements such as AI-driven traffic management, autonomous vehicles, and Mobility-as-a-Service (MaaS) have shown significant potential, their effectiveness is highly dependent on industry adaptability, infrastructure readiness, and supportive policies. The research highlights the need for a structured evaluation framework, such as MCDM methodologies, to assess digital integration levels and guide decision-making in transportation development. We used two MCDM methods such as BWM to compute the criteria weights and the MOOSRA method to rank the alternatives. We used the TreeSoft set to deal with criteria and sub criteria.

Furthermore, the study reveals that disparities in digital adoption pose a major challenge, with some sectors rapidly advancing while others lag due to financial constraints, regulatory barriers, and technological gaps. Addressing these challenges requires collaborative efforts between governments, industry stakeholders, and technology providers to create scalable and adaptable digital transportation solutions. Investments in digital infrastructure, cybersecurity measures, and workforce training are essential to ensure a smooth transition toward smart mobility systems.

A key takeaway from this evaluation is the necessity of a long-term strategic approach that aligns technological innovation with industry needs. Without proper synchronization, digital transformation efforts may result in inefficiencies, fragmented systems, and missed opportunities for sustainable development. Therefore, a holistic, data-driven decision-making approach must be adopted to ensure the seamless integration of digital technology into transportation networks.

In conclusion, coordinated development between digital transportation technology and the transportation industry is a vital factor for achieving a smarter, more sustainable, and efficient mobility ecosystem. Policymakers and industry leaders must leverage systematic evaluation frameworks to identify strengths, weaknesses, and opportunities, ensuring that digital transformation in transportation leads to long-term economic, social, and environmental benefits.

Acknowledgements

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Neutrosophic Inverse Gaussian Distribution in Economic Policy Design under Indeterminacy

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Abstract: Uncertainty and indeterminacy are central concepts in economic policy design. The inherent uncertainty is difficult to incorporate into traditional statistical models. To overcome the limitation of classical inverse Gaussian (IG) distribution for managing the imprecise data, we examine the structure of neutrosophic inverse Gaussian distribution (NIGD) which is an expansion of the classical IG distribution under neutrosophic framework. The IG distribution, which is commonly employed for reliability analysis and in financial modeling, features a positively skewed curve which makes it is an appropriate index for modeling asymmetric economic data, including risk assessments, investment returns, as well as financial duration. That model includes degrees of truth, lack of information and degrees of falsity, elements that enable policymakers to compute economic variables with incomplete / imprecise information. The mean, variance, skewness and kurtosis of NIGD are derived in neutrosophic environment. The quantile function of the proposed is derived which is further utilized to generate random samples from the proposed model. The utilization of the distribution in economic uncertainty modeling is described using two numerical examples.

Keywords: Neutrosophic probability, inverse Gaussian distribution, economic policy, indeterminacy, uncertainty modeling, financial risk.

1. Introduction

Probability distributions constitute a fundamental tool of economic policy making to recent years; their application comprises uncertainties analysis, economic indicators modelling and decision support in various correlation terms [1]. Recession predictions are essentially complex; there are stochastic factors driving the economy that are beyond economic theory, such as inflation, interest rates, stock market dynamics, and others [2]. For example, policymakers use probability models to assess risk, allocate resources, and predict economic trends. However, one of the well-known models applied in financial modeling is normal distribution, particularly used in the studies of portfolio optimization and risk and the returns of the asset are mostly assumed to have a normal distribution [3]. The exponential and Weibull distributions are also used commonly in analyzing economic data, financial data and macroeconomic data modelling, which enables policymakers to assess the lifespan

of economic policies and business lifespans [4]. The Poisson distribution is another useful distribution in modeling rare economic phenomenon, like financial panics, and spikes in unemployment, helping governments plan their response to such crises [5]. The same goes for Bayesian inference with probability distributions, allowing policymakers to revise economic forecasts when new information comes into the world, thus improving the readiness of the policy structures [6]. In this way statistical distributions have long applications history in the field of economic data. One of the most useful among several other probability distributions is the IG distribution for modeling economic and financial data with positive skewness, meaning that the tail of the distribution is longer on the right side [7]. Unlike the normal distribution, which is symmetric by nature, the IG is appropriate for modelling asymmetric economic phenomena like financial returns, insurance claims, and the time-to-default events used in risk assessment of credit [8]. It is commonly used to study waiting times, so it is relevant for investigating business cycle duration, economic recovery duration, and market stabilization duration. In risk management for finance, IG distribution quantifies the likelihood of extreme losses, which are critical to banks as well as regulators, in terms of reserving capital. In industrial economics, it analyzes production processes, helping find efficient investment strategies and reducing risks of economic downturns [9]. The diverse nature of the IG distribution plays a central role in producing robust economic policies applications [10]. Fuzzy theory is an active and critical component of modern statistical inferences and data sets in programs of uncertainty, obscurity, and precision [11]. While classical statistical methods depend on exact values and point estimates of probabilities, fuzzy statistical methods provide degrees of membership rather than strictly yes or no outcomes. This is one of the reasons fuzzy statistics are particularly useful in fields such as economic, financial, or social sciences where data typically contain fuzzy or inadequate information [12]. Classical probability attempts to assign probabilities on the basis of frequency or axiomatic rules, treating all uncertainty as quantifiable in precise terms [13]. But in real-world economics issues like the prediction of inflation, modeling consumer behavior or assessing financial risk, the input for decisions often consists of linguistic or imprecise data (e.g. "high inflation", "moderate risk") that is not appropriately generated by traditional probability distributions. Fuzzy statistics are mathematical frameworks that enhance classical statistical techniques by utilizing fuzzy numbers, fuzzy sets, and theories of fuzzy logic based on inference, making it easier for policymakers and analysts to handle ambiguous data [14]. For example, in financial risk management, instead of defining rigid limits for investment risk (e.g., "low risk" below 5 percent volatility and "high risk" above 15 percent), fuzzy statistics allow for gradual transitions between levels of risk that reflect more realistic and flexible decision-making. Fittingly, also in economic policy design, especially when labor-employment rates or GDP growth needs to be forecasted, which are multiclient related uncertainties, fuzzy regression models present a better approach compared to classical regression, being able to capture the vague nature of the relationship between economic variables [15]. Another application is analyzing consumer sentiment, where responses like “somewhat satisfied” or “moderately optimistic” result in something more nuanced than fitting a data point into one of three or four arguments. Since fuzziness reflects uncertainty and linguistics captures human stories, fuzzy statistical methods are thus extremely useful for practical applicability as human implications of economic phenomena are inherently complex, subjective, and incomplete at best, providing more flexible yet stronger tools for economic policymaking under uncertainty.

The neutrosophic theory proposed by Smarandache generalizes the classical and the fuzzy logic by adding an additional component known as the “indeterminacy” to model [16-19]. Neutrosophic sets extend beyond fuzzy logic that is constrained by a single degree of membership (between 0 and 1) by introducing three independent parameters, truth (T), indeterminacy (I), and falsity (F) whose values are in the range 0 and 1 [20-22]. Neutrosophic theory has many applied areas applications including neutrosophic statistical methods. Neutrosophic statistics are more effective to study real world complex problems while data which provide solutions are ambiguous, feet and/or information

missing exist in economic policymaking. For example, in finance forecasting, a market trend may be true to an extent, but indeterminate to an extent due to the impact of unforeseen geopolitical factors and false to an extent due to obsolete data sources. Fuzzy logic provides a method for reasoning with uncertain attributes, but is limited in its ability to accommodate contradictions and incomplete knowledge. Fuzzy methods overestimate the fuzziness of their inputs, under the assumption that only small-scale, gradual transitions exist between pieces of knowledge; however, in practice, there are situations in which some information is actually lost or contradictory, and these do not fade away into a smooth gradient. An example of such a situation is outlined in economic risk assessment, where the uncertainty of the financial health of a company corresponds not even to information being deficient in scholarly documentation, but also to experts presenting contradictory opinions, which neutrosophic methods can deal with much better than traditional fuzzy models can (in this sense).

Neutrosophic statistics, on the other hand, is a generalization of classical methods of statistics composed of three components: truth, indeterminacy, and falsity, which provides it with a greater capacity for analyzing data that is incomplete, inconsistent, and uncertain [23]. Neutrosophic statistics accept uncertainty, vagueness, and paradoxes, which is a natural phenomenon that occurs in the real world due to the fuzzy setup of economic and financial decision making. Neutrosophic statistical methods differ the classical statistics in terms of facilitating indeterminate and indecisive information data [24]. In macroeconomic modeling for example, it is more appropriate to handle the data inconsistencies arising from missing or unreliable sources when using neutrosophic probability distributions. This flexibility improves uncertain decision-making and gives policymakers more robust adaptive tools than classical statistical approaches can provide. Under neutrosophic probability distributions generalize the classical probability models adding new coordinates: truth (T), indeterminacy (I) and falsity (F), making it a suitable representation of random variables with uncertainty and inconsistency data [25-26]. Neutrosophic distributions which are a generalization of the traditional probability distributions capture the ambiguity related to parameters, thus are a great tool to use in fields like economic and financial forecasting, financial risk analysis, reliability modeling, etc., [27-28]. For instance, in stock market predictions, the price increase probability is sometimes true and sometimes false and partially known knowledge to be indeterminate because of externals factors and partially is false (e.g. old information).

In this work, we study a neutrosophic variant of the classical IG distribution that assumes precise data and well-defined probabilities, which might not be the case in the real world under economic conditions. Uncertainty, ambiguity, and missing information are the hallmarks of designing economic policy due to volatile markets, incomplete datasets, and diverse views from experts. The IG distribution is a special case of NIGD where: NIG is given by the complete truth T, indeterminacy (I) and complete falsity F. Such current-to-future projection dynamic captures conditionality of structural dynamics of policy environment under risk and uncertainty and contributes to risk assessment and policy optimization, as well as robustness of decision making in the face of uncertainty.

The paper is organized as follows. The description of classical IG distribution is presented in Section 2. Section 3 discusses the proposed model. Random data generating from the proposed model is presented in Section 4. Sec. Finally, the findings of the study are presented in Section 5.

2. Preliminary Results

In this section, we will view the basic results of the classical structure of the inverse Gaussian distribution and their related functions.

A random variable is said to follow the Gaussian model if it has the following PDF and CDF functions:

$$h_Y(y \mid \beta, \theta) = \sqrt{\frac{\theta}{2\pi y^3}} \exp\left(-\frac{\theta(y-\beta)^2}{2\beta^2 y}\right), y > 0 \quad (1)$$

$$H_Y(y \mid \beta, \theta) = \Phi\left(\sqrt{\frac{\theta}{y}}\left(\frac{y}{\beta} - 1\right)\right) + \exp\left(\frac{2\theta}{\beta}\right)\Phi\left(-\sqrt{\frac{\theta}{y}}\left(\frac{y}{\beta} + 1\right)\right) \quad (2)$$

The shapes of the PDF and CDF of the NIGD are given in Figure 1 and Figure 2.

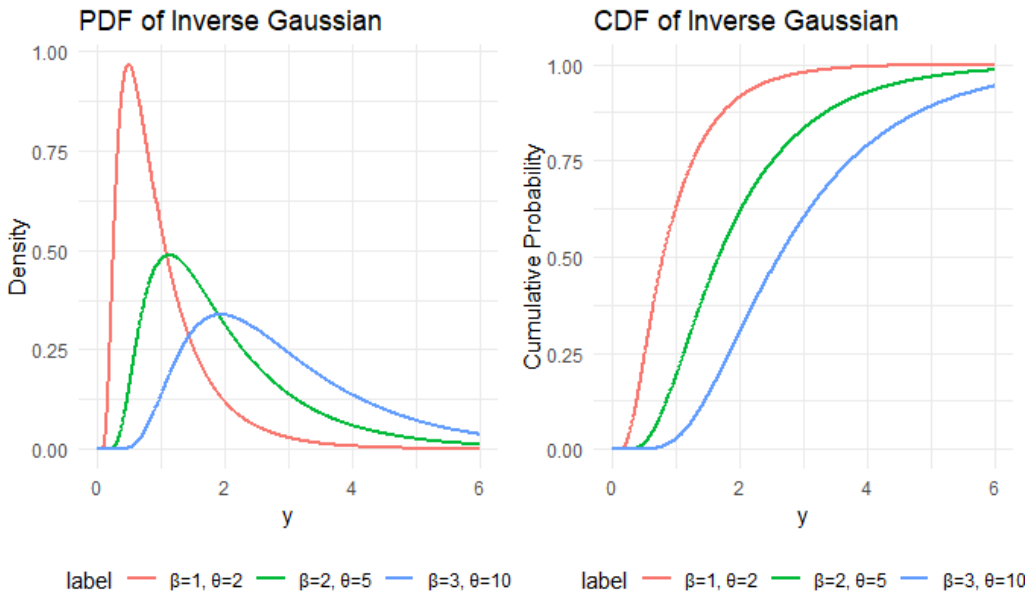


Figure 1 PDF and CDF of the NIGD with different parameters setting

Figure 1 shows the PDF and CDF function with different values of shape and scale parameters. The NIGD is a continuous probability distribution that is widely used for positively skewed data, especially as applied in reliability analysis, survival analysis, and stochastic processes. It is defined by two parameters which are the shape parameter θ and the scale parameter β. The shape parameter θ controls the skewness and spread of the distribution. For bigger values of θ the distribution is symmetric and resembles that of a normal distribution, while lower θ generates a highly skewed distribution with a long right tail. The scale parameter β defines the trajectory of central tendency, indicating how the peak moves along the domain. A larger beta shifts the distribution to the right, yielding a higher expected value, while a smaller β shifts it to the left. Because its shape is determined by both parameters, the PDF defines the likelihood of different values occurring. Knowing that, we also know from CDF that the CDF is the probability of a random variable event being less than a threshold, contributing to decision-making process scenarios.

Now other key statistical characteristics of the classical IG distribution can be established from literature as:

The mean which denotes an expected value of the random variable Y is given by:

$$E[Y] = \beta \quad (3)$$

This quantity provides essential information about the central tendency of the distribution.

To know about the spread of the distribution, variance is commonly used measure of dispersion which is given by:

$$\mathrm{Var}(Y) = \frac{\beta^3}{\theta} \quad (4)$$

The mode is also a useful measure to measure the most frequent of any distribution. It is value where we get the maximum peak of the PDF. Unlike the mean, it is not affected by the extreme values that commonly exist in the macroeconomic data. Mode of the IG distribution is given by:

$$\text{Mode} = \beta\left(\sqrt{1+\frac{9\beta}{4\theta}}-\frac{3\beta}{2\theta}\right) \quad (5)$$

Median also plays a significant role in data modelling. Median of the IG distribution can be written as:

$$\text{Median} \approx \beta\left(1-\frac{1}{3}\sqrt{\frac{\beta}{\theta}}\right) \quad (6)$$

Median is also a very important measure which is commonly recommended measure in case of skewed mode because it is also not highly affected by extreme values. It is particularly useful when data is expected to contain outliers such as real estate prices, income level making it key measure in decision-making and descriptive statistics.

Shape coefficients are also very important measures for any distribution. The skewness coefficient of the IG distribution is given by:

$$\text{Skewness} = 3\sqrt{\frac{\beta}{\theta}} \quad (7)$$

This coefficient provides information about the symmetry of the distribution. Another measure related to shape is kurtosis which can be defined as:

$$\text{Kurtosis} = \frac{15\beta}{\theta} \quad (8)$$

Kurtosis coefficient provides information about the tail thickness of the distribution.

Similarly, the moment generating function (MGF) of the IG distribution can be written as:

$$M_Y(t) = \exp\left(\frac{\theta}{\beta}\left(1-\sqrt{1-\frac{2\beta^2 t}{\theta}}\right)\right), t < \frac{\theta}{2\beta^2} \quad (9)$$

MGF has a key role in mathematical statistics and is commonly derived for generating moments. These moments can then easily be obtained by differentiating the MGF, so this is particularly handy in theoretical derivations. Also, when it exists, the MGF uniquely determines the distribution, so it is possible to identify, and compare two probability distributions. Moreover, MGF also allows the distributions of sum of independent random variables which makes it a versatile tool in applied statistics and probability inference. If we assume the different values of the shape parameter such as θ =[2.5, 5, 7.5, 10] and location parameter $\beta = [1, 2, 3, 4]$ then the mean, variance, mode and median can be computed which is provided in Table 1.

Table 1 Statistical characteristics of the IG distribution

Location	Shape	Mean	Variance	Mode	Median
1	2.5	1	0.4	0.77	0.79
2	5	2	1.6	1.55	1.57
3	7.5	3	3.6	2.33	2.36
4	10	4	6.4	3.11	3.15

Results in Table 1 show the statistical key properties of the IG distribution. Results indicate that all the values are crips numbers because we have exact knowledge about the distributional parameters. Now in the next section, statistical model of the IG distribution with key statistical properties under neutrosophic framework are presented.

3. Neutrosophic framework for IG Model

In this section, the statistical framework for the IG distribution under the neutrosophic environment is described. Under the neutrosophic environment we assume that the parameters of the distribution are interval forms and not the crisp values. The PDF and CDF of the NIGD can be written as:

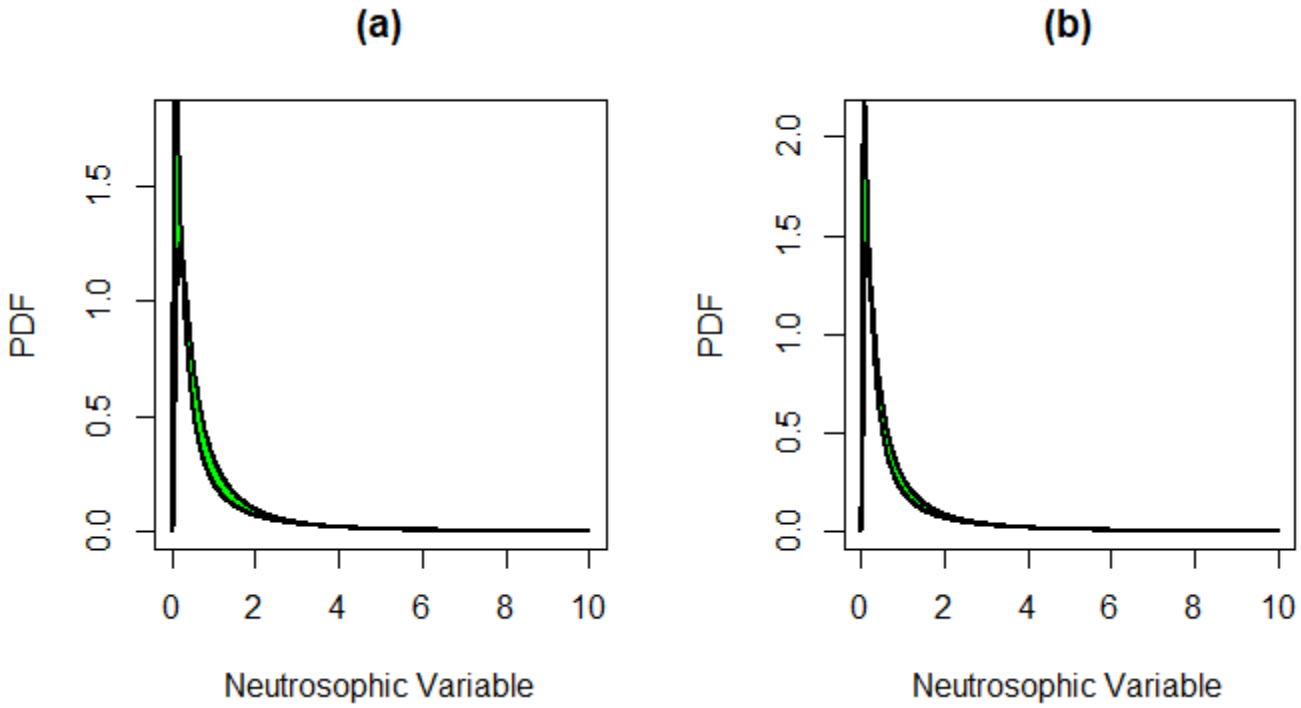
$$h_Y(y \mid \beta_N, \theta_N) = \sqrt{\frac{\theta_N}{2\pi y^3}} \exp\left(-\frac{\theta_N(y-\beta_N)^2}{2\beta_N^2 y}\right), y > 0 \quad (10)$$

$$H_Y(y \mid \beta_N, \theta_N) = \Phi\left(\sqrt{\frac{\theta_N}{y}}\left(\frac{y}{\beta_N} - 1\right)\right) + \exp\left(\frac{2\theta_N}{\beta_N}\right)\Phi\left(-\sqrt{\frac{\theta_N}{y}}\left(\frac{y}{\beta_N} + 1\right)\right) \quad (11)$$

where $\beta_N = [\beta_l, \beta_u]$ and $\theta_N = [\theta_l, \theta_u]$.
If we assume the different values of β_N and θ_N, plots of PDF and CDF are given in Figure 2 and Figure 3 respectively.

Figure 2 PDF curves of the NIGD with parameters (a) $\beta_N = [1,1]$ and $\theta_N = [1.5, 3.5]$ (b) $\beta_N = [1,1]$ and $\theta_N = [2, 3.75]$.

In Figure 2, we depict the PDF of the proposed model with varying values for the neutrosophic distributional parameters. The PDF curves are similarly right skewed in shape, consistent with the IG distribution. Then as shape parameter increases, the peak of the distribution shifts slightly to the left, and the curve becomes more spread out indicating increased variability. Also notice that the density function reaches a maximum for lower values of Y and then asymptotically decreases, resulting in a



heavy tail in the distribution. This shows that for larger location parameters, the probability of getting extreme values increases, and the distribution becomes more spread.

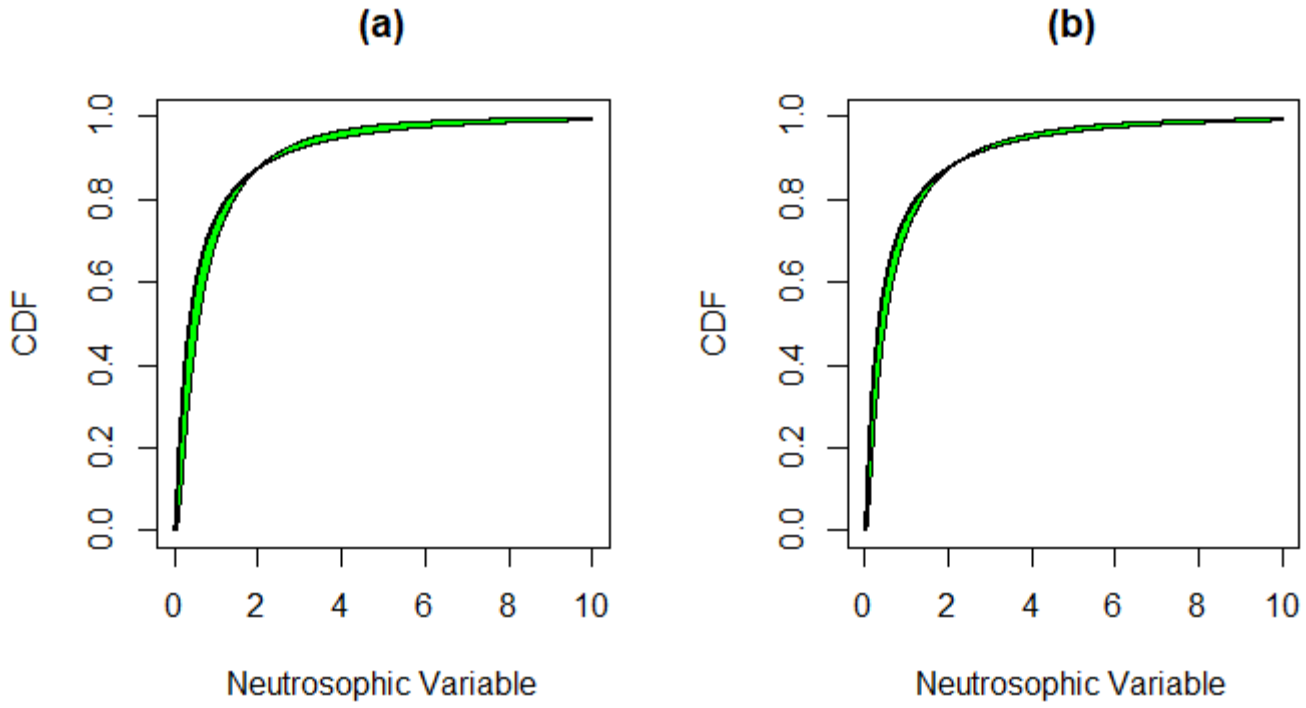


Figure 3 CDF curves of the NIGD with parameters (a) $\beta_N = [1,1]$ and $\theta_N = [1.5, 3.5]$ (b) $\beta_N = [1,1]$ and $\theta_N = [2, 3.75]$.

In Figure 3, we show the CDF of the proposed model, for the same neutrosophic parameter values. The CDF curves showing the probability that the random variable Y will be less than or equal to some given value y. As we can see, CDF is an increasing function, going to 1 when y tends to infinity. For larger shape parameter values, the CDF curve rises more slowly, indicating the greater dispersion of the distribution. Differences in CDF curves indicate that the value of shape parameters affect the distribution in overall behavior.

Mean and variance of the NIGD can be structured as follows:

$$E[Y] = \beta_N \tag{12}$$

$$\mathrm{Var}(Y) = \frac{\beta_N^3}{\theta_N} \tag{13}$$

where $\beta_N = [\beta_l, \beta_u]$ and $\theta_N = [\theta_l, \theta_u]$.

The mean and variance of the NIGD offers crucial information about the central tendency and dispersion of the distribution along with indeterminacy and uncertainty. Mean value is not fixed as it is dependent from the location parameter (β_N) which takes neutrosophic values that may be intervals or uncertain values. This versatility enables modeling of imprecise or incomplete data. The variance, defined by both the shape parameter (θ_N) and location parameter (β_N), outlines how wide the distribution is. Unlike classical IG distribution, which have well-defined mean and variance, NIGD allows for interval-based or fuzzy representations, where corresponding replications of uncertainty owned by real-world data. This makes it especially powerful in cases where data is less than perfect, open to interpretation or where more than one solution exists.

The following two examples describe the utility of the proposed model in economic policy design.

Example 1

A policy maker is interested in studying how much time it will take for an economic crises to happen in different areas. Economic conditions are still shrouded in uncertainty, and the behavior of inflation rates, interest rates as well as foreign exchange reserves varies over time, which diffuses usefulness of classical statistical models. The neutrosophic NIGD is a more flexible model that integrates truth, indeterminacy, and falsity of economic data. Let the average time until an economic crisis occurs (in years) is uncertain and estimated as given as $\theta_N = [5, 6]\, years$ and the uncertainty in economic fluctuations is modeled using the shape parameter $\beta_N = [2, 3]\, years$. Suppose the economist is interested in finding the mean and variance of the fitted model under an uncertain environment.

Using the properties of the proposed model mean and variance can be calculated as follows:

$$E(T) = \theta_N = [5, 6]\, years$$
$$var\ (T) = [41.67, 108]\, years^2$$

The time before the economic crisis is expected to be around 5 to 6 years, but because of the economic uncertainties, the predictions may vary, this variability in the predictions is captured by neutrosophic variance. High variance indicates uncertainty, so economic policy needs to be flexible in the face of fluctuations of financial stability. Because of the need to explain this imprecision in economic forecasting to policymakers, the NIGD based approach may provide important benefits to achieving their goals of risk management and responsive policymaking.

Example 2

A bank wants to know how long its customers, on average, will take to fully repay their loan. But volatile economic circumstances cast uncertainty across factors such as inflation, job security and interest rate movements and mean the period of payment is indefinite. Rather than a classical model, the bank uses the neutrosophic NIGD to account for this uncertainty. Assume that repayment time data follow the neutrosophic NIGD with $\theta_N = [7, 9]\, years$ and $\beta_N = [3, 4]\, years$.

Using the characteristics of the proposed model, mean and variance can be computed as:

$$E(T) = \theta_N = [7, 9]\, years$$
$$var\ (T) = [85.75, 243]\, years^2$$

It could take 7 to 9 years to pay back, although predicting it is imprecise because of economic uncertainties. Higher variance (in the 85.75 to 243 years²) indicates large swings which would imply risk adjusted interest rates for borrowers to the bank. This methodology aids in improved risk assessment, lending to financial institutions to be flexible about policies by adjusting them around economic uncertainty.

Likewise, the mean, other measures used for central location are median and mode which can be defined as in context of neutrosophic environment.

$$\text{Median}(Y) \approx \beta_N \left(1 - \frac{1}{3}\sqrt{\frac{\beta_N}{\theta_N}}\right) \quad (14)$$

$$\text{Mode}(Y) = \beta_N \left(\sqrt{1 + \frac{9\beta_N}{4\theta_N}} - \frac{3\beta_N}{2\theta_N}\right) (\quad 15)$$

Understanding the characteristics of data through the neutrosophic median and neutrosophic mode of the NIGD is particularly useful in cases involving uncertainty or incomplete information. The expression for median indicates that it is controlled by both the scale (θ_N) and shape (β_N) parameter. The neutrosophic median, unlike the mean which could be influenced by extreme values, is more resistant to outliers and indeterminate deviations, and so well suited for cases where data is not easy to acquire with accuracy or involve extreme values such as economic forecasting or financial market Analysis. The mode, on other hand, is the highest value of the probability density function, can be tuned by β_N and θ_N. In context of economic policy both median and mode can assess in decision making under uncertainty environment. Neutrosophic extensions enable more flexible, adaptive, and uncertainty-

resilient policies for policymakers to mitigate risk, assess impacts more comprehensively, and plan for economically robust responses in unpredictable market environments.
If the neutrosophic values of the distribution parameters are assumed, then key statistics of the NIGD can be calculated as given in Table 2.

Table 2 Statistical characteristics of the NIGD under imprecise parameters settings

β_N	θ_N	Mean	Variance	Mode	Median
[0.5, 0.5]	[2, 2.5]	[0.5, 0.5]	[0.05,0.062]	[0.452,0.463]	[0.425,0.455]
[0.5, 0.5]	[4, 75]	[0.5, 0.5]	[0.026,0.031	[0.477, 0.477]	[0.445, 0.472]
[0.5, 0.5	[6, 6.68]	[0.5, 0.5	[0.018, 0.021]	[0.482, 0.484]	[0.454, 0.479]
[0.5, 0.5	[8, 8.95]	[0.5, 0.5	[0.013,0.015]	[0.485,0.488]	[0.461, 0.483]

Results in Table 2 show that when we have imprecision in the underlying parameters, all the statistical characteristics are involved in imprecision.
Shape coefficients of the proposed model can be defined as:

$$\text{Skewness} = 3\sqrt{\frac{\beta_N}{\theta_N}} \quad 16)$$

$$\text{Kurtosis} = 15\frac{\beta_N}{\theta_N} \quad (17)$$

As we can see NIGD is like the classic form and the coefficients of skewness and coefficient of kurtosis describes it asymmetric and peakedness nature along with elements of uncertainty, indeterminacy and inconsistency. These coefficients become interval-based or fuzzy in accordance with the incomplete or imprecise conditions of data in a neutrosophic environment. The skewness coefficient measures asymmetry around central tendency. Indeed, skewness coefficient in classical form shows a specific number, whereas, in the neutrosophic setting, they can assume several possible values as there exists the uncertainty of data here, and thus, this is helpful in evaluating economic inequality. In income distribution, for example, neutrosophic skewness can provide decision-makers with the extent of potential income differences under uncertain conditions and can lead to more flexible tax policies or the means-tested adjustment of welfare programs. On the contrary, kurtosis is the coefficient of the heaviness of tails of the distribution. In standard statistics, this is an exact number, but in neutrosophic domain it considers indeterminate variations in the number, thus presenting a more valid perspective of real-world instances like financial crisis or market crash. This is incredibly useful in economic risk management, where decision-makers need to prepare for a variety of potential extreme scenarios, rather than relying on a single expected outcome. Including neutrosophic measures of skewness and kurtosis allows for improved responsiveness of economic policies and financial models to unpredictable environmental conditions, facilitating more effective risk management and sustainable growth strategies.

4 Quantile Function

In probability theory, the quantile function of a probability distribution is a function that describes the distribution in terms of an arbitrary quantile. In other words, it is useful for ascertaining the value of a random variable for a certain cumulative probability. Essentially, the quantile function provides us with thresholds or cut-off points (eg. the value below which 10% of the data sit or the value above 25% of the data). It is particularly useful in areas like risk management, economics, and decision-making, where it is important to understand the behavior of data at its extreme values at its worst-case or best-case scenario. It allows you to know the corresponding value for a desired percentile, in an inverse

relationship to the CDF, which is vital to calculate measures based on percentiles that are enormously useful in making decisions in uncertain environments. The quantile function of the proposed model can be written as:

$$Q(p \mid \beta_N, \theta_N) = \beta_N \left(1 + \sqrt{\frac{9\beta_N^2}{4\theta_N} + \left(1 - \frac{4p}{\theta_N}\right)^2}\right) \quad (18)$$

This quantile function can be used to generate random samples that assumed to follow NIGD by using inverse CDF method. In this method, inverse transformation approach is used, and we generate uniformly distributed random numbers between 0 and 1 and then use the quantile function with respect to the following CDF of random samples from NIGD. We generate random numerical values between 0 and 1. These probability values are then applied to the quantile function to obtain neutrosophic NIGD samples. Doing this several times, we can produce many samples of NIGD random variable. This is particularly useful because it provides a way to generate samples from the distribution of a random variable when the distribution itself is complex and difficult to sample from. Monte Carlo simulations are beneficial when closed-form expressions of the random variable are not accessible or when the distribution is complex (e.g., in the case of the NIGD).

To generate the random samples, we have used a program written in R making use of some functions from the R library "statmod". Using these functions, we can obtain samples for the NIGD using the quantile function method. Random samples obtained from this approach are shown in Table 3.

Table 3 Random numbers generated from the proposed model using Monte Carlo experiment

Random numbers
[0.854, 1.123], [0.765, 1.045], [0.923, 1.204], [0.812, 1.112], [0.876, 1.156], [0.794, 1.098], [0.873, 1.134]
[0.902, 1.187], [0.789, 1.043], [0.865, 1.152], [0.847, 1.129], [0.888, 1.203], [0.823, 1.097], [0.904, 1.214]
[0.911, 1.176], [0.765, 1.054], [0.856, 1.126], [0.822, 1.093], [0.931, 1.223], [0.875, 1.153], [0.897, 1.190]
[0.877, 1.145], [0.853, 1.118], [0.928, 1.217], [0.802, 1.075], [0.872, 1.138], [0.794, 1.056], [0.883, 1.164]
[0.894, 1.194], [0.872, 1.136], [0.910, 1.199], [0.765, 1.051], [0.842, 1.115], [0.801, 1.089], [0.932, 1.228]
[0.919, 1.204], [0.812, 1.097], [0.866, 1.143], [0.839, 1.110], [0.926, 1.211], [0.794, 1.068], [0.903, 1.190]
[0.884, 1.172], [0.803, 1.085], [0.918, 1.201], [0.866, 1.150], [0.879, 1.146], [0.799, 1.081], [0.902, 1.185]

Results in Table 3 show interval value samples from the proposed model at parameters setting $\beta_N = [1,1]$ and $\theta_N = [1.5, 2.5]$.These random samples are generated using specific seed value 222 in the program. If we assume that indeterminacy in the dispersion parameter is zero, yielded samples would coincide with the classical approach of the IG distribution.

5 Conclusions

Based on classical inverse Gaussian distribution and neutrosophic logic, the neutrosophic inverse Gaussian distribution (NIGD) has been described in this study. The suggested model presents a powerful alternative to its classical statistical variant with a view to capture uncertainty and imprecision in economic data. This allows the neutrosophic NIGD to characterize asymmetric processes by truth, information deficit, and falsity, leading to a versatile continuum to analyze economic processes with a focus on risk measures, return on investment, and financial time scales. The key statistical properties of the proposed model have been discussed both in theoretical and practical aspects. The proposed model extended the classical IG distribution to better account for partial or imprecise information when trying to model economic variables which are intimately linked to proper policy design. The quantile function of the proposed model has been derived further

utilized to random samples from the proposed model. Numerical findings indicated that the proposed model is a versatile model to capture uncertainty data that often occurs in financial risk management and economic growth data modeling.

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A New SDIMSIM Methods with Optimized Region of Stability and Sustainable Development with Applications On Neutrosophic Data

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Abstract: General Linear Methods (GLMs) were introduced as logical extensions of the conventional Runge–Kutta and linear multistep methods. In the scenario when second derivatives, in addition to first derivatives, can be computed, a GLM modification known as SGLMs (GLM with second derivative) was created. The development of SDIMSIMs for the second type is presented in this study. For SDIMSIMs construction, we give examples with parameters (p, q, r, s), namely p = q = r = s = 1..., 4, where r is the number of external stages, s is the number of internal stages, p is the order, and q is the stage order. Furthermore, we will prove that those approaches are the largest area contained by the A-Stability and L-Stability properties; in addition to their mathematical importance, SDIMSIMs contribute widely to sustainable development by unlocking complex systems in climate modeling, renewable energy, ecological monitoring, epidemiological studies, etc. SDIMSIMs address this challenge by enhancing the numerical performance of these simulations and thus contributing to decreased resource consumption, lower energy demand and better computational sustainability. In this paper, we conduct construction and stability certificate for SDIMSIMs, and the supporting capacity of SDIMSIM for supporting sustainable solutions to global issues is certified. Also, we apply our study on some neutrosophic data with indeterminacy element (I), and compare

the results.

Keywords: SDIMSIMs; L-stability; SGLM, neutrosophic data.

1. Introduction

The second derivative Various variants of the classical Runge–Kutta and linear multistep methods were suggested to be naturally generalized by the General Linear Methods (SGLMs) ([1], [3], [6], and [7]), e.g., integro-differential terms, differential terms within the main formula, and so forth, alternative methods were also suggested in terms of solving differential algebraic systems ([9],[11],[12],[13]). In medical applications such as ([4],[5]), a different approach was used. They are the cutting edge with wide applications in diverse fields important for sustainable development. In climate modeling, it improves the precision and efficacy of the simulations, allowing for the better predictions of long-term climate trends, and aiding climate adaptation strategies (Allen & Ingram, 2002) in [2]. In Ecological research, SDIMSIM offer advanced simulation of complex interactions within ecosystems, facilitating biodiversity conservation. Neutrosophic sets are very useful in the study of real life problems [15], Smarandache presented those sets in [14], and then, they were applied in many different areas of science and theoretical studies [16-19].

The SGLMs extend the framework of the GLMs for numerically solving autonomous ordinary differential equations:

$$y' = f\big(y(x)\big), \quad y: \mathbb{R} \to \mathbb{R}^m, \quad f: \mathbb{R}^m \to \mathbb{R}^m$$

An SGLM for the system is expressed as:

$$\begin{aligned} Y^{[n]} &= h(A \otimes I_m) f\big(Y^{[n]}\big) + h^2(A' \otimes I_m) g\big(Y^{[n]}\big) + (U \otimes I_m) y^{[n-1]}, \\ y^{[n]} &= h(B \otimes I_m) f\big(Y^{[n]}\big) + h^2(B' \otimes I_m) g\big(Y^{[n]}\big) + (V \otimes I_m) y^{[n-1]}, \end{aligned}$$

with coefficient matrices represented as:

$$\begin{bmatrix} A & A' & U \\ B & B' & V \end{bmatrix}$$

For stability analysis, we consider the linear test problem:

$$y' = \xi y$$

where $\xi \in \mathbb{C}$, and stability is governed by the matrix $M(z)$:

$$M(z) = V + z(B + zB')(I_s - zA - z^2A')^{-1}U$$

The $M(z)$ characteristic polynomial is given by:

$$p(w, z) = \det\big(wI_r - M(z)\big) \qquad (1)$$

2. Order Condition Structure

This section revisits the order conditions for the methods, as presented in [5]. The key approach involves an input vector in the following form:

$$y_i^{[n-1]} = \sum_{k=0}^{p} h^k \alpha_{ik} y^{(k)}(x_{n-1}) + O(h^{p+1}), \quad i = 1, 2, \dots, r \qquad (2)$$

where α_{ik} are real parameters, and $y_i^{[n]}$ represents the approximation at stage i for the integration point n. The stage values $Y_i^{[n]}$ at step size h are needed to be of order q approximations of the solution at $x_{n-1} + c_i h$:

$$Y_i^{[n]} = \sum_{k=0}^{p} \frac{c_i^k}{k!} h^k y^{(k)}(x_{n-1}) + O(h^{q+1}), \quad i = 1,2, \dots, s \qquad (3)$$

Additionally, the computed values at the end of the current step for output should satisfy:

$$y_i^{[n]} = \sum_{k=0}^{p} h^k \alpha_{ik} y^{(k)}(x_n) + O(h^{p+1}), \quad i = 1,2, \dots, r$$

Let $\alpha_k = [\alpha_{1k}\ \alpha_{2k}\ \dots\ \alpha_{rk}]^T$ for $k = 0,1, \dots, p$. Here, α_0 and α_1 are recognized as the pre-consistency and consistency vectors, respectively (refer to [4]). Let $Z := [1\ z\ \dots\ z^p]^T \in \mathbb{C}^{p+1}$, and define the matrix W as:

$$W = [\alpha_0\ \alpha_1\ \dots\ \alpha_p]$$

We now present the following theorem:

Theorem 1.

Assume that equation (1) is satisfied by $y^{[n-1]}$. Equations (2) and (3) are then satisfied by the SGLM system of order p and stage order q=p if and only if:

$$\exp(cz) = zA\exp(cz) + z^2 A'\exp(cz) + UWZ + O(z^{p+1}),$$

$$\exp(z)WZ = zB\exp(cz) + z^2 B'\exp(cz) + VWZ + O(z^{p+1}).$$

The product WZ is the same vector w(z) with a polynomial-valued shape as explained in [5], where the exponential function is applied element-wise to a vector.

When $U = I_s$, in Theorem 1 the stage order conditions will be satisfied if and only if:

$$W = C - ACK - A'CK^2,$$

Where the matrix of Vandermonde is $C = \left(C_{ij}\right) \in \mathbb{R}^{s\times(p+1)}$, with coefficients given by:

$$C_{ij} = \frac{c_i^{j-1}}{(j-1)!}, \quad 1 \le i \le s,\ 1 \le j \le p+1,$$

the matrix of shifting $K \in \mathbb{R}^{(p+1)\times(p+1)}$ is described by $K = \left[0\ e_1\ \ldots\ e_p\right]$, where e_j is the j-th vector of unity (see [5]).

3. Construction of SDIMSIMs

Let $r = s$, and $U = I_s$. The coefficient matrix B is computed as:

$$B = B_0 - AB_1 - AB_2 - VB_3 - (B - VA)B_4 + VA$$

The matrices B_0, B_1, B_2, B_3, and B_4 are defined as:

$B_0 = \int_{1+c_i}^{1} \phi_j\,(x)dx, \quad B_1 = \phi_j(1+c_i), \quad B_2 = \phi_j{}'(1+c_i), \quad B_3 = \int_0^{c_i} \phi_j\,(x)dx,$
As in [1] and [6].

a) SDIMSIMs of Type 2 (with $p = q = r = s = 1$)

As given in Theorem 1, the SDIMSIMs order of type 2 is less than 3. The matrix of Coefficients for the method of order 2 is given by:

$$\begin{bmatrix} \lambda & \mu & 1 \\ 1 & \mu & 1 \end{bmatrix}.$$

With abscissae vector $c = [1]^T$, coefficients are:

$$\begin{bmatrix} 0.131631473311373 & 0.004702134734068 & 1 \\ 1 & 0.004702134734068 & 1 \end{bmatrix}.$$

The area of the stability region is $S = -175.7794$, where the sign is negative because the area is on the left side. Figure 1 displays it in contrast to Figures 2, 3 the works of A. Abdi and G. Hojjati etcetera in ([6], [8]).

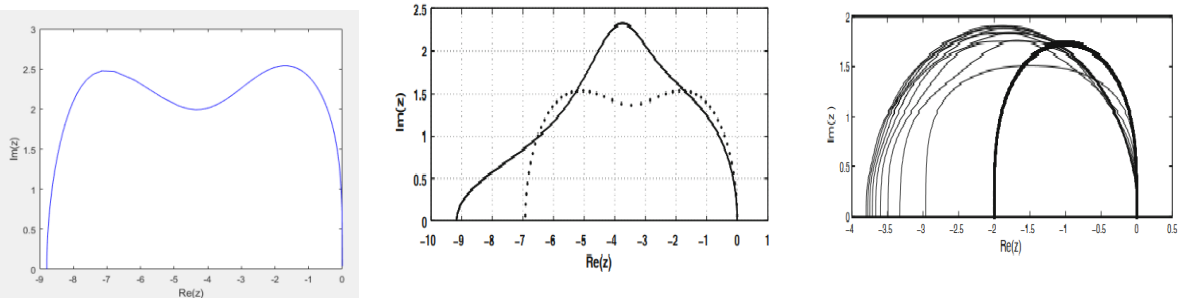

Figure 1: We compare the maximal area of the stability region for this paper's method of order 1 on the left side, the method at [6] in the middle and [8] on the right.

b) SDIMSIMs of Type 2 (with $p = q = r = s = 2$)

By selecting the abscissae vector $c = [-1,1]^T$, coefficients for the method of order 2 at least are:

$$A = \begin{pmatrix} 0.275562183474736 & 0 \\ 3.193741017519553 & 0.275562183474736 \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 0.0124050750 & 0 \\ 0.191395813953893 & 0.0124050750 \end{pmatrix}$$

$$V = (0.836540100273188 \quad 0.1634598997)$$

$$\hat{B} = (0.04166288324 \quad 0.002027732315)$$

$$B = \begin{pmatrix} 1.171861588 & -0.0124001620 \\ 0.05535242576 & 0.1285704909 \end{pmatrix}$$

The area of the stability region is $S = -164.5267$. Figure 4 displays the stability region in contrast to Figures 5 and 6 from the works ([6],[8]).

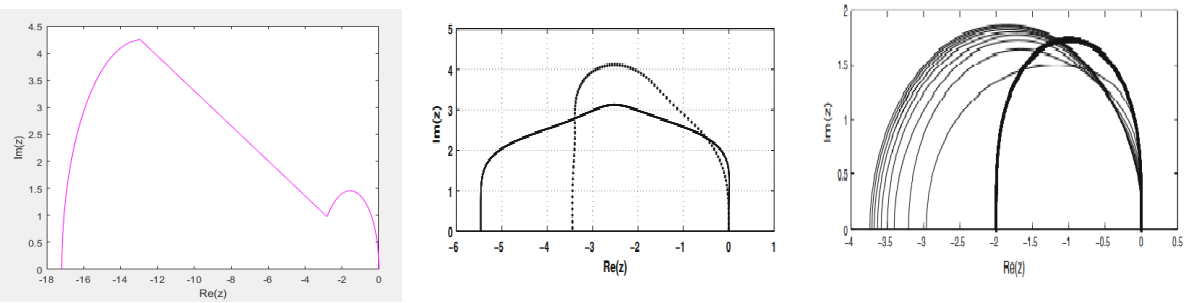

Figure 2: The method of order 2's maximum region of stability on the left side, [6] of the same order

at the middle and [8] to the right side.

c) SDIMSIMs of Type 2 (with $p = q = r = s = 3$)

By choosing abscissae vector $c = [-1, 0.222610896326282, 0.997211035225582]^T$, coefficients of the method of order 3 are given by:

$$A = \begin{pmatrix} 0.185949019058918 & 0 & 0 \\ -0.029251166425091 & 0.185949019058918 & 0 \\ 0.180266217726395 & 0.762611999618729 & 0.185949019058918 \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 0.015294685296723 & 0 & 0 \\ -0.004313489973732 & 0.015294685296723 & 0 \\ 0.069788314022775 & 0.175636485019311 & 0.015294685296723 \end{pmatrix}$$

$$V = (0.227767612412112 \quad 0.188598515058830 \quad 0.583633872529058)$$

$$\hat{B} = (0.04340094012 \quad 0.1053919568 \quad 0.008926496410)$$

$$B = \begin{pmatrix} -0.4401257211 & -0.1020157260 & -0.4736454833 \\ 0.8117363413 & 1.149846336 & 0.7782165791 \\ 0.614207097 & 0.952317092 & 0.580687335 \end{pmatrix}$$

Area: $S = -65.6410$. See Figure 7 in compare to 8,9 from ([1],[7]).

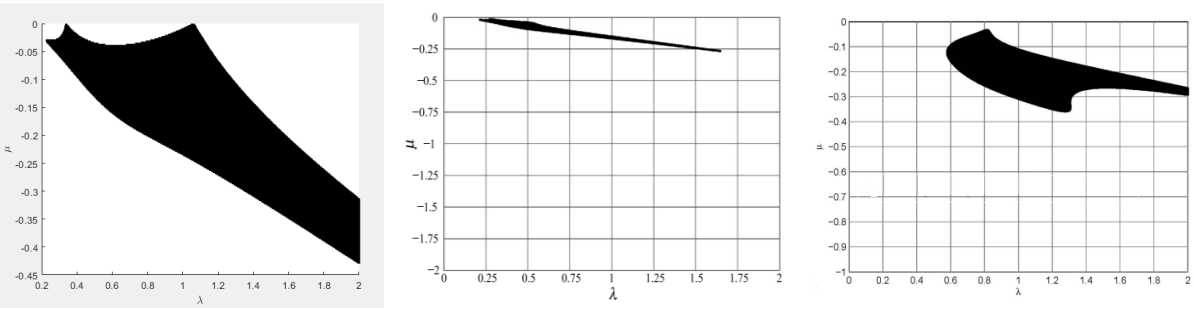

Figure 3: The area of L-stability for methods of order 3 to the left side, [1] at the middle and the work of [7] to the right side.

d) SDIMSIMs of Type 2 (with $p = q = r = s = 4$)

By choosing abscissae vector $c = [-1, 0, 0.5, 1]^T$, coefficients of the method of order 4 are given by:

$$A = \begin{pmatrix} -0.047332764332787 & 0 & 0 & 0 \\ 0.991453136844788 & -0.047332764332787 & 0 & 0 \\ 0.759226970197592 & 0.537348733117443 & -0.047332764332787 & 0 \\ 0.680653072026738 & 0.838351438070254 & 0.194437929981731 & -0.047332764332787 \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 0.105956042213680 & 0 & 0 & 0 \\ 0.640558958225514 & 0.105956042213680 & 0 & 0 \\ 0.711684069906730 & 0.130936627137292 & 0.105956042213680 & 0 \\ 0.771001762263965 & 0.271738039373422 & 0.166677332416951 & 0.105956042213680 \end{pmatrix}$$

$$V = (0.352677084796130 \quad 0.754516757290226 \quad 0.097257807025064 \quad -0.204451649)$$

$$\hat{B} = (0.4322649863 \quad 0.03712292834 \quad -0.02377240315 \quad -0.02166288755)$$

$$B = \begin{pmatrix} 1.221887526 & 0.4009782561 & 0.5114759583 & 0.5655098563 \\ 1.230434390 & 0.4095251193 & 0.5200228215 & 0.5740567195 \\ 1.425311823 & 0.6044025528 & 0.7149002550 & 0.7689341530 \\ 1.508445086 & 0.687535816 & 0.798033518 & 0.852067416 \end{pmatrix}$$

Area: $S = -37.8217$. See Figure 10 in compare to 11, 12 from ([1], [7]).

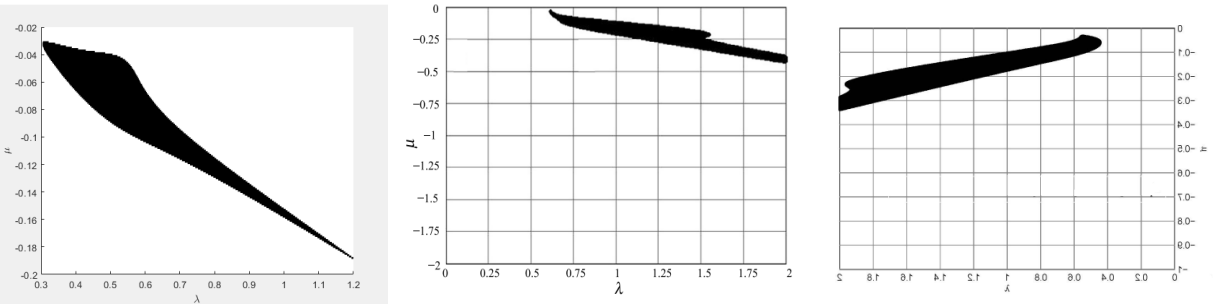

Figure 4: The area of L-stability for methods of order 4 to the left side, the work [1] at the middle and [7] to the right side.

Application on Neutrosophic Data

a) SDIMSIMs of neutrosophic Type 2 (with $p = q = r = s = 2 + I$)

By selecting the abscissae vector $c = [-1 + I, 1 + I]^T$, coefficients for the method of order 2 at least are:

$$A = \begin{pmatrix} 0.275562183474736 + I & I \\ 3.193741017519553 + I & 0.275562183474736 + I \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 0.0124050750 + I & I \\ 0.191395813953893 + I & 0.0124050750 + I \end{pmatrix}$$

$$V = (0.836540100273188 + I \quad 0.1634598997 + I)$$

$$\hat{B} = (0.04166288324 + I \quad 0.002027732315 + I)$$

$$B = \begin{pmatrix} 1.171861588 + I & -0.0124001620 + I \\ 0.05535242576 + I & 0.1285704909 + I \end{pmatrix}$$

The area of the stability region is $S = -164.5267 + I$.

b) SDIMSIMs of Type 2 (with $p = q = r = s = 3 + I$)

By choosing abscissae vector $c = [-1, 0.222610896326282 + I, 0.997211035225582 + I]^T$, coefficients of the method of order 3 are given by:

$$A = \begin{pmatrix} 0.185949019058918 + I & 1 + I & 2 + I \\ -0.029251166425091 + I & 0.185949019058918 + I & 1 + I \\ 0.180266217726395 + I & 0.762611999618729 + I & 0.185949019058918 + I \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 0.015294685296723 + I & I & I \\ -0.004313489973732 + I & 0.015294685296723 + I & I \\ 0.069788314022775 + I & 0.175636485019311 + I & 0.015294685296723 + I \end{pmatrix}$$

$$V = (0.227767612412112 + I \quad 0.188598515058830 + I \quad 0.583633872529058 + I)$$

$$\hat{B} = (0.04340094012 + I \quad 0.1053919568 + 3I \quad 0.008926496410 + 2I)$$

$$B = \begin{pmatrix} -0.4401257211 + 3I & -0.1020157260 + I & -0.4736454833 + I \\ 0.8117363413 + I & 1.149846336 + 3I & 0.7782165791 + 2I \\ 0.614207097 + 5I & 0.952317092 + I & 0.580687335 + 5I \end{pmatrix}$$

Area: $S = -65.6410 + I$.

c) SDIMSIMs of Type 2 (with $p = q = r = s = 4 + I$)

By choosing abscissae vector $c = [-1, 0, 0.5, 1]^T$, coefficients of the method of order 4 are given by:

$$A = \begin{pmatrix} -0.047332764332787 + I & 2I & 2I & 6I \\ 0.991453136844788 + I & -0.047332764332787 + I & 4I & 2I \\ 0.759226970197592 + I & 0.537348733117443 + I & -0.047332764332787 + I & 9I \\ 0.680653072026738 + I & 0.838351438070254 + I & 0.194437929981731 + I & -0.047332764332787 + I \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 0.105956042213680 & 3 + 2I & 1 + 2I & 7I \\ 0.640558958225514 & 0.105956042213680 & 5 + 2I & 2I \\ 0.711684069906730 & 0.130936627137292 & 0.105956042213680 & 7 + 2I \\ 0.771001762263965 & 0.271738039373422 & 0.166677332416951 & 0.105956042213680 \end{pmatrix}$$

$$V = (0.352677084796130 + I \quad 0.754516757290226 + I \quad 0.097257807025064 + I \quad -0.204451649 + I)$$

$$\hat{B} = (0.4322649863 + I \quad 0.03712292834 + I \quad -0.02377240315 + I \quad -0.02166288755 + I)$$

$$B = \begin{pmatrix} 1.221887526 + I & 0.4009782561 + 3I & 0.5114759583 + 4I & 0.5655098563 + I \\ 1.230434390 + 2I & 0.4095251193 + I & 0.5200228215 + I & 0.5740567195 + 5I \\ 1.425311823 + I & 0.6044025528 + 13I & 0.7149002550 + 7I & 0.7689341530 + I \\ 1.508445086 + 7I & 0.687535816 + I & 0.798033518 + I & 0.852067416 + 9I \end{pmatrix}$$

Area: $S = -37.8217 + I$

4. Conclusion

This paper introduces a new class of second derivative methods, the SDIMSIMs, within the framework of General Linear Methods (GLMs), emphasizing their potential to contribute to sustainable development. By offering enhanced stability and computational efficiency, SDIMSIMs are vital for solving complex differential equations across various fields, from climate science and renewable energy to ecology and public health. The ability to simulate large-scale systems more efficiently directly contributes to sustainability goals by reducing energy consumption and optimizing resources.

The integration of SDIMSIMs into these critical domains supports the development of sustainable solutions to pressing global challenges. Moving forward, these methods will continue to play a crucial role in advancing scientific research, particularly in the quest for sustainable solutions to environmental and health crises.

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Advanced Combined Multi-Criteria Decision-Making Methodology with IndetermSoft Set for Comprehensive Analysis of Blended Physical Education Teaching in Colleges Framework

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Abstract: Prioritizing and weighing criteria is a crucial step in the decision-making process for many qualities. The stepwise weight assessment ratio analysis (SWARA) is the most often used multi-criteria decision-making (MCDM) weighting methods. The SWARA approach is the most effective way to evaluate criteria when policies are at the center of decision-making. Expert consultation is used to prioritize the criteria in the first phase, and the weighing procedure is the second crucial step. By adding the reliability assessment of experts' ideas as the first stage, this study aims to use SWARA method to compute the criteria weights of Blended Physical Education Teaching in Colleges. Then we use the root assessment method (RAM) to rank the alternatives. IndetermSoft set is used in this study to deal with indeterminacy and uncertainty in the values of the criteria. This study uses five criteria and ten alternatives to be evaluated.

Keywords: Multi-Criteria Decision Making; IndetermSoft Set; Blended Physical; Education; Colleges Teaching.

1. Introduction

Blended physical education (PE) teaching in colleges has emerged as a modern instructional approach, integrating both digital learning tools and traditional in-person training to enhance student engagement and performance. This hybrid model aims to combine the benefits of online instructional materials, interactive simulations, and virtual coaching with hands-on physical training, teamwork, and skill development. The rise of digital technology in education has enabled instructors to extend their learning beyond the gymnasium, allowing students to review techniques, track progress, and receive personalized feedback through digital platforms. However, the effectiveness of this approach varies significantly based on factors such as course design, technological infrastructure, instructor proficiency, and student adaptability[1], [2].

Evaluating the effectiveness of blended PE teaching requires a comprehensive framework that assesses both cognitive and physical learning outcomes. Unlike conventional PE programs that

primarily focus on physical activities, blended models emphasize conceptual understanding, self-paced learning, and technology-assisted performance tracking. Key evaluation criteria include student engagement, retention of practical skills, the adaptability of digital resources, physical fitness improvement, and instructor competency in integrating technology. A well-structured evaluation ensures that students not only develop their physical abilities but also acquire theoretical knowledge that enhances their overall understanding of exercise science, sports psychology, and biomechanics[3], [4].

Despite its advantages, blended PE teaching faces several challenges that need to be addressed for effective implementation. Variability in student participation, digital literacy, and access to high-quality online resources can create disparities in learning experiences. Additionally, not all sports and physical activities can be effectively taught through digital platforms, as some require direct instructor supervision and real-time feedback. The transition from traditional, instructor-led coaching to a hybrid approach demands that educators develop new teaching strategies, integrate emerging technologies, and balance online learning with hands-on training[5], [6]. Proper assessment of these challenges allows institutions to refine their blended PE programs, ensuring a well-rounded, inclusive, and engaging learning environment for all students.

The effective evaluation of blended PE teaching plays a crucial role in shaping future educational strategies. By using data-driven evaluation methods, student feedback, and performance metrics, institutions can identify areas for improvement and optimize curriculum design. As blended learning continues to evolve, colleges must embrace innovative assessment frameworks, including Multi-Criteria Decision-Making (MCDM) models, to quantify teaching effectiveness and enhance student outcomes. A well-structured evaluation system will ensure that blended PE teaching remains a dynamic, impactful, and sustainable approach, fostering both academic excellence and physical well-being in higher education[7], [8].

The MCDM theories unquestionably began with the development of the simple additive weighting (SAW) approach. To choose the best alternative or alternatives, the number of options is assessed based on several criteria. A decision matrix with criteria and options is the simplest way to represent this kind of decision-making problem. It is necessary to weigh and normalize the matrix. Ranking the options and choosing the best one, that is, the most significant or desirable—is the final goal. How to determine the weights of the choice criteria is one of the most crucial elements of every decision-making issue.

Therefore, several approaches have been developed to solve various issues with the decision-making process. Developing existing techniques in various aspects, developing new ways to increase the capacity to adapt to various demands, and generating new concepts in the field, such dynamic MCDM created in the previous ten years, are the three principles that comprise this growth[9], [10].

The contributions of this study are organized as follows:

We proposed an MCDM model for Analysis Blended Physical Education Teaching in Colleges.

Two MCDM methods are used such as SWARA methodology to compute the criteria weights and the RAM method to rank the alternatives.

IndetermSoft Set is used to dela with indeterminacy in the criteria values of this study.

This study uses five criteria and ten alternatives to be evaluated by three experts and decision makers.

1.1 Research Motivation

Blended Physical Education (PE) teaching has gained popularity in colleges due to its ability to combine traditional face-to-face instruction with digital learning tools. This hybrid approach enhances student engagement, provides flexibility, and allows for personalized learning experiences. However, evaluating the effectiveness of blended PE programs remains a challenge. Many factors, such as student participation, digital adaptability, skill retention, physical fitness improvements, and instructor competency, influence learning outcomes. Traditional assessment methods lack a structured framework to measure these factors effectively. Additionally, decision-making in education often involves uncertainty and subjective judgments, making it difficult to derive clear conclusions. This research aims to bridge this gap by introducing a structured, data-driven evaluation method to assess the success of blended PE teaching in colleges.

1.2 Research Objective

The goal of this study is to develop a Multi-Criteria Decision-Making (MCDM) framework to assess and improve blended PE teaching in colleges. This framework integrates two powerful decision-making techniques:

1. Stepwise Weight Assessment Ratio Analysis (SWARA) – Used to determine the importance of different evaluation criteria.
2. Root Assessment Method (RAM) – Applied to rank different teaching alternatives based on expert evaluations.

Additionally, the study incorporates the IndetermSoft Set theory to handle uncertainties and imprecise values in the decision-making process. By combining these techniques, the proposed framework provides a structured and reliable method for evaluating blended PE programs, helping educators and policymakers make informed improvements to teaching strategies.

1.3 Gaps in Existing Methods

Although blended PE teaching offers numerous benefits, current evaluation methods face several limitations:

1. Lack of a systematic framework – Most existing studies do not provide a clear decision-making structure to assess multiple influencing factors.
2. Challenges in dealing with uncertainty – Many evaluation models struggle with ambiguous or incomplete data, leading to less reliable results.
3. Limited integration of hybrid decision-making techniques – Traditional methods often focus on a single evaluation technique, which limits their ability to provide accurate and well-balanced rankings.

4. Difficulty in assessing both qualitative and quantitative aspects – Most models focus on numerical performance metrics while ignoring subjective factors such as teaching effectiveness and student adaptability.

To overcome these challenges, this study introduces a hybrid approach that integrates SWARA, RAM, and IndetermSoft Set theory to create a more comprehensive and adaptable evaluation model.

1.4 Research Contribution

This study makes several key contributions to the field of educational decision-making and blended learning evaluation:

1. Proposes a hybrid MCDM-based evaluation framework specifically designed for assessing Blended PE Teaching in Colleges.
2. Introduces IndetermSoft Set theory to handle uncertainty in expert judgments, improving the reliability of the evaluation process.
3. Combines SWARA and RAM methodologies to provide a more accurate and structured ranking system for different teaching approaches.
4. Develops a new assessment model that evaluates five key criteria and compares ten different teaching alternatives, offering a comprehensive performance analysis.
5. Demonstrates the effectiveness of the proposed framework through real-world case studies, validating its ability to enhance educational decision-making in higher education.
6. Provides valuable insights for educators, administrators, and policymakers to refine blended PE curriculum design and improve teaching effectiveness.

2. IndetermSoft Set

P(H) is the powerset of H, H is a non-empty subset of U, and U is a universe of discourse. Assume that and is an attribute and that A is a collection of its values. An IndetermSoft Set (Function) is a function F: A → (H) regarding the values of one or more attributes, there is some ambiguity in set A.

There is some indeterminacy in P(H).

- ✓ Alternatively, F(v) = indeterminate (uncertain, ambiguous, or not unique) if there is at least one attribute value $v \in A$.
- ✓ Or any combination of the three circumstances.

Smarandache [11], [12] defined an IndetermSoft Set as a soft set that has a specific amount of indeterminate (ambiguous, uncertain, alternative, contradictory) data or technique.

3. SWARA-RAM Methodology

This section is divided into two main steps. In the first step, we show the steps of the SWARA method to compute the criteria weights. In the second step, we show the steps of the RAM method to rank the alternatives.

First Step

In this step, we apply the SWARA method [13], [14].

Initial ranking of the criteria.

The criteria are ranked based on their importance by experts and decision makers.

Determine the coefficient of the criteria for each expert and decision maker.

$$Y_j = \begin{cases} 1 & if\ j = 1 \\ x_j + 1 & if\ j > 1 \end{cases} \quad (1)$$

Compute the initial weight of criteria

$$r_j = \begin{cases} 1 & if\ j = 1 \\ \frac{r_j}{y_j} & if\ j > 1 \end{cases} \quad (2)$$

Compute the relative weights of criteria.

$$w_j = \frac{r_j}{\sum_{j=1}^{n} r_j} \quad (3)$$

Second Step

In this step, we apply the RAM method [15], [16].

Build the decision matrix

$$D = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}; i = 1, \dots m; j = 1, \dots, n \quad (4)$$

Normalize the decision matrix

$$q_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \quad (5)$$

Compute the weighted decision matrix.

$$s_j = q_{ij} w_j \quad (6)$$

Compute the sum of weighted values for beneficial and non-beneficial criteria such as:

$$Z_{+i} = \sum_{j=1}^{n} S_{+ij} \quad (7)$$

$$Z_{-i} = \sum_{j=1}^{n} S_{-ij} \quad (8)$$

Compute the overall value of each alternative.

$$F_i = \sqrt[2+Z_{-i}]{2 + Z_{+i}} \quad (9)$$

Rank the alternatives using the value of F_i

4. Implementation Process of MCDM Approach

This section shows the implementation of two MCDM methods. We have three experts to evaluate the criteria and alternatives. They use the scale between 0.1 to 0.9. They evaluate the criteria and alternatives based on their experience. We use five criteria of this study such as: Interactive Learning Engagement (High), Adaptability of Digital Learning Resources (Excellent), Practical Skill Retention (Strong), Physical Fitness Improvement (Significant), Instructor's Digital Teaching Competency (Expert, Basic). The alternatives show ten colleges.

First Step

We show the results of the SWARA method to compute the criteria weights.

We initial ranking the criteria.

We determine the coefficient of the criteria for each expert and decision maker using Eq. (1).

Eq. (2) is used to compute the initial weight of criteria

Eq. (3) is used to compute the relative weights of criteria as shown in Fig 1.

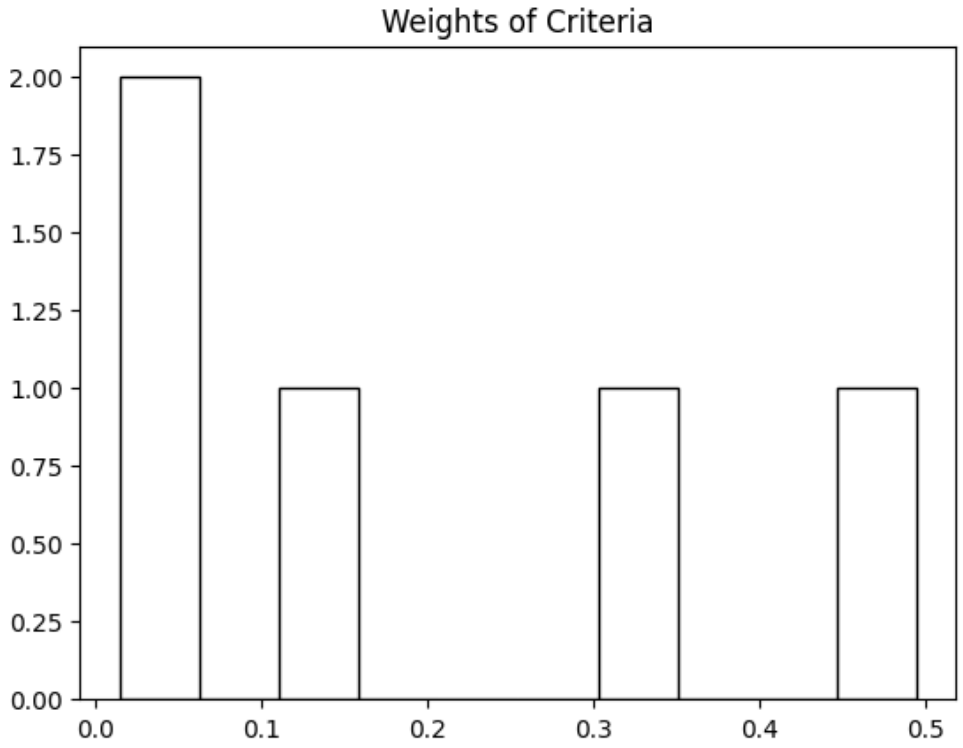

Fig 1. The importance of each criterion.

Second Step

In this step, we show results of the RAM method. This study uses the IndetermSoft set to deal with indeterminacy of the sub criteria. This study has indeterminacy in fifth criterion only. So, we apply the RAM method under these values in each step. In the first stage, we selected the Expert as a sub criterion. In the second stage, we select Basic as a sub criterion. In the third stage, we select Expert and Basic as sub criterion.

In the first stage

Eq. (4) is used to build the decision matrix. Then we combine the decision matrix as shown in Fig 2.

Eq. (5) is used to normalize the decision matrix as shown in Fig 3.

Eq. (6) is used to compute the weighted decision matrix as shown in Fig 4.

Eq. (7 and 8) are used to compute the sum of weighted values for beneficial and non-beneficial criteria.

Eq. (9) is used to compute the overall value of each alternative as shown in Fig 5.

Then we rank the alternatives using the value of F_i.

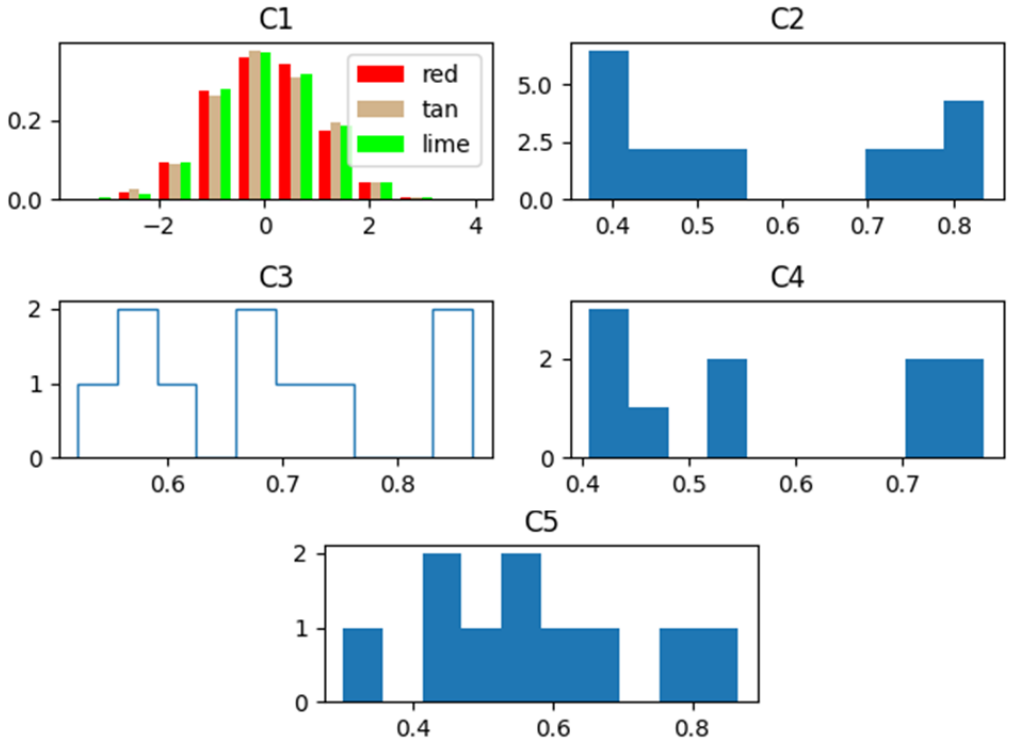

Fig 2. The decision matrix.

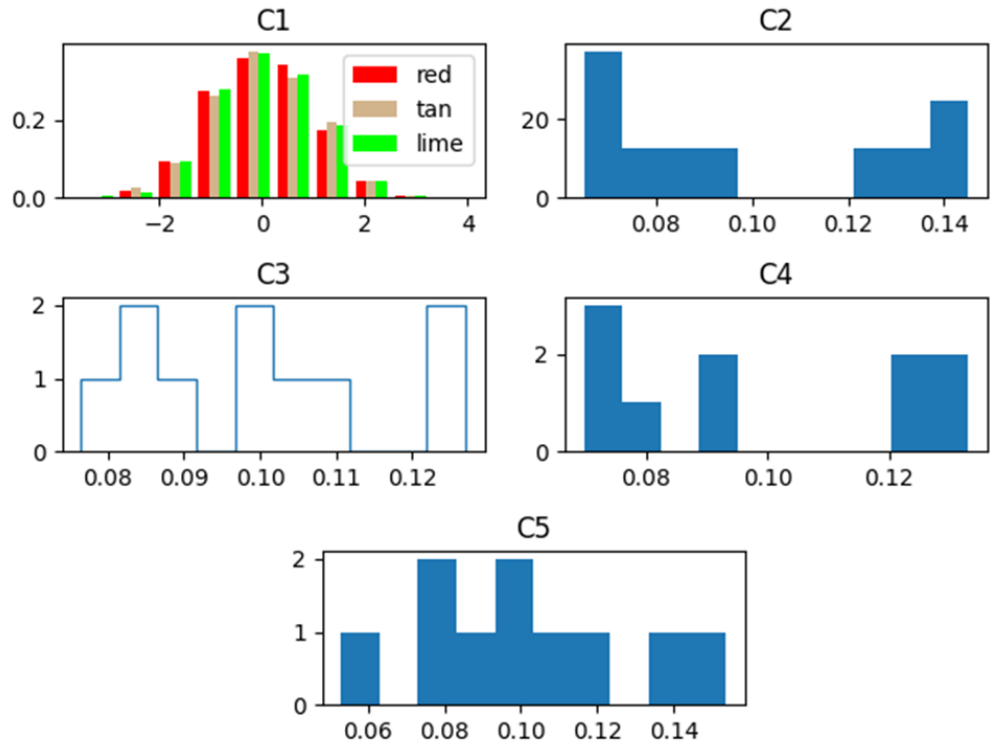

Fig 3. Normalized values.

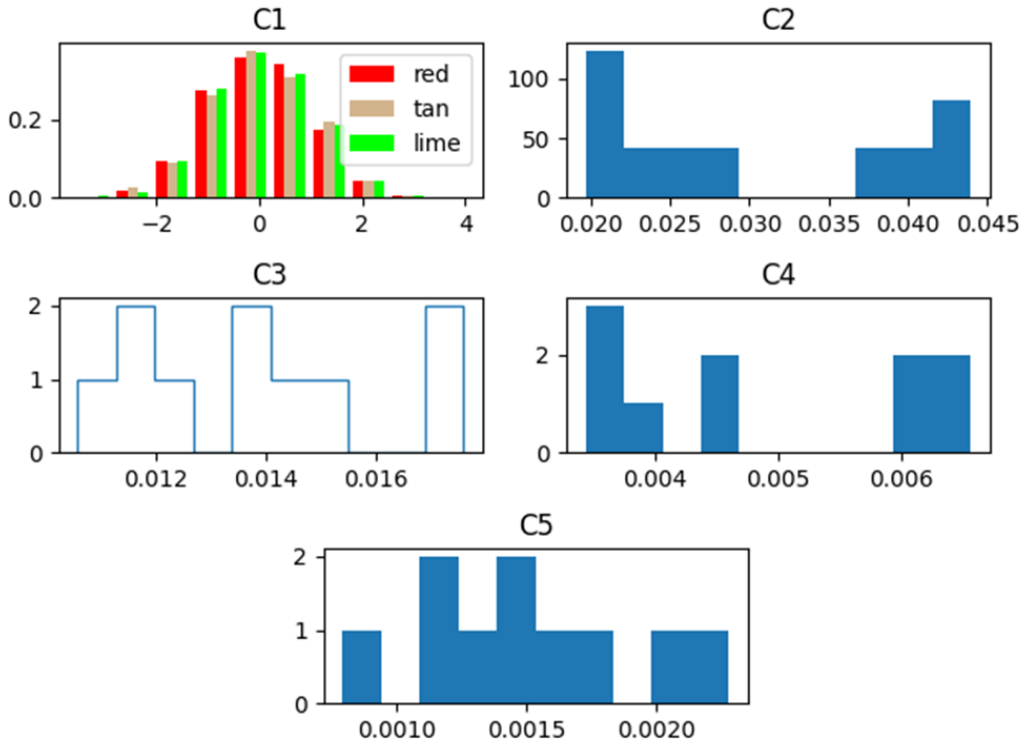

Fig 4. Weighted decision matrix.

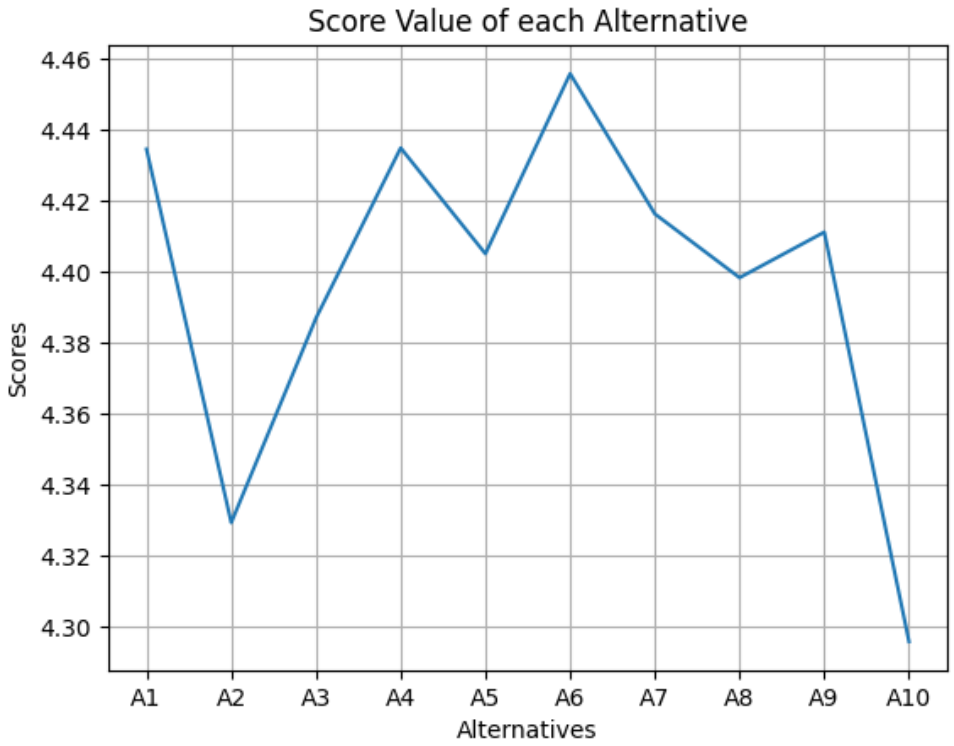

Fig 5. Overall score of each alternative.

In the second stage

We normalize the decision matrix as shown in Fig 6.

We compute the weighted decision matrix as shown in Fig 7.

We compute the overall value of each alternative as shown in Fig 8.

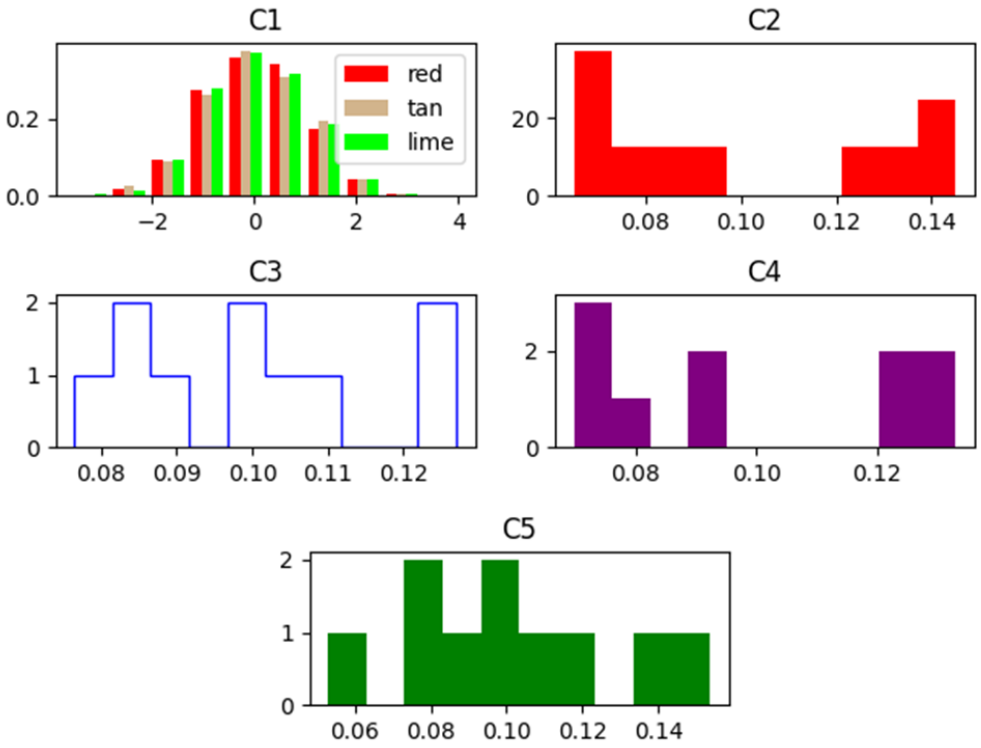

Fig 6. Normalized values.

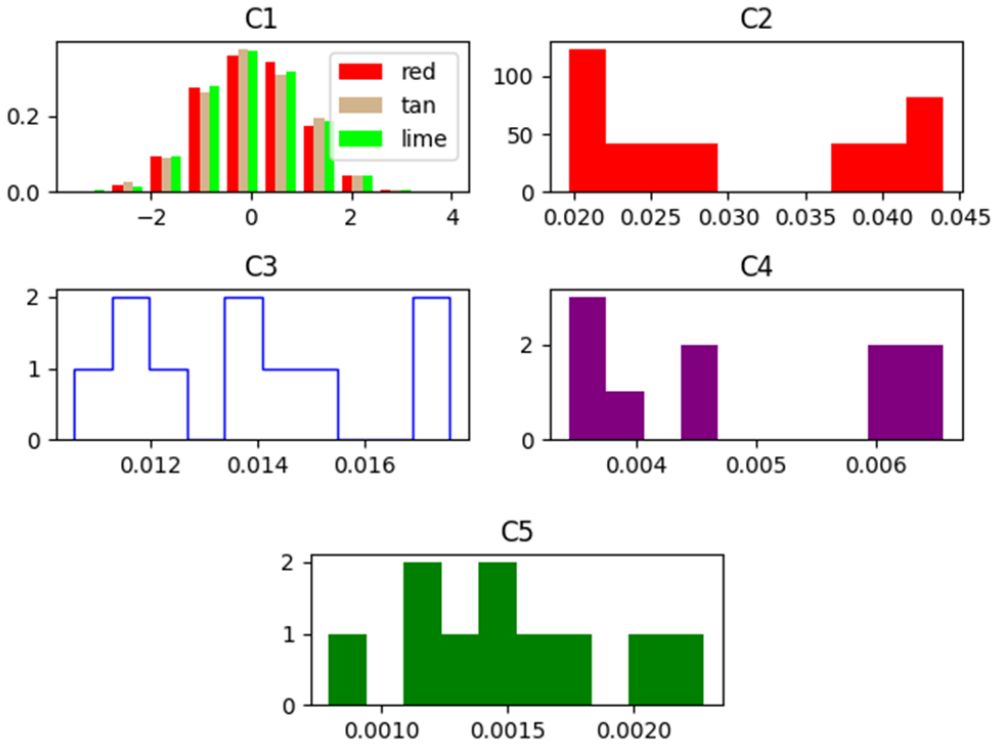

Fig 7. Weighted decision matrix.

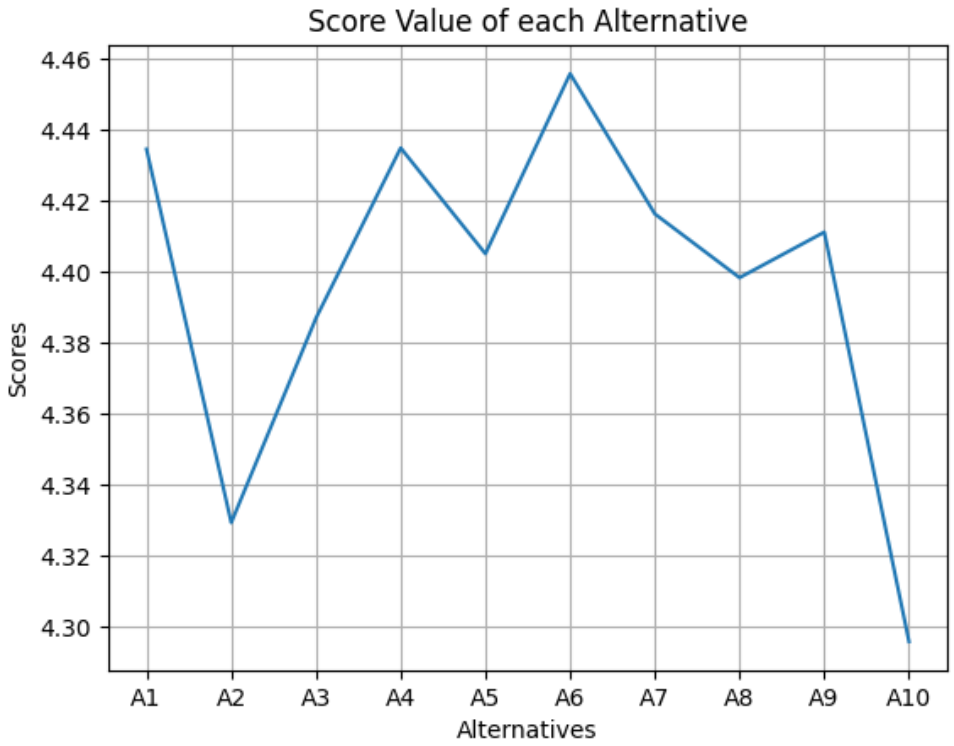

Fig 8. Overall score of each alternative.

In the third stage

We normalize the decision matrix as shown in Fig 9.

We compute the weighted decision matrix as shown in Fig 10.

We compute the overall value of each alternative as shown in Fig 11.

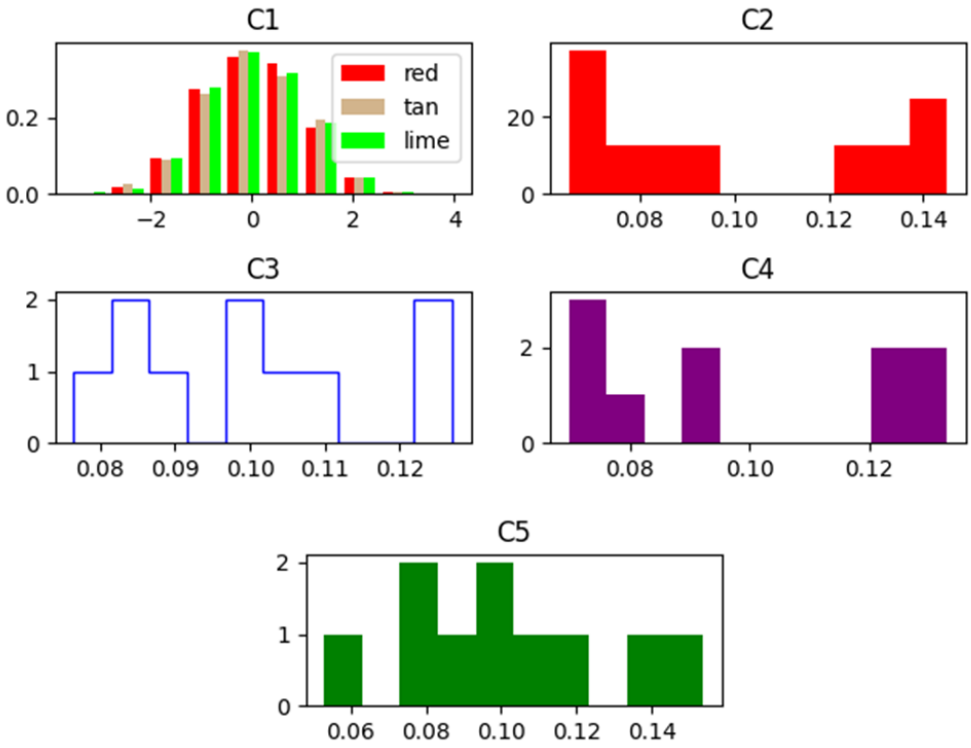

Fig 9. Normalized values.

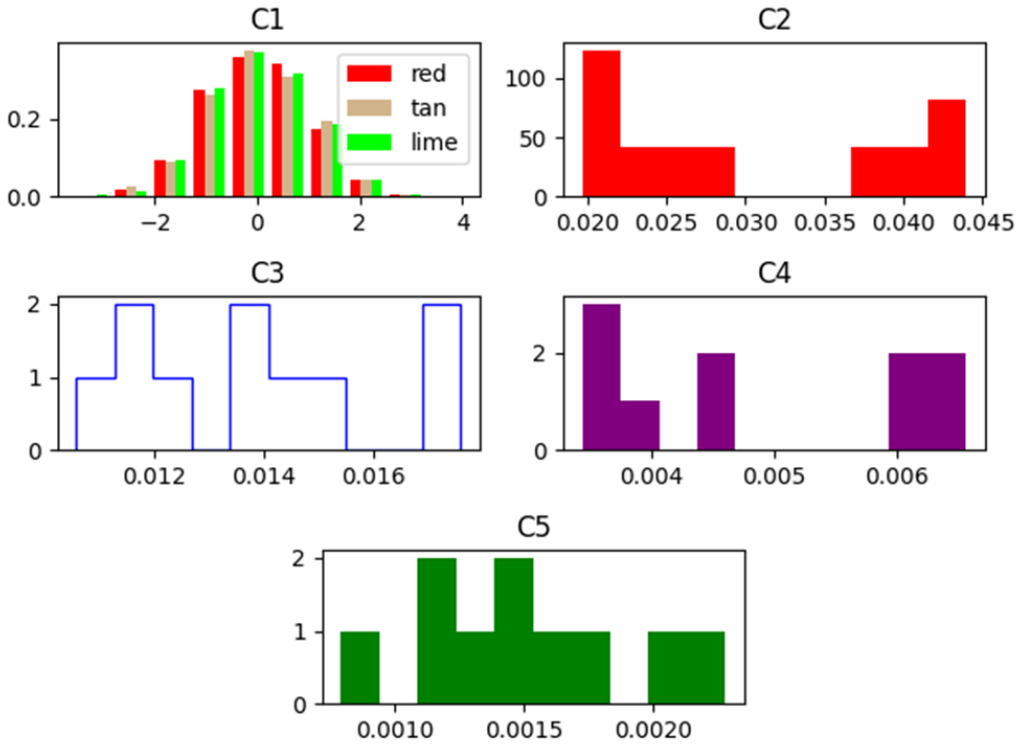

Fig 10. Weighted decision matrix.

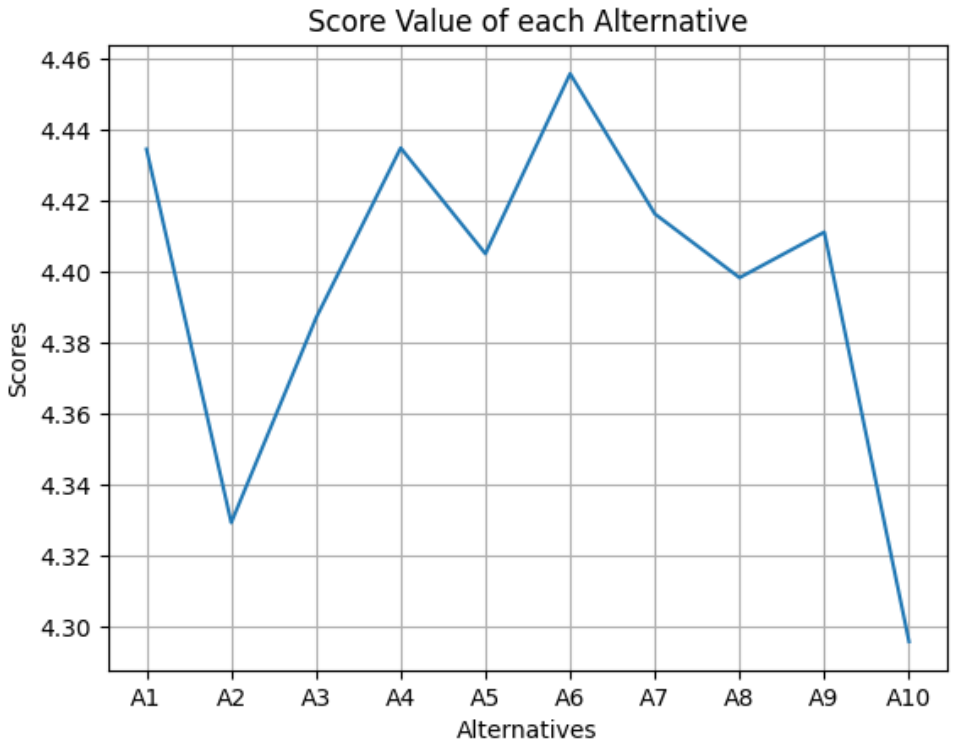

Fig 11. Overall score of each alternative.

Finally, we ranked the alternatives based on three stages as shown in Fig 12. We show the alternative 6 is the best and alternative 10 is the worst.

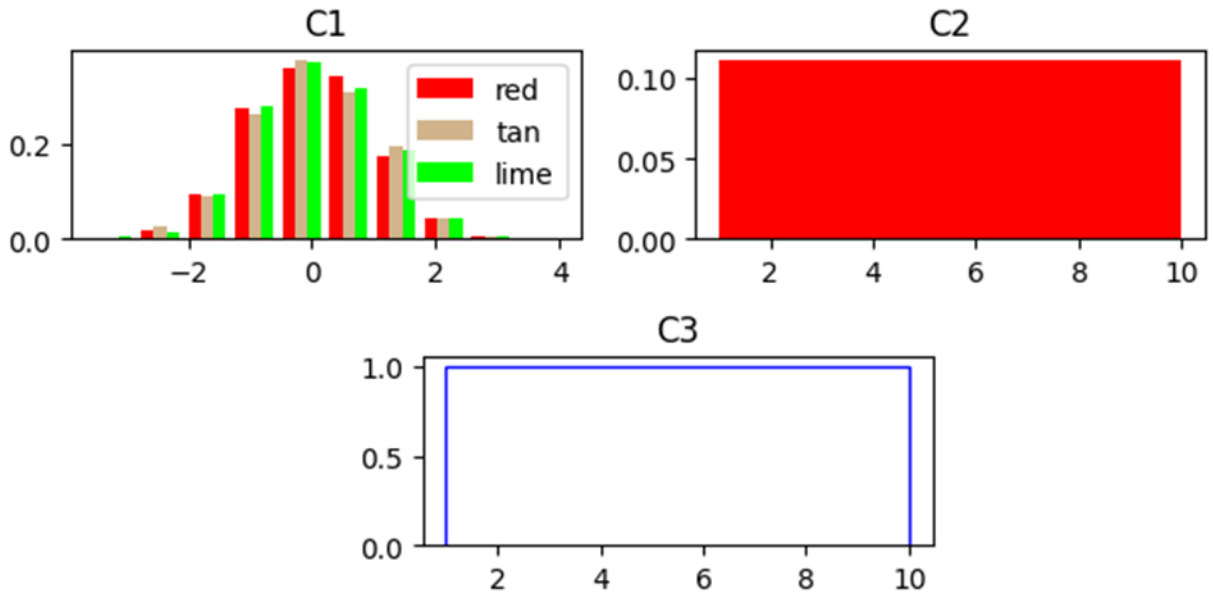

Fig 12. Final ranks of alternatives.

5. Conclusions and Future Directions

Making important decisions and setting policies are top priorities for any organization. A key challenge has always been establishing a systematic approach to decision-making and applying effective techniques to resolve complex issues. To achieve successful outcomes, structured and well-planned methods should be used. Multi-Criteria Decision-Making (MCDM) is considered a valuable tool for making critical decisions, especially in complex environments. Among MCDM techniques, Stepwise Weight Assessment Ratio Analysis (SWARA) is recognized as a practical and effective method for weighing and prioritizing criteria in decision-making. In this study, we applied the Root Assessment Method (RAM) to rank alternatives in the evaluation of Blended Physical Education Teaching in Colleges. Additionally, the IndetermSoft Set was used to handle uncertainty in sub-criteria, ensuring a more reliable and robust decision-making process. The study was conducted using five evaluation criteria and ten alternatives, with results indicating that alternative 6 performed the best, while alternative 10 had the lowest ranking. Future Directions are:

- Expanding the model to evaluate other educational disciplines and hybrid learning programs.
- Integrating artificial intelligence and machine learning to improve decision-making accuracy.
- Testing the framework across multiple institutions to enhance its scalability and adaptability.
- Incorporating real-time data analytics to refine decision-making based on live student performance metrics.
- Exploring additional hybrid decision-making methods to further optimize evaluation processes.

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Neutrosophic Quasi-XLindley distribution with applications of COVID-19 data

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Abstract: The Quasi-XLindley distribution (QXLD) is widely used in the field of survival and reliability engineering to simulate lifespan data in different fields of human, electronic designs and other fields. However, when dealing with uncertain data, a more generalized version of this distribution is needed. To address this, a neutrosophic Quasi-XLindley distribution (NQXLD) is developed in this paper. The NQXLD is particularly useful for representing skewed uncertain data. In this study, we present some statistical characteristics of the NQXL distribution, including the neutrosophic mean time failure, neutrosophic hazard rate, neutrosophic moments, and neutrosophic survival function. We also evaluate the parameters using the maximum likelihood (ML) estimation technique in a neutrosophic context based on a simulation study. Finally, applications of three different real data sets are considered to investigate the applicability of the suggested NQXL distribution. The results show the flexibility of the NQXL distribution in fitting various types of COVID-19 data as compared to the QXLD.

Keywords: Neutrosophic; COVID-19; Maximum likelihood estimation; Probability distribution; Quasi-XLindley distribution, Uncertainty

1. Introduction

The use of lifetime probability models has revolutionized analytical tools and is now widespread across various fields., such as renewable energies, health sciences, biology and engineering. Over the past century, numerous vital distributions have been derived, as the Quasi-XLindley distribution suggested by [20] being one of the most widely used regarding reliability. Its usefulness has transformed the applied sciences, inspiring researchers to discover new and innovative ways to leverage this powerful analytical toolset. The probability density function (PDF) of the QXLD is given by:

$$f(x;\alpha,\theta)=\left\{\frac{\theta}{1+\alpha}\left(\alpha+\frac{\theta(1+x)}{1+\theta}\right)e^{-\theta x}\right\},\ x>0,\alpha,\theta>0. \tag{1}$$

The cumulative distribution function (CDF) of the QXLD is defined as:

$$F(x;\alpha,\theta)=1-\left(1+\frac{\theta x}{(1+\theta)(1+\alpha)}\right)e^{-\theta x}. \tag{2}$$

Statistical distributions are an essential tool for modeling data with exact and determinate observations, but they become unsuitable when there is uncertainty in the observations or distribution parameters. In such cases, fuzzy-logic-based distributions can be applied to datasets.

The fuzzy Rayleigh distribution suggested by Pak et al. [23] is a perfect example of this. It has been extensively studied and successfully employed for reliability analysis and biomass pyrolysis by Dhaundiyal and Singh [14]. Van Hecke has explored the estimation of a fuzzy Rayleigh distribution in [31], and Pak et al. [24] have estimated its parameter for fuzzy lifetime data. Shafiq et al. [27] provided comprehensive work on distribution reliability concerns using the fuzzy method. Chaturvedi et al. [11] have used a fuzzy method to analyze hybrid censored data. Therefore, it is safe to conclude that fuzzy-logic-based distributions offer a powerful alternative to classical probability models when uncertainty exists in the observations or distribution parameters.

The rise of neutrosophic distributions has been a significant development in recent years. These distributions are designed to simulate phenomena with ambiguous observations. Several researchers have proposed the best neutrosophic probability distributions to analyze these data sets. Alhabib et al. [5] have proposed neutrosophic Poisson, neutrosophic exponential, and neutrosophic uniform distributions. Alhasan and Smarandache [18] have introduced a wide range of neutrosophic distributions, including the neutrosophic Weibull, Rayleigh, three-parameter Weibull, five-parameter Weibull, beta Weibull, and inverse Weibull distributions. Patro and Smarandache [24] proposed neutrosophic normal and binomial distributions. Aslam [10] suggested the neutrosophic Raleigh distribution, which has been used to model wind speed data. The neutrosophic Beta distribution was proposed by Sherwani et al. [28], while the neutrosophic Kumaraswamy distribution was proposed by Ahsan-ul-Haq [1]. Aslam [7] studied the Weibull distribution under indeterminacy and used it to construct the sampling strategy for testing average wind speed. More recently, Ahsan-ul-Haq and Zafar [2] proposed a neutrosophic discrete Ramos-Louzada distribution. Duan et al. [16] suggested neutrosophic exponential distribution and investigated it in modeling and applications for complex data. Khan et al. [19] considered a development of the neutrosophic gamma distribution. Eassa et al. [17] proposed neutrosophic generalized Pareto distribution. Algamal et al. [4] introduced the neutrosophic Beta-Lindley distribution with an application to bladder cancer data. Sherwani et al. [29] offered the neutrosophic discrete geometric distribution. Rao [26] suggested the neutrosophic log-logistic distribution. Ahsan-ul-Haq et al. [3] suggested neutrosophic Topp-Leone distribution considered for interval-valued data analysis. The increasing number of neutrosophic distributions available makes it easier to simulate complex phenomena with ambiguous observations, which is crucial for solving real-world problems.

The neutrosophic logic provides information regarding determinacy, indeterminacy, and falseness as Neutrosophy [21]. Neutronsophic logic is, therefore, more effective than interval-based analysis and fuzzy logic. Later on, several writers demonstrated their work on neutrosophic logic for various issues as Pratihar et al. [25]. The introduction of neutrosophic logic is investigated by Smarandache [30] has led to the development of neutrosophic statistics, which provide a means of determining indeterminacy and determinacy (as elaborated in Aslam [8]). In the absence of data relating to the measure of indeterminacy, classical statistics can be employed as an alternative to neutrosophic statistics.

The rest of the paper is organized as follows: The suggested NQXLD is presented in Section 1 with its cdf. Some statistical properties are given in Section 3. Section 4 shows the estimation of the distribution parameters. In Section 5, three real data sets are investigated to illustrate the distribution applicability. The paper is concluded in Section 6.

2. Neutrosophic Quasi-XLindley Distribution

The authors [30] and [9] introduced neutrosophic statistics and propose the following formula for the neutrosophic variable: $X_N = X_L + I_N X_U$, $I_N \in [I_L, I_U]$ where X_L and $I_N X_U$ represent

definite and indeterminate portions, respectively, and $I_N \in [I_L, I_U]$ is an arbitrary interval. Assume that the NQXLD, with neutrosophic scale parameters $\alpha_N \in [\alpha_L, \alpha_U]$ and $\theta_N \in [\theta_L, \theta_U]$, follows the neutrosophic random variable $I_N \in [I_L, I_U]$. Remember that the lower and upper values are, respectively, represented by the symbols X_L, α_L, θ_L and X_U, α_U, θ_U. The PDF of the NQXLD is provided by:

$$f(x_N; \alpha_N, \theta_N) = \left\{ \frac{\theta_L}{1+\alpha_L} \left(\alpha_L + \frac{\theta_L (1+x_L)}{1+\theta_L} \right) e^{-\theta_L x_L} \right\} + \left\{ \frac{\theta_U}{1+\alpha_U} \left(\alpha_U + \frac{\theta_U (1+x_U)}{1+\theta_U} \right) e^{-\theta_U x_U} \right\} I_N. \quad (3)$$

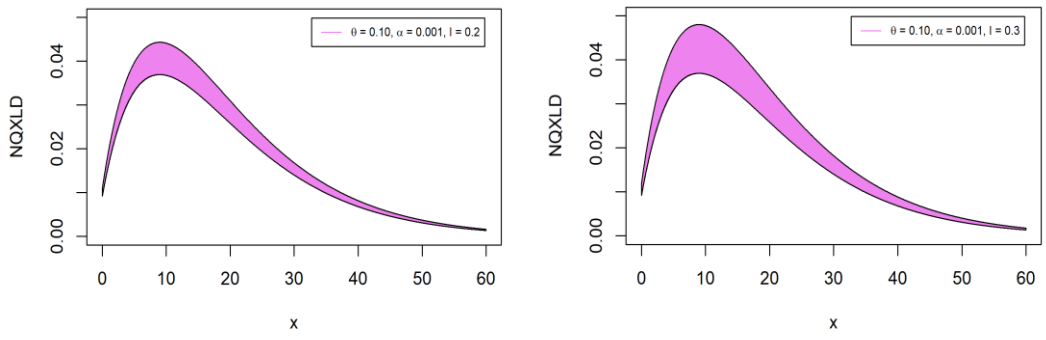
If $x_L = x_U = x_N$, then the pdf can be written as

$$f(x_N; \alpha_N, \theta_N) = \left\{ \frac{\theta_N}{1+\alpha_N} \left(\alpha_N + \frac{\theta_N (1+x_N)}{1+\theta_N} \right) e^{-\theta_N x_N} \right\} (1+I_N), \quad I_N \in [I_L, I_U] \quad (4)$$

where $x_N \geq 0, \alpha_N, \theta_N > 0, I_N \in [I_L, I_U]$. Note that when $I_L = 0$, the neutrosophic quantities reduce to classical statistics. The corresponding CDF of the NQXLD is given by:

$$F(x_N; \alpha_N, \theta_N) = \left\{ 1 - \left(1 + \frac{\theta_N x_N}{(1+\theta_N)(1+\alpha_N)} \right) e^{-\theta_N x_N} \right\} (1+I_N), \; I_N \in [I_L, I_U]. \quad (5)$$

Figure 1 shows the pdf plots for the NQXL distribution for some values of the distribution parameters. It can be noted that the pdf has several shapes which make the distribution more flexible in fitting the real data. Figure 2 presents some CDF plots of the NQXLD for some parameters.



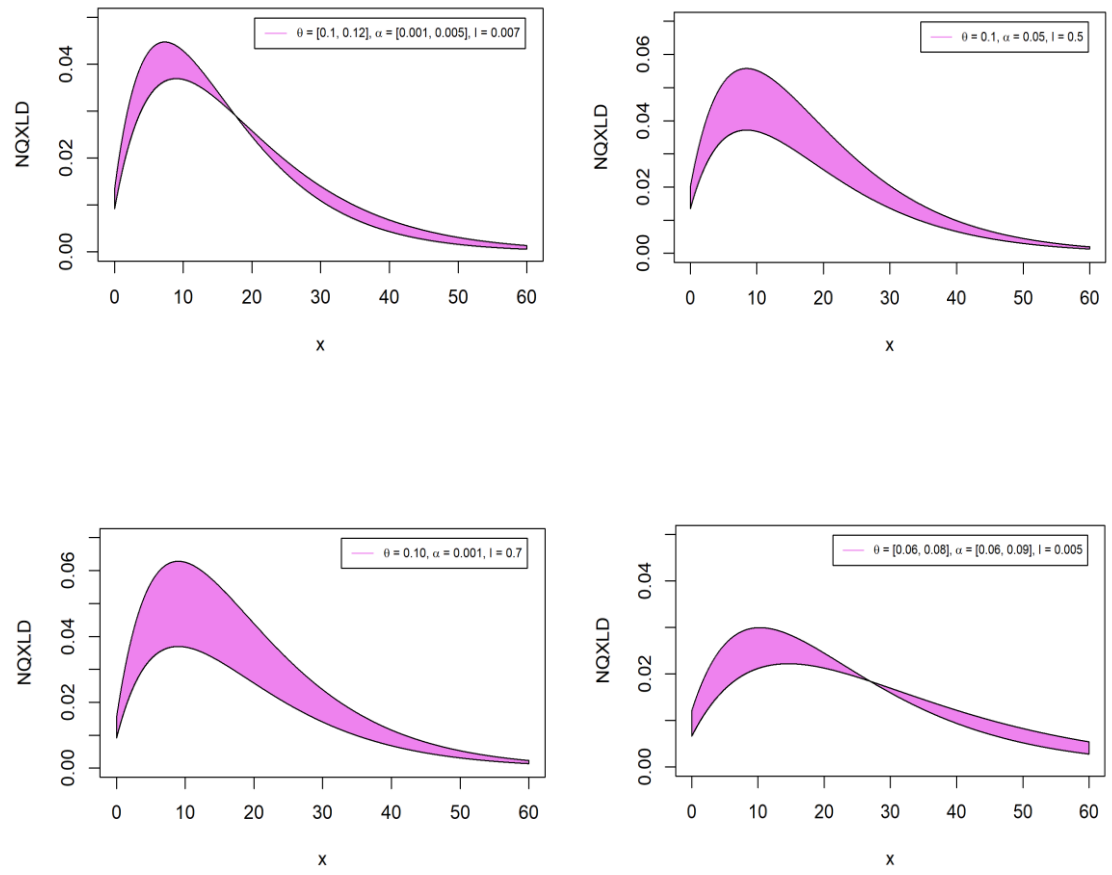


Figure 1: The pdf curves for NQXL distribution for various values of the parameters

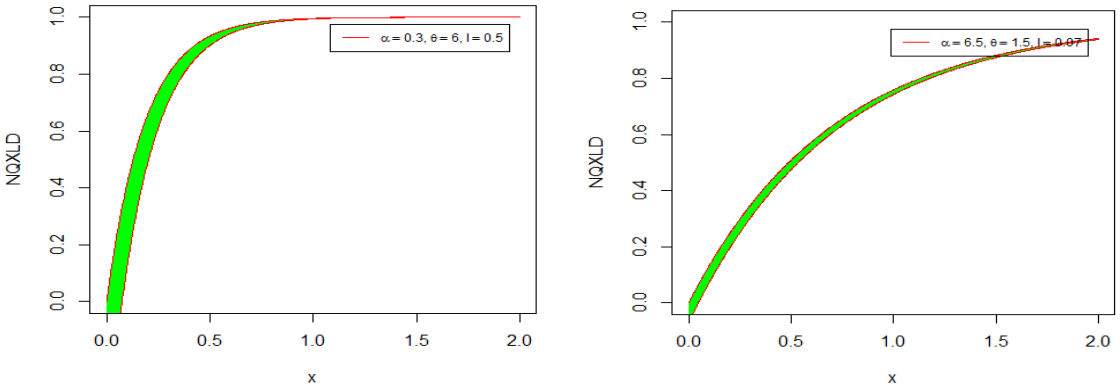


Figure 2: The cdf curves for NQXL distribution for various values of the parameters

3. Statistical Properties

The NQXLD statistical features covered in this section are now visible and available.

3.1 **Moments and associated measures**

Calculating statistical measures like variance, skewness, kurtosis, and central tendency can be simplified by utilizing the moments of a specific density function. To find the rth moment of random variable X_N around the origin, Equation (3) can be applied.

$$\begin{aligned} E\left(X_N^r\right) = \mu_{rN}^{'} &= \int_0 x_N^r f\left(x_N\right) dx \\ &= \int x_N^r \left\{ \frac{\theta_N}{1+\alpha_N} \left(\alpha_N + \frac{\theta_N \left(1+x_N\right)}{1+\theta_N} \right) e^{-\theta_N x_N} \right\} \left(1+I_N\right) dx_N \\ &= \left\{ \frac{\theta_L}{1+\alpha_L} \int x_L^r \left(\alpha_L + \frac{\theta_L \left(1+x_L\right)}{1+\theta_L} \right) e^{-\theta_L x_L} \right\} + \left\{ \frac{\theta_U}{1+\alpha_U} \int x_U^r \left(\alpha_U + \frac{\theta_U \left(1+x_U\right)}{1+\theta_U} \right) e^{-\theta_U x_U} \right\} I_N dx_U . \end{aligned}$$

The following is the final expression of ordinary moments.

$$E\left(X_N^r\right) = \mu_{rN}^{'} = \left\{ \frac{1}{\theta_N^r \left(1+\alpha_N\right)} \left(\left(\alpha_N + \frac{\theta_N}{1+\theta_N} \right) \Gamma(r+1) + \frac{1}{1+\theta_N} \Gamma(r+2) \right) \right\} \left(1+I_N\right). \tag{6}$$

Equation (4) can be used to get the first four moments of the random variable X_N about the origin by inserting $r = 1,2,3,4$ as:

$$E(X_N) = \mu'_{1N} = \frac{\alpha_N(1+\theta_N) + (\theta_N + 2)}{\theta_N(1+\alpha_N)(1+\theta_N)} (1+I_N)$$

$$E(X_N^2) = \mu'_{2N} = \frac{2\left(\alpha_N(1+\theta_N) + (\theta_N + 3)\right)}{\theta_N^2(1+\alpha_N)(1+\theta_N)} (1+I_N)$$

$$E(X_N^3) = \mu'_{3N} = \frac{6\left(\alpha_N(1+\theta_N) + (\theta_N + 4)\right)}{\theta_N^3(1+\alpha_N)(1+\theta_N)} (1+I_N)$$

$$E(X^4) = \mu'_{4N} = \frac{24\left(\alpha_N(1+\theta_N) + (\theta_N + 5)\right)}{\theta_N^4(1+\alpha_N)(1+\theta_N)} (1+I_N)$$

Variance (Var) and index of dispersion (ID) of the NQXLD distribution are given below as:

$$\begin{aligned}\mathrm{Var}\left(X_N\right)=\sigma_N^2&=\mu_{2N}^{'}-\left(\mu_{1N}^{'}\right)^2\\&=\frac{\alpha_N^2\left(\theta_N+1\right)^2+2\alpha_N\left(\theta_N+1\right)\left(\theta_N+2\right)+\theta_N\left(\theta_N+4\right)+2}{\theta_N\left(\alpha_N+1\right)^2\left(\theta_N+1\right)^2},\sigma_N^2\in\left[\sigma_L^2,\sigma_U^2\right],\end{aligned}\tag{7}$$

$$ID(X)=\frac{\mathrm{Var}\left(X_N\right)}{\mu_{1N}^{'}}=\frac{\alpha_N^2\left(\theta_N+1\right)^2+2\alpha_N\left(\theta_N+1\right)\left(\theta_N+2\right)+\theta_N\left(\theta_N+4\right)+2}{\theta_N\left(\alpha_N+1\right)\left(\theta_N+1\right)\left(2+\alpha_N+\theta_N+\alpha_N\theta_N\right)}.\tag{8}$$

The following relations can be utilized to determine the coefficient of kurtosis, represented by β_{2N}, and the coefficient of skewness, represented by $\sqrt{\beta_{1N}}$, are given by

$$\beta_{2N}=\frac{\mu_{4N}^{'}-4\mu_{3N}^{'}\mu_{1N}^{'}+6\mu_{2N}^{'}\left(\mu_{1N}^{'}\right)^2-3\left(\mu_{1N}^{'}\right)^4}{\sigma_N^4}\left(1+I_N\right),$$

$$\sqrt{\beta_{1N}}=\frac{\mu_{3N}^{'}-3\mu_{2N}^{'}\mu_{1N}^{'}+2\left(\mu_{1N}^{'}\right)^3}{\sigma_N^3}\left(1+I_N\right).$$

The following relations can be utilized to determine the coefficient of kurtosis, represented by β_{2N} and the coefficient of skewness, represented by β_{1N}. The neutrosophic median, neutrosophic skewness, neutrosophic kurtosis, standard deviation, and neutrosophic mean are displayed in Table (1). The results shown in Table 1 indicate that, for a given set of neutrosophic alpha parameters, different statistic values decrease when neutrosophic theta parametric values increase.

Table 1: The statistical measures of neutrosophic statistic NQXLD

Statistic	$\alpha_N=[1.5,\ 2]$	$\alpha_N=[1.5,\ 2]$	$\alpha_N=[1.5,\ 2]$	$\alpha_N=[1.5,\ 2]$	$\alpha_N=[1.5,\ 2]$
	$\theta_N=[0.5,0.75]$	$\theta_N=[1,1.25]$	$\theta_N=[1.75,2]$	$\theta_N=[2,3]$	$\theta_N=[1,1.5]$
Mean	[2.5333,1.5873]	[1.2000,0.9185]	[0.6546,0.5556	[0.5667,0.3611]	[1.1667,0.7429]
S.D.	[2.4185,1.5461]	[1.1662,0.9031]	[0.6439,0.5500]	[0.5588,0.35897]	[1.1426,0.7350]
Median	[1.8210 1.1225]	[0.8502 0.6447]	[0.4593,0.3879]	[0.3969,0.25136]	[0.8215, 0.5189]
Skewness	[1.8162,1.8848]	[1.8764,1.9211]	[1.9234,1.9502]	[1.9332,1.9693]	[1.9056,1.9479]
Kurtosis	[7.8009,8.2116]	[8.1592,8.4438]	[8.4584,8.6389]	[8.5234,8.7724]	[8.3427,8.6228]

3.2 Features of reliability

The NQXL distribution's reliability attributes, such as the mean residual life function, stress strength reliability, neutrosophic survival and hazard functions, and neutrosophic reversed hazard function, are derived in this section.

3.2.1 ***Neutrosophic survival function***

$$S(x_N) = 1 - F(x_N) = \left(1 + \frac{x_N\theta_N}{(\alpha_N+1)(1+\theta_N)}\right) e^{-\theta_N x}(1 + I_N) + I_N, x_N > 0. \quad (9)$$

3.2.2 Hazard rate function

$$h(x_N) = \left\{\frac{\theta_N(\alpha_N(1+\theta_N)+\theta_N(1+x_N))}{(\alpha_N+1)(1+\theta_N)+x_N\theta_N}\right\}(1 + I_N), x_N > 0. \quad (10)$$

3.2.3 Neutrosophic mean residual life function

$$m(t) = \left\{\frac{1}{\theta_N\left(1+\frac{t_N\theta_N}{(\alpha_N+1)(1+\theta_N)}\right)}\left[1 + \frac{1+t_N\theta_N}{(\alpha_N+1)(1+\theta_N)}\right]\right\}(1 + I_N). \quad (11)$$

3.2.4 Measures of actuarial

Assessing market risk is crucial to actuarial science, mainly when dealing with asset portfolios. It is essential to evaluate the risks of any market transaction, whether it involves buying or selling. This evaluation calculates two significant actuarial measures for the NQXL distribution- Value at Risk (VaR) and Tail Value at Risk (TVaR). To determine the VaR of the NQXL model, we utilize a precise mathematical tool called the formula $x_p = F^{-1}(x_N)$, where x_N represents the direct solution of the subsequent non-linear equation.

$$1 - \frac{p}{(1+I_N)} = \left(1 + \frac{x_N\theta_N}{(\alpha_N+1)(1+\theta_N)}\right) e^{-\theta_N x},\ 0 < p < 1. \quad (12)$$

By following this formula, we can accurately calculate the conditional tail expectation, which represents the entire value at risk. It's essential to compute this value to make informed decisions that mitigate risk and ensure the safety of your investments.

$$\begin{aligned} TVaR_{Np} &= \frac{1}{1-p}\int_{VaR_{Np}}^{\infty} x f\left(x_N;\alpha_N,\theta_N\right)dx \\ &= \frac{1}{1-p}\int_{VaR_{Np}}^{\infty} x\left\{\frac{\theta_L}{1+\alpha_L}\left(\alpha_L + \frac{\theta_L\left(1+x_L\right)}{1+\theta_L}\right)e^{-\theta_L x_L}\right\} + \left\{\frac{\theta_U}{1+\alpha_U}\left(\alpha_U + \frac{\theta_U\left(1+x_U\right)}{1+\theta_U}\right)e^{-\theta_U x_U} I_N\right\}dx \\ &= \frac{e^{-\theta_N VaR}\left(2+\alpha_N+\theta_N VaR\left(2+\alpha_N\right)+\theta_N^2 VaR(\alpha_N+VaR+1)\right)}{(1-p)(\alpha_N+1)(\theta_N+1)} \end{aligned} \quad (13)$$

4. Estimation of Parameters

The neutrosophic parameters of the created NQXLD are derived by the utilization of the maximum likelihood estimation (MLE) technique. Let $X_{N1}, X_{N2},...,X_{Nn}$ be a random sample of NQXLD with neutrosophic characteristics. The log- likelihood function can be represented as:

$$L(\theta_0,\alpha_0)=n\ln(\theta_0)+\sum_{i=1}^{n}\ln\left(\alpha_0+\frac{\theta_0 x_{0i}(1+\alpha_0 x_{0i})}{1+\theta_0 x_{0i}}\right)-\alpha_0\sum_{i=1}^{n}x_{0i}-n\ln(1+\alpha_0). \tag{14}$$

The derivatives of Equation (14) relative to θ_N,α_N, are respectively defined as:

$$\frac{\partial L(\theta_N,\alpha_N)}{\partial\theta_N}=\frac{n}{\theta_N}+\sum_{i=1}^{n}\frac{1+x_{Ni}}{(1+\theta_N x_{Ni})\left(\alpha_N+\frac{\theta_N x_{Ni}(1+\alpha_N x_{Ni})}{1+\theta_N x_{Ni}}\right)}-\sum_{i=1}^{n}x_{Ni}=0, \tag{15}$$

$$\frac{\partial L(\theta_N,\alpha_N)}{\partial\theta_N}=\frac{n}{\theta_N}+\sum_{i=1}^{n}\frac{1+x_{Ni}}{(1+\theta_N x_{Ni})\left(\alpha_N+\frac{\theta_N x_{Ni}(1+\alpha_N x_{Ni})}{1+\theta_N x_{Ni}}\right)}-\sum_{i=1}^{n}x_{Ni}=0. \tag{16}$$

The MLE for (θ_N,α_N), $\theta_N\in[\theta_L,\theta_U]$ and $\alpha_N\in[\alpha_L,\alpha_U]$ can be found by solving Equations (15) and (16).

This section presents a comprehensive simulation study for the maximum likelihood (ML) estimation approach. The study takes into account both small and large sample sizes. The mean square error (MSE) and average bias are calculated to compare each estimator's performance. All the results in this section wer computed using the R programming language Core Team [13].

We consider the NQXL model and simulate data with $N = 10,000$. We also consider different sample sizes, including $n=50$ 100, 250, and 350 for certain parameter values. Table 2 provides the MSEs and simulated averages for all the simulations.

Table 2. Simulation results of the ML estimators for the NQXLD.

Sample	Parameter			MSE		NAB	
n	α	β	I_N	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
50	0.1	0.5	0	0.79970	0.00500	0.34785	0.01298
100				0.39806	0.00292	0.21414	0.01021
250				0.08422	0.00124	0.08187	0.00541
350				0.04436	0.00091	0.05619	0.00404
50	0.050	0.50	0.1	0.01387	0.23702	0.04991	0.47295
100				0.00356	0.01444	0.04941	0.11447
250				0.00250	0.01359	0.04082	0.11417

350				0.00249	0.01340	0.02292	0.11409
50	0.50	0.50	0.1	0.25532	0.03994	0.49297	0.19562
100				0.24396	0.03891	0.48718	0.19529
250				0.23966	0.03891	0.45142	0.19214
350				0.22328	0.03884	0.39219	0.18946
50	1.50	1.50	0.50	0.24992	0.30193	0.49989	0.54635
100				0.01000	0.16996	0.09995	0.40942
250				0.01000	0.16711	0.09999	0.40764
350				0.00999	0.16601	0.09888	0.40664

The average bias diminishes as sample size increases, according to the simulation results. It indicates that as sample size grows, practice and theory agree better. A larger sample size also results in a decrease in the estimators' MSEs. It is clear that the obtained estimators exhibit consistency, and the parameters' maximum likelihood estimator operates effectively, producing precise and accurate findings.

6 Applications

This section showcases the potential of the NQXLD distribution for three real-life datasets as compared to the QXLD. Some summary measures of this data set are given in Table 3. To compare the distribution, we utilize the suggested NQXLD with the classical QXL distribution. Model parameters are estimated through maximum likelihood estimation. The optimal model is chosen based on the smallest values of criteria such as log-likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC), $AIC = 2k - 2\ln(\hat{L})$ and $BIC = k\ln(n) - 2\ln(\hat{L})$, where k is the number of parameters, $\hat{L}$ is the maximized log-likelihood value, and n is the sample size. The results are listed in Table 4. These data can be described as follows.

Data 1: Between April 15, 2020, and June 30, 2020, COVID-19 data was collected in the United Kingdom and it is presented in Amaal and Ehab [6]. The data observations are: 0.0587, 0.0863, 0.1165, 0.1247, 0.1277, 0.1303, 0.1652, 0.2079, 0.2395, 0.2751, 0.2845 0.2992, 0.3188, 0.3317, 0.3446, 0.3553, 0.3622, 0.3926, 0.3926, 0.4110, 0.4633, 0.4690 0.4954, 0.5139, 0.5696, 0.5837, 0.6197, 0.6365, 0.7096, 0.7193, 0.7444, 0.8590, 1.0438 1.0602, 1.1305, 1.1468, 1.1533, 1.2260, 1.2707, 1.3423, 1.4149, 1.5709, 1.6017, 1.6083 1.6324, 1.6998, 1.8164, 1.8392, 1.8721, 1.9844, 2.1360, 2.3987, 2.4153, 2.5225, 2.7087 2.7946, 3.3609, 3.3715, 3.7840, 3.9042, 4.1969, 4.3451, 4.4627, 4.6477, 5.3664, 5.4500 5.7522, 6.4241, 7.0657, 7.4456, 8.2307, 9.6315, 10.1870, 11.1429, 11.2019, 11.4584.

Data II: Number of deaths per day due to COVID-19 in Nepal see Dhungana and Kumar [15]. The data are as follows: 2, 2, 2, 2, 2, 2, 3, 2, 3, 3, 4, 2, 5, 5, 3, 2, 4, 4, 8, 4, 4, 3, 2, 3, 7, 6, 6, 11, 9, 3, 8, 7, 11, 8, 12, 12, 14, 7, 11, 12, 6, 14, 9, 9, 11, 6, 6, 5, 5, 14, 9, 15, 11, 8, 4, 7, 11, 10, 16, 2, 7, 17, 6, 8, 10, 4, 10, 7, 11, 11, 8, 7, 19, 9, 15, 12, 10, 14, 22, 9, 18, 12, 19, 21, 12, 12, 18, 8, 26, 21, 17, 13, 5, 15, 14, 11, 17, 16, 17, 23, 24, 20, 30, 18, 18, 17, 21, 18, 22, 26, 15, 13, 13, 6, 9, 17, 12, 17, 22, 7, 16, 16, 24, 28, 23, 23,19, 25, 29, 21, 9, 13, 16, 10, 17, 20, 23, 14, 12, 11, 15, 9, 18, 14, 13, 6, 16, 12, 11, 7, 3, 5, 5.

Data III: The third dataset displays the number of COVID-19 deaths in China from January 23, 2020, to March 28, 2020. You can find more details about this dataset at https://www.worldometers.info/coronavirus/country/china. The numbers are as follows: 16, 15, 24, 26, 26, 38, 43, 46, 45, 57, 64, 65, 73, 73, 86, 89, 97, 108, 97, 146, 121, 143, 142, 105, 98, 136, 114, 118, 109, 97, 150, 71, 52, 29, 44, 47, 35, 42, 31, 38, 31, 30, 28, 27, 22, 17, 22, 11, 7, 13, 10, 14, 13, 11, 8, 3, 7, 6, 9, 7, 6, 6, 5, 3, 5, 5, 5, and 8.

Table 3: Some descriptive measures for the data sets

Statistic	N	Mean	Median	Q_1	Q_2	Q_3	S.D.	Min	Max
Data I	70	2.9228	1.2483	0.3926	1.2260	3.7840	4.6033	0.0587	21.1906
Data II	153	11.6144	11	6	11	16	6.7591	2	30
Data III	68	48.4559	31	11	31	73	43.8496	3	150

Table 4. The MLEs and goodness-of-fit based on the data sets

Data	Model	$\hat{\alpha}$	$\hat{\theta}$	I_N	-LogLik	AIC	BIC
I	QXLD	1124.02179	0.41056	0	287.4088	291.4088	296.0702
	NQXLD	1236.42400	0.45162	0.1	272.9216	276.9216	281.5831
		1348.82600	0.49268	0.2	259.6959	263.6959	268.3574
		1461.22800	0.53374	0.3	247.5294	251.5294	256.1909
II	QXLD	0.00001	0.16031	0	1011.2060	1015.2060	1021.2660
	NQXLD	1e-06	0.01603	0.1	982.0407	986.0407	992.1016
		2e-06	0.03206	0.2	955.4152	959.4152	965.4761
		3e-06	0.04809	0.3	930.9222	934.9222	940.9830
III	QXLD	2.91390	0.02579	0	663.5121	667.5121	671.9511
	NQXLD	0.29139	0.00258	0.1	650.5499	654.5499	658.9889
		0.58278	0.00516	0.2	638.7163	642.7163	647.1554
		0.87417	0.00774	0.3	627.8305	631.8305	636.2696

The results given in Table 4 are significant and demonstrate the importance of the indeterminacy parameter in improving the fitting of the distribution. Based on the maximum likelihood estimates and goodness-of-fit metrics for the base QXLD and NQXLD, it is evident that the NQXL distribution is a superior fit in comparison to the classical QXLD for all data sets considered in this study. Therefore, it is essential to consider the indeterminacy parameter when modeling distributions to ensure a better fit for the data considered in this study.

7. Conclusions

The neutrosophic Quasi-XLindley distribution is more flexible than the Quasi-XLindley distribution under classical statistics, as revealed by a study that examined its fundamental characteristics. Using

three COVID-19 data sets, the research found that decision-makers can benefit from this distribution. This distribution could be used to construct other statistical distributions that will be useful in future research. The study suggests exploring additional properties of the neutrosophic Quasi-XLindley distribution and expanding on the multivariate distributions in subsequent analyses. Furthermore, G-families of distributions under neutrosophic statistics can be a promising area for future research based on the suggested NQXLD.

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MCDM Approach to Assess Innovation and Entrepreneurship Education in Higher Vocational Colleges under IndetermSoft Set

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Abstract: Innovation and entrepreneurship education has become an essential component of higher vocational colleges, aiming to cultivate students' creativity, problem-solving abilities, and business acumen. As the global economy increasingly prioritizes innovation-driven development, vocational institutions must ensure that their educational programs align with industry demands and entrepreneurial ecosystems. However, assessing the quality of innovation and entrepreneurship education remains a complex challenge due to the interdisciplinary nature of the field, the diverse skill sets required, and the rapidly evolving business environment. We use the concept of multi-criteria decision-making (MCDM) to deal with different criteria. Two MCDM methods are used in this study such as Entropy method to compute the criteria weights and the EDAS method to rank the alternatives. we use the IndetermSoft set to deal with indeterminacy in criteria values. We propose an application to show the validation of the proposed approach.

Keywords: MCDM Methodology; Innovation and Entrepreneurship; IndetermSoft Set; Education.

1. Introduction

Innovation and entrepreneurship education has emerged as a critical pillar in higher vocational colleges, bridging the gap between academic learning and real-world business practices. With the rapid advancement of technology and evolving market needs, vocational institutions must equip students with the necessary entrepreneurial mindset, technical skills, and problem-solving capabilities. Unlike traditional education, which primarily focuses on theoretical knowledge, innovation and entrepreneurship education emphasizes experiential learning, hands-on business projects, and industry collaboration[1], [2]. However, the effectiveness of such programs varies significantly, necessitating a structured evaluation framework to ensure quality and relevance. The assessment of innovation and entrepreneurship education in vocational colleges involves

multiple dimensions, including curriculum design, faculty competency, access to entrepreneurial resources, and the integration of industry expertise. A well-structured curriculum must balance theoretical foundations with practical applications, offering students exposure to case studies, startup incubation programs, and real-world business challenges[3], [4]. Moreover, faculty expertise plays a crucial role in guiding students through the complexities of entrepreneurship, requiring educators to possess not only academic knowledge but also hands-on business experience. The absence of qualified instructors can significantly hinder the effectiveness of entrepreneurial education, limiting students' ability to transform innovative ideas into viable business ventures[5], [6]. Industry collaboration is another critical factor in evaluating the quality of innovation and entrepreneurship education. Strong partnerships between vocational colleges and businesses provide students with mentorship opportunities, internship placements, and access to industry networks. Without direct engagement with the entrepreneurial ecosystem, students may struggle to develop market-ready skills, reducing their employability and business success rates. Additionally, technological integration in entrepreneurship education, including digital tools, business simulation software, and online learning platforms, enhances the learning experience and prepares students for the digital economy[7], [8]. A lack of modern technological resources can create a disconnect between classroom learning and industry realities. Given the complexity of evaluating entrepreneurship education, Multi-Criteria Decision-Making (MCDM) methodologies offer a structured approach to ranking and assessing various programs. By considering multiple qualitative and quantitative criteria, MCDM techniques help in identifying the strengths and weaknesses of different vocational institutions[9], [10]. This systematic evaluation provides valuable insights for policymakers, educators, and industry stakeholders, facilitating the continuous improvement of innovation and entrepreneurship education.

Ultimately, enhancing the quality of entrepreneurship education in vocational colleges is essential for fostering a new generation of innovative thinkers and business leaders. As economies become increasingly reliant on startups and technological advancements, vocational institutions must refine their educational strategies to meet industry expectations. This study contributes to the discourse on vocational education quality assessment by offering a robust framework for evaluating entrepreneurship programs, ensuring that students receive the necessary skills and resources to thrive in a competitive business landscape

The main contributions of this study are organized as follows:

- We use the concept of IndetermSoft set to deal with indeterminacy in each value of criteria and sub criteria.
- Three cases of indeterminacy are used in this study to solve the indeterminacy.
- Two MCDM methods are used in this study.
- We use the Entropy method to compute the criteria weights.
- We use the EDAS method to rank the alternatives.
- Ranks of alternatives are analyzed by different cases.

The rest of this study is organized as follows:

- Section 2 shows the steps of the proposed approach with the IndetermSoft set.
- Section 3 shows the results of the proposed approach with the IndetermSoft set and three cases.
- Section 4 shows the conclusions of this study.

2. IndetermSoft-Entropy-EDAS Methodology

Smarandache [11], [12] presented an IndetermSoft Set to deal with indeterminacy in the values of the criteria. Let U be a universe discourse, H a non-empty subset of U, and P(H) the power set of H. let a be a criterion and A be a set of criteria values.

Then $F: A \to P(H)$ is called an IndetermSoft set if:

The set A, P(H) has some indeterminacy and there is a criteria value has indeterminacy value.

IndetermSoft set has some indeterminacy and it is an extension of determinate set[13], [14].

Entropy Method

Create a decision matrix between the criteria and alternatives[15], [16].

Normalize the decision matrix

$$y_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \tag{1}$$

Compute the entropy value

$$e_j = -h \sum_{i=1}^{m} y_{ij} \ln y_{ij} \tag{2}$$

$$h = \frac{1}{\ln m} \tag{3}$$

Compute the criteria weights.

$$w_j = \frac{1-e_j}{\sum_{j=1}^{n} (1-e_j)} \tag{4}$$

EDAS Method

The EDAS Method is used to rank the alternatives[13], [17].

Compute the average solution

$$B_j = \frac{\sum_{i=1}^{m} x_{ij}}{m} \tag{5}$$

Compute the positive and negative distance from the average solution

$$PR_{ij} = \frac{\max\left(0,(y_{ij}-B_j)\right)}{B_j} \tag{6}$$

$$NR_{ij} = \frac{\max\left(0,(B_j - y_{ij})\right)}{B_j} \quad (7)$$

$$PR_{ij} = \frac{\max\left(0,(B_j - y_{ij})\right)}{B_j} \quad (8)$$

$$NR_{ij} = \frac{\max\left(0,(y_{ij} - B_j)\right)}{B_j} \quad (9)$$

Compute the weighted PR and NR

$$SPR_i = \sum_{j=1}^{n} PR_{ij}\, w_j \quad (10)$$

$$SNR_i = \sum_{j=1}^{n} NR_{ij}\, w_j \quad (11)$$

Compute the weighted normalized SPR and SNR

$$NSPR_i = \frac{SPR_i}{\max\limits_i SPR_i} \quad (12)$$

$$NSNR_i = \frac{SNR_i}{\max\limits_i SNR_i} \quad (13)$$

Compute the appraisal score

$$F_i = \frac{NSPR_i + NSNR_i}{2} \quad (14)$$

3. Results and Discussion

This section shows the results of the proposed approach. Three experts and decision makers are invited to evaluate the criteria and alternatives. Nine criteria and ten alternatives are used in this study such as: Curriculum Design (Comprehensive, Basic), Entrepreneurial Resources Support (Extensive), Faculty Competency (Highly Skille), Practical Training Opportunities (Sufficient), Industry Collaboration (Strong), Graduate Employment and Startup Rate (High), Technology Integration (Advanced), Funding and Financial Support (High), Student Innovation Ability (High). The alternatives of this study are Cross-Disciplinary Innovation and Startup Support, Comprehensive Entrepreneurial Program, Student-Led Business Project Platform, Vocational College Startup Fund Program, Online and Offline Blended Entrepreneurship Course, Hands-On Entrepreneurial Training Initiative, Industry-Focused Innovation Hub, Government-Supported Startup Incubator, Technology-Driven Business Accelerator, Public-Private Partnership for Entrepreneurship

In this study, we show indeterminacy in the first criterion. So, we use the concept of the IndetermSoft soft set to deal with this indeterminacy. So, we apply the MCDM methods with this indeterminacy with three cases.

In the first case, we select the Comprehensive.

In the second case, we select the Basic.

In the third case, we select Comprehensive and Basic.

The first case:

We create a decision matrix between the criteria and alternatives.

We normalize the decision matrix using Eq. (1) as shown in Table 1.

Then we compute the entropy value using Eq. (2).

Then we compute the criteria weights using Eq. (4).

Table 1. Normalized values by Entropy method.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.10152	0.078799	0.121052	0.108129	0.100532	0.105299	0.100216	0.101312	0.1128
A_2	0.145741	0.08917	0.065636	0.081256	0.11071	0.105033	0.096898	0.068121	0.0845
A_3	0.080226	0.093587	0.121052	0.108129	0.092389	0.093742	0.105062	0.09997	0.1267
A_4	0.108927	0.06896	0.089665	0.108403	0.05388	0.110227	0.072817	0.122512	0.1125
A_5	0.11333	0.116648	0.111998	0.091951	0.111291	0.083902	0.135109	0.117725	0.0775
A_6	0.078547	0.132337	0.085789	0.084623	0.120043	0.095829	0.100216	0.082828	0.0846
A_7	0.145741	0.105007	0.079581	0.072802	0.086144	0.100756	0.10169	0.114181	0.0775
A_8	0.072053	0.091903	0.082488	0.141018	0.128185	0.105181	0.084926	0.084716	0.1128
A_9	0.063836	0.118731	0.126215	0.122433	0.098288	0.091404	0.105062	0.088393	0.0845
A_{10}	0.090078	0.104859	0.116525	0.081256	0.098538	0.108628	0.098004	0.120242	0.1267

Then we compute the average solution using Eq. (5).

Then we compute the positive and negative distance from the average solution suing Eqs. (6-9) as shown in Table 2 and 3.

Then we compute the weighted PR and NR using Eqs. (10 and 11).

Then we compute the weighted normalized SPR and SNR using Eqs. (12 and 13).

Then we compute the appraisal score using Eq. (14).

Table 2. PR_{ij} values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.015197	0	0.21052	0.081287	0.005317	0.05299	0.002157	0.013117	0.127537
A_2	0.457413	0	0	0	0.107095	0.050326	0	0	0
A_3	0	0	0.21052	0.081287	0	0	0.05062	0	0.266816
A_4	0.089268	0	0	0.084029	0	0.102265	0	0.225115	0.124685
A_5	0.133298	0.166477	0.119977	0	0.112911	0	0.351089	0.17725	0
A_6	0	0.323366	0	0	0.200426	0	0.002157	0	0
A_7	0.457413	0.050066	0	0	0	0.007562	0.016901	0.141807	0
A_8	0	0	0	0.410182	0.281849	0.051806	0	0	0.127537

A_9	0	0.187307	0.262151	0.224331	0	0	0.05062	0	0
A_{10}	0	0.048588	0.165248	0	0	0.086284	0	0.202425	0.266816

Table 3. NR_{ij} values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0	0.212008	0	0	0	0	0	0	0
A_2	0	0.108301	0.343639	0.187435	0	0	0.031016	0.318792	0.154822
A_3	0.197738	0.06413	0	0	0.076105	0.062579	0	0.000298	0
A_4	0	0.310396	0.103352	0	0.4612	0	0.271829	0	0
A_5	0	0	0	0.080495	0	0.160982	0	0	0.224699
A_6	0.214526	0	0.142113	0.153769	0	0.041714	0	0.171718	0.154347
A_7	0	0	0.204191	0.271982	0.138557	0	0	0	0.224699
A_8	0.279466	0.080971	0.17512	0	0	0	0.15074	0.152837	0
A_9	0.361636	0	0	0	0.017115	0.085959	0	0.116069	0.154822
A_{10}	0.099222	0	0	0.187435	0.014623	0	0.019958	0	0

The second case:

We create a decision matrix between the criteria and alternatives.

We normalize the decision matrix using Eq. (1) as shown in Table 4.

Then we compute the entropy value using Eq. (2).

Then we compute the criteria weights using Eq. (4).

Table 4. Normalized values by Entropy method.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.10152	0.102272	0.121052	0.108129	0.100532	0.105299	0.100216	0.101312	0.112754
A_2	0.145741	0.11921	0.065636	0.081256	0.11071	0.105033	0.096898	0.068121	0.084518
A_3	0.080226	0.115868	0.121052	0.108129	0.092389	0.093742	0.105062	0.09997	0.126682
A_4	0.108927	0.075002	0.089665	0.108403	0.05388	0.110227	0.072817	0.122512	0.112469
A_5	0.11333	0.075002	0.111998	0.091951	0.111291	0.083902	0.135109	0.117725	0.07753
A_6	0.078547	0.112526	0.085789	0.084623	0.120043	0.095829	0.100216	0.082828	0.084565
A_7	0.145741	0.122551	0.079581	0.072802	0.086144	0.100756	0.10169	0.114181	0.07753
A_8	0.072053	0.112526	0.082488	0.141018	0.128185	0.105181	0.084926	0.084716	0.112754
A_9	0.063836	0.05624	0.126215	0.122433	0.098288	0.091404	0.105062	0.088393	0.084518
A_{10}	0.090078	0.108802	0.116525	0.081256	0.098538	0.108628	0.098004	0.120242	0.126682

Then we compute the average solution using Eq. (5).

Then we compute the positive and negative distance from the average solution suing Eqs. (6-9) as shown in Table 5 and 6.

Then we compute the weighted PR and NR using Eqs. (10 and 11).

Then we compute the weighted normalized SPR and SNR using Eqs. (12 and 13).

Then we compute the appraisal score using Eq. (14).

Table 5. PR_{ij} values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.015197	0.022717	0.21052	0.081287	0.005317	0.05299	0.002157	0.013117	0.127537
A_2	0.457413	0.192097	0	0	0.107095	0.050326	0	0	0
A_3	0	0.158681	0.21052	0.081287	0	0	0.05062	0	0.266816
A_4	0.089268	0	0	0.084029	0	0.102265	0	0.225115	0.124685
A_5	0.133298	0	0.119977	0	0.112911	0	0.351089	0.17725	0
A_6	0	0.125264	0	0	0.200426	0	0.002157	0	0
A_7	0.457413	0.225513	0	0	0	0.007562	0.016901	0.141807	0
A_8	0	0.125264	0	0.410182	0.281849	0.051806	0	0	0.127537
A_9	0	0	0.262151	0.224331	0	0	0.05062	0	0
A_{10}	0	0.088016	0.165248	0	0	0.086284	0	0.202425	0.266816

Table 6. NR_{ij} values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0	0	0	0	0	0	0	0	0
A_2	0	0	0.343639	0.187435	0	0	0.031016	0.318792	0.154822
A_3	0.197738	0	0	0	0.076105	0.062579	0	0.000298	0
A_4	0	0.249977	0.103352	0	0.4612	0	0.271829	0	0
A_5	0	0.249977	0	0.080495	0	0.160982	0	0	0.224699
A_6	0.214526	0	0.142113	0.153769	0	0.041714	0	0.171718	0.154347
A_7	0	0	0.204191	0.271982	0.138557	0	0	0	0.224699
A_8	0.279466	0	0.17512	0	0	0	0.15074	0.152837	0
A_9	0.361636	0.437598	0	0	0.017115	0.085959	0	0.116069	0.154822
A_{10}	0.099222	0	0	0.187435	0.014623	0	0.019958	0	0

The third case:

We create a decision matrix between the criteria and alternatives.

We normalize the decision matrix using Eq. (1) as shown in Table 7.

Then we compute the entropy value using Eq. (2).

Then we compute the criteria weights using Eq. (4).

Table 7. Normalized values by Entropy method.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9

A_1	0.098649	0.078799	0.121052	0.108129	0.100532	0.105299	0.100216	0.101312	0.112754
A_2	0.078834	0.08917	0.065636	0.081256	0.11071	0.105033	0.096898	0.068121	0.084518
A_3	0.103362	0.093587	0.121052	0.108129	0.092389	0.093742	0.105062	0.09997	0.126682
A_4	0.102621	0.06896	0.089665	0.108403	0.05388	0.110227	0.072817	0.122512	0.112469
A_5	0.103362	0.116648	0.111998	0.091951	0.111291	0.083902	0.135109	0.117725	0.07753
A_6	0.088103	0.132337	0.085789	0.084623	0.120043	0.095829	0.100216	0.082828	0.084565
A_7	0.147201	0.105007	0.079581	0.072802	0.086144	0.100756	0.10169	0.114181	0.07753
A_8	0.093873	0.091903	0.082488	0.141018	0.128185	0.105181	0.084926	0.084716	0.112754
A_9	0.087567	0.118731	0.126215	0.122433	0.098288	0.091404	0.105062	0.088393	0.084518
A_{10}	0.096426	0.104859	0.116525	0.081256	0.098538	0.108628	0.098004	0.120242	0.126682

Then we compute the average solution using Eq. (5).

Then we compute the positive and negative distance from the average solution suing Eqs. (6-9) as shown in Table 8 and 9.

Then we compute the weighted PR and NR using Eqs. (10 and 11).

Then we compute the weighted normalized SPR and SNR using Eqs. (12 and 13).

Then we compute the appraisal score using Eq. (14).

Table 8. PR_{ij} values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0	0	0.21052	0.081287	0.005317	0.05299	0.002157	0.013117	0.127537
A_2	0	0	0	0	0.107095	0.050326	0	0	0
A_3	0.033624	0	0.21052	0.081287	0	0	0.05062	0	0.266816
A_4	0.026215	0	0	0.084029	0	0.102265	0	0.225115	0.124685
A_5	0.033624	0.166477	0.119977	0	0.112911	0	0.351089	0.17725	0
A_6	0	0.323366	0	0	0.200426	0	0.002157	0	0
A_7	0.472012	0.050066	0	0	0	0.007562	0.016901	0.141807	0
A_8	0	0	0	0.410182	0.281849	0.051806	0	0	0.127537
A_9	0	0.187307	0.262151	0.224331	0	0	0.05062	0	0
A_{10}	0	0.048588	0.165248	0	0	0.086284	0	0.202425	0.266816

Table 9. NR_{ij} values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A_1	0.013509	0.212008	0	0	0	0	0	0	0
A_2	0.211659	0.108301	0.343639	0.187435	0	0	0.031016	0.318792	0.154822
A_3	0	0.06413	0	0	0.076105	0.062579	0	0.000298	0
A_4	0	0.310396	0.103352	0	0.4612	0	0.271829	0	0
A_5	0	0	0	0.080495	0	0.160982	0	0	0.224699
A_6	0.118968	0	0.142113	0.153769	0	0.041714	0	0.171718	0.154347
A_7	0	0	0.204191	0.271982	0.138557	0	0	0	0.224699

A_8	0.061273	0.080971	0.17512	0	0	0	0.15074	0.152837	0
A_9	0.124328	0	0	0	0.017115	0.085959	0	0.116069	0.154822
A_{10}	0.035736	0	0	0.187435	0.014623	0	0.019958	0	0

Then we obtain the criteria weighs in three cases as shown in Table 10. We obtained the final ranks in Table 11.

Table 10. Criteria weights.

	First case	Second case	Third case
C_1	0.222	0.211	0.099
C_2	0.1029447	0.1470498	0.1191708
C_3	0.130798	0.124367	0.151414
C_4	0.122812	0.116774	0.14217
C_5	0.126572	0.120349	0.146523
C_6	0.020085	0.019098	0.023251
C_7	0.067835	0.0645	0.078528
C_8	0.097036	0.092265	0.112331
C_9	0.110178	0.104761	0.127544

Table 10. Final Ranks.

	First case	Second case	Third case	Final ranks
A_1	1	1	1	1
A_2	10	10	6	10
A_3	3	3	2	3
A_4	6	6	9	6
A_5	4	5	7	7
A_6	5	4	5	5
A_7	9	9	8	9
A_8	8	7	10	8
A_9	7	8	4	4
A_{10}	2	2	3	2

4. Conclusions

The evaluation of innovation and entrepreneurship education in higher vocational colleges is essential for ensuring that students are equipped with the necessary skills to navigate the complexities of modern business environments. This study highlights the importance of a multi-dimensional assessment framework, considering factors such as curriculum relevance, faculty expertise, industry collaboration, and technological integration. By employing MCDM methodologies, this research provides a structured and objective analysis of entrepreneurship programs, helping institutions identify areas for improvement and best practices. We use the

IndetermSoft set to deal with indeterminacy in criteria values. This study proposed three cases to compute the criteria weights by Entropy and ranking the alternatives by EDAS method.

Findings suggest that vocational colleges with strong industry linkages, hands-on training opportunities, and up-to-date curricula perform better in preparing students for entrepreneurial success. The study underscores the need for institutions to foster dynamic learning environments where students can engage with real-world business challenges, participate in startup incubators, and leverage digital tools for innovation. Additionally, continuous faculty development and investment in modern educational infrastructure are necessary to enhance the overall effectiveness of entrepreneurship education.

5. Study Implications

By implementing the proposed evaluation framework, policymakers, educators, and industry stakeholders can make informed decisions to improve the quality of entrepreneurship education in vocational institutions. Strengthening partnerships between academia and industry, integrating digital resources, and fostering experiential learning will ensure that students graduate with the skills, knowledge, and confidence required to succeed in the entrepreneurial landscape. The development of high-quality entrepreneurship education programs in vocational colleges is crucial for promoting innovation-driven economic growth. By continuously refining evaluation models and aligning educational strategies with industry trends, vocational institutions can produce a workforce that is not only employable but also capable of driving business innovation and creating sustainable economic opportunities

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Evaluation of the implementation quality of the Augmented Reality project at the National University of Chimborazo, based on the Neutrosophic SERVQUAL model

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Abstract. This work explores the integration of contemporary educational practices, highlighting Augmented Reality (AR) as an innovative tool at the National University of Chimborazo, La Dolorosa campus, Ecuador. An AR environment based on QR codes has been developed to improve campus navigation. The state of the art was studied to obtain location requirements, create the environment with CoSpaces and Autodesk Maya, and verify user satisfaction through a survey. A usability test was conducted at the end of the product. The ADDIE methodology (Analyze, Design, Develop, Implement, and Evaluate) was used to create the product, where each phase contributed to the implementation of Augmented Reality. In analysis, innovative ideas were investigated and generated; in design, the immersive environment was structured; in development, Autodesk Maya was used for 3D modeling and CoSpaces for programming; in implementation, a QR code was generated to share the product, and it was evaluated through user feedback. The proposal offered an immersive environment that allowed a better understanding of the blocks of the UNACH La Dolorosa campus. Specifically, in this article we address the problem of evaluating the quality of the product. To do so, we conducted a survey of 30 students about their satisfaction with this product and its usefulness for student life. We agreed that this presents the challenges (1) There is indeterminacy and uncertainty in the opinions, (2) It is necessary to use a widely accepted method of quality evaluation. That is why we applied the Neutrosophic SERVQUAL model for higher education found in the literature.

Keywords: Higher Education, Augmented Reality (AR), SERVQUAL, single-valued triangular neutrosophic number, Neutrosophic SERVQUAL model, Likert Scale.

1 Introduction

Traditional education has only been founded on the principle of teaching for centuries, but how does it compare to our current practices? Traditional education focuses on teaching, not learning. It assumes that the student learns as they are taught, but this is not always right. There are often difficulties in applying strategies and there is a lack of understanding of the content given in class.

Current education has changed and step by step we must choose to take the benefits and incorporate them into education. The current reality responds to a complex system that works in a network, and in which a multitude of processes, learning, and discoveries occur in a combined way between a human and a non-human entity.

Nowadays, it is clear that the digital world surrounds us and our entire environment is surrounded by technology. We see a computer, tablet, or smartphone everywhere. This makes it increasingly necessary for these types of tools to be considered and incorporated into educational systems.

Immersive environments have become part of education. Experiential immersive learning is represented by various activities that allow students to immerse themselves in a place that resembles reality and is artificially

constructed (virtual world).

Augmented Reality (AR) has become an innovative technology with the potential to transform the way we interact with our environment. AR continues to evolve in different areas every day. AR is a technology that is being presented as truly useful and with different possibilities to facilitate learning by students in different curricular areas, but it must be recognized that more technological analyses are being carried out than research on its application in the educational field.

The use of markers such as QR codes utilizing innovative technologies such as AR has provided us with several applications. QR codes are an easy way to read information on mobile devices and allow direct or indirect access to information resources. For libraries, this element allows extending services and access to mobile users simply and inexpensively; resource guides, practical information, direct download of documentation, and access to multimedia information.

This project had as its first phase the development of an Augmented Reality environment based on markers for the La Dolorosa Campus of the National University of Chimborazo (UNACH in Spanish). Through the use of this technology, the aim is to provide an interactive and enriched experience that improves orientation, facilitates the location of classrooms, laboratories, and services, and provides information about the offices and classrooms located in the different blocks. In addition, it is expected that this project will promote the development of technological skills and foster innovation in the educational environment of UNACH.

Our purpose with this paper is to assess the quality of the project implementation or, in other words, the satisfaction of students and workers with the results obtained. As an evaluation tool, we use the Neutrosophic SERVQUAL model that appears in [1]. The SERVQUAL model is used to evaluate the quality of services. However, if Higher Education is considered a service to students, and more specifically the service provided to them by using Augmented Reality on the university campus, then it makes sense to evaluate the quality of this service based on the satisfaction of students with this implementation [2-6]. The use of Neutrosophy associated with SERVQUAL has to do with the presence of uncertainty and indeterminacy that is part of any evaluation of human behavior, which is full of subjectivity [1, 7, 8].

This paper is divided into three main sections; first, Preliminaries contains the basic notions of the SERVQUAL model and Neutrosophy. The Results section contains the fundamental elements used in the implementation of Augmented Reality at the La Dolorosa Campus of the National University of Chimborazo, in Ecuador. Then, in this section, we explain the results of applying the Neutrosophic SERVQUAL model that appears in [1], in addition, we recall the fundamental theoretical elements of this model. The last section contains the conclusions.

2 Preliminaries

This section contains the basic concepts about the SERVQUAL model and Neutrosophy, in particular, the single-valued triangular neutrosophic numbers. We will base ourselves on the Neutrosophic SERVQUAL model that appears in [1].

2.1. SERVQUAL model

Two models are known that can be used to measure the quality of service. The first of them is the SERVPERF model, which only works under the results of perceptions, and the second one is the SERVQUAL model, which allows expectations to be compared with perceptions [2-6].

The SERVQUAL model is one of the main tools for measuring service quality. This model attempts to answer three essential questions: When is a service perceived as being of quality? What dimensions make up quality? And, what questions should be included in the quality questionnaire? The model was illustrated so that consumers' perceptions of quality are influenced by five gaps that occur in organizations as described further.

The SERVQUAL model comes from business research, which aims to measure customer satisfaction regarding a service provided [2-6]. This measurement is made as the difference between the customer's perception of the quality of the service and their expectation before receiving it. This method is based on the results of a survey carried out, where customers are asked to evaluate the quality of the service about five dimensions. They are asked to state their degree of agreement or disagreement with each of these dimensions.

The gaps identified in SERVQUAL indicate differences between important aspects of a service, such as customer needs, the service experience itself, and the perceptions that company employees have regarding customer requirements. In the case studied, students and teaching staff are considered equivalent to the business sector's customers. It is clear that they are not the same, however, this method is feasible to use, since the quality of teaching is also a desired state for all actors who are part of the teaching-learning process. In addition, it is an aspect that needs to be measured to advance the effectiveness of this process.

The five dimensions that make up the method are: response capacity, reliability, security, empathy, and tangibility. Reliability refers to the ability to provide a careful and reliable service. Response Capacity is the willingness of the staff providing the service to quickly help with customer requirements. Security consists of the trust

generated by the service received. Empathy focuses on identifying alternatives to provide the customer with personalized attention. Tangibility has to do with the appearance and functionality of physical facilities, equipment, personnel, material, and communication channels.

2.2. The single-valued triangular neutrosophic numbers

Definition 1: ([1, 9-15]) The *Neutrosophic set* N is characterized by three membership functions, which are the truth-membership function T_A, indeterminacy-membership function I_A, and falsity-membership function F_A, where U is the Universe of Discourse, $\forall x \in U$, $T_A(x), I_A(x), F_A(x) \subseteq]^{-}0, 1^{+}[$, and $^{-}0 \leq \inf T_A(x) + \inf I_A(x) + \inf F_A(x) \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^{+}$.

See that according to Definition 1, $T_A(x), I_A(x), F_A(x)$ are real standard or non-standard subsets of $]^{-}0, 1^{+}[$ and hence, $T_A(x), I_A(x), F_A(x)$ can be subintervals of $[0, 1]$.

Definition 2: ([1, 9-15]) The *Single-Valued Neutrosophic Set* (SVNS) N over U is $A = \{< x; T_A(x), I_A(x), F_A(x) > : x \in U\}$, where $T_A: U \to [0, 1]$, $I_A: U \to [0, 1]$, and $F_A: U \to [0, 1]$, $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

The *Single-Valued Neutrosophic number* (SVNN) is symbolized by $N = (t, i, f)$, such that $0 \leq t, i, f \leq 1$ and $0 \leq t + i + f \leq 3$.

Definition 3: ([1, 9-15]) The *single-valued triangular neutrosophic number* $\tilde{a} = \langle (a_1, a_2. a_3); \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle$, is a neutrosophic set on $\mathbb{R}$, whose truth, indeterminacy, and falsity membership functions are defined as follows, respectively:

$$T_{\tilde{a}}(x) = \begin{cases} \alpha_{\tilde{a}}\left(\frac{x - a_1}{a_2 - a_1}\right), & a_1 \leq x \leq a_2 \\ \alpha_{\tilde{a}}, & x = a_2 \\ \alpha_{\tilde{a}}\left(\frac{a_3 - x}{a_3 - a_2}\right), & a_2 < x \leq a_3 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

$$I_{\tilde{a}}(x) = \begin{cases} \frac{(a_2 - x + \beta_{\tilde{a}}(x - a_1))}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \beta_{\tilde{a}}, & x = a_2 \\ \frac{(x - a_2 + \beta_{\tilde{a}}(a_3 - x))}{a_3 - a_2}, & a_2 < x \leq a_3 \\ 1, & \text{otherwise} \end{cases} \quad (2)$$

$$F_{\tilde{a}}(x) = \begin{cases} \frac{(a_2 - x + \gamma_{\tilde{a}}(x - a_1))}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \gamma_{\tilde{a}}, & x = a_2 \\ \frac{(x - a_2 + \gamma_{\tilde{a}}(a_3 - x))}{a_3 - a_2}, & a_2 < x \leq a_3 \\ 1, & \text{otherwise} \end{cases} \quad (3)$$

Where $\alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \in [0, 1]$, $a_1, a_2, a_3 \in \mathbb{R}$ and $a_1 \leq a_2 \leq a_3$.

Definition 4: ([1, 9-15]) Given $\tilde{a} = \langle (a_1, a_2, a_3); \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle$ and $\tilde{b} = \langle (b_1, b_2, b_3); \alpha_{\tilde{b}}, \beta_{\tilde{b}}, \gamma_{\tilde{b}} \rangle$ two single-valued triangular neutrosophic numbers and λ any non-null number in the real line. Then, the following operations are defined:

1. Addition: $\tilde{a} + \tilde{b} = \langle (a_1 + b_1, a_2 + b_2, a_3 + b_3); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}} \rangle$,
2. Subtraction: $\tilde{a} - \tilde{b} = \langle (a_1 - b_3, a_2 - b_2, a_3 - b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}} \rangle$,
3. Inversion: $\tilde{a}^{-1} = \langle (a_3^{-1}, a_2^{-1}, a_1^{-1}); \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle$, where $a_1, a_2, a_3 \neq 0$,
4. Multiplication by a scalar number:
   $\lambda\tilde{a} = \begin{cases} \langle (\lambda a_1, \lambda a_2, \lambda a_3); \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle, & \lambda > 0 \\ \langle (\lambda a_3, \lambda a_2, \lambda a_1); \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle, & \lambda < 0 \end{cases}$
5. Division of two triangular neutrosophic numbers:

$$\frac{\tilde{a}}{\tilde{b}} = \begin{cases} \langle\left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1}\right); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}}\rangle, a_3 > 0 \; and \; b_3 > 0 \\ \langle\left(\frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}\right); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}}\rangle, a_3 < 0 \; and \; b_3 > 0 \\ \langle\left(\frac{a_3}{b_1}, \frac{a_2}{b_2}, \frac{a_1}{b_3}\right); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}}\rangle, a_3 < 0 \; and \; b_3 < 0 \end{cases}$$

6. Multiplication of two triangular neutrosophic numbers:

$$\tilde{a}\tilde{b} = \begin{cases} \langle(a_1b_1, a_2b_2, a_3b_3); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}}\rangle, & a_3 > 0 \; and \; b_3 > 0 \\ \langle(a_1b_3, a_2b_2, a_3b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}}\rangle, & a_3 < 0 \; and \; b_3 > 0 \\ \langle(a_3b_3, a_2b_2, a_1b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}}\rangle, & a_3 < 0 \; and \; b_3 < 0 \end{cases}$$

Where, $\wedge$ is a t-norm and $\vee$ is a t-conorm.

Let $\tilde{a} = \langle(a_1, a_2, a_3); \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}}\rangle$ be a single-valued triangular neutrosophic number, then,

$$S(\tilde{a}) = \frac{1}{8}[a_1 + a_2 + a_3](2 + \alpha_{\tilde{a}} - \beta_{\tilde{a}} - \gamma_{\tilde{a}}) \quad (4)$$

$$\mathbf{A}(\tilde{a}) = \frac{1}{8}[a_1 + a_2 + a_3](2 + \alpha_{\tilde{a}} - \beta_{\tilde{a}} + \gamma_{\tilde{a}}) \quad (5)$$

They are called the score and accuracy degrees of $\tilde{a}$, respectively.

Let $\{\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_n\}$ be a set of n SVTNNs, where $\widetilde{A}_j = \langle(a_j, b_j, c_j); \alpha_{\tilde{a}_j}, \beta_{\tilde{a}_j}, \gamma_{\tilde{a}_j}\rangle$(j = 1, 2 , …, n), then the *weighted mean of the SVTNNs* is calculated with the following Equation:

$$\widetilde{A} = \sum_{j=1}^{n} \lambda_j \widetilde{A}_j \quad (6)$$

Where λ_j is the weight of A_j , $\lambda_j \in [0, 1]$ and $\sum_{j=1}^{n} \lambda_j = 1$.

3 Results

This section contains the results obtained in this project. Especially the evaluation of the quality of the implementation of the Augmented Reality product carried out at UNACH. The following subsection contains some elements of the obtained final Augmented Reality product. Subsection 3.2 contains the details of the Neutrosophic SERVQUAL model proposed in [1]. In addition, we offer the results of the application of such a model in the problem we address in this article.

3.1. Details of the developed Augmented Reality Project

Initially, studies on Intelligent Systems were minimal, because the implementation of these alternative systems had high costs due to the devices required, but they grew thanks to the massification of technology and technological advances in mobile devices and cloud resources, as well as solutions to reduce the cost of the devices such as Google Cardboard, where the affordable price has allowed to increase the possibilities of applying the resources for educational purposes.

Augmented Reality is an interactive technology that enriches the physical environment with superimposed virtual elements. This virtual layer, which is placed between the physical environment and the user, adds textual information, images, videos, or other virtual elements to the person's perception of the physical environment.

Virtual Reality (VR) is an immersive technology that creates simulated environments in which users can interact intuitively and realistically. This technology is transforming diverse fields, from entertainment to education and vocational training.

VR is one in which the participating observer is fully immersed in a completely synthetic world, which may or may not mimic the properties of a real-world environment, whether existing or fictional, but which may also surpass the limits of physical reality by creating a world in which the physical laws governing gravity, time and material properties no longer apply.

Some authors propose a classification method for mixed reality based on three criteria: immersion, interaction, and information. A paradigm that combines technologies that, by mapping the user's space, display virtual content embedded in 3D and recorded in space and time. Virtual objects can be positioned around the real environment, the user, or any other virtual or physical object. In addition, the mixed reality experience must be user-centered and offer natural and immediate interactions.

Figure 1 represents these relationships schematically:

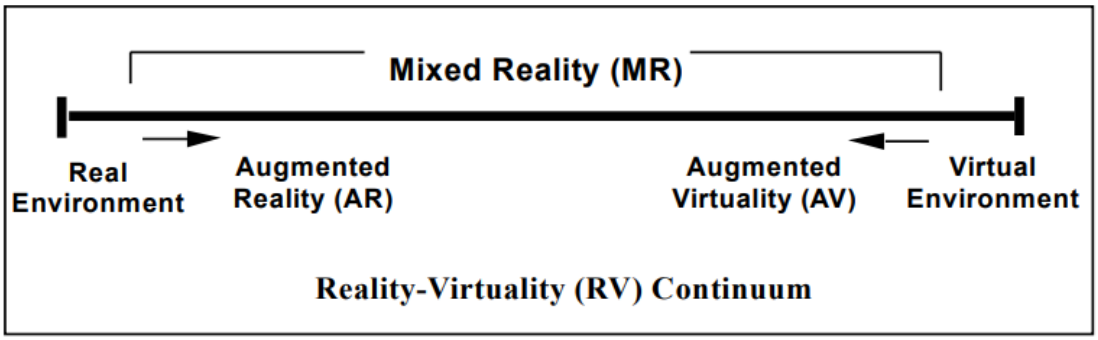

Figure 1. Augmented reality: a class of displays on the reality-virtuality continuum.

Note that the graph in Figure 1 represents what the process of mixed reality, VR, and AR looks like. Adapted from "Augmented Reality: A class of displays on the reality- virtuality continuum" (p.283), by Milgram et al.

An immersive environment has been created in augmented reality, which consists of sound and images and a representation of the UNACH campus La Dolorosa in 3D with a pleasant interface, thus capturing the attention of the student. On the other hand, the use of QR codes allows the augmented reality product to be easily accessible to both internal and external users of the campus.

Figure 2 contains the proposed QR code:

Figure 2. Code to share the final product.

Some visual elements of the final product obtained are shown in Figures 3, 4, and 5.

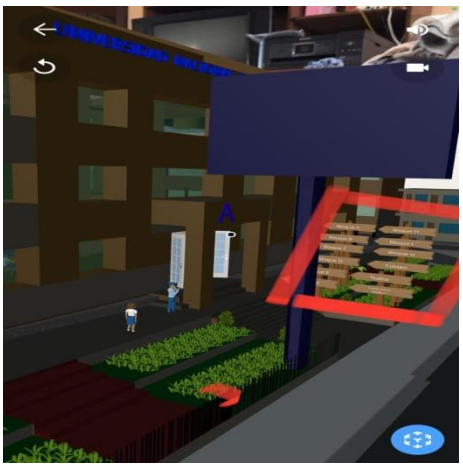

Figure 3. Block A shows the campus location signs according to the product obtained from AR.

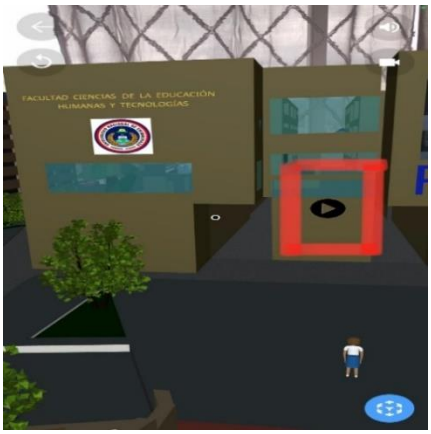

Figure 4. Play black to listen to the offices of the selected block.

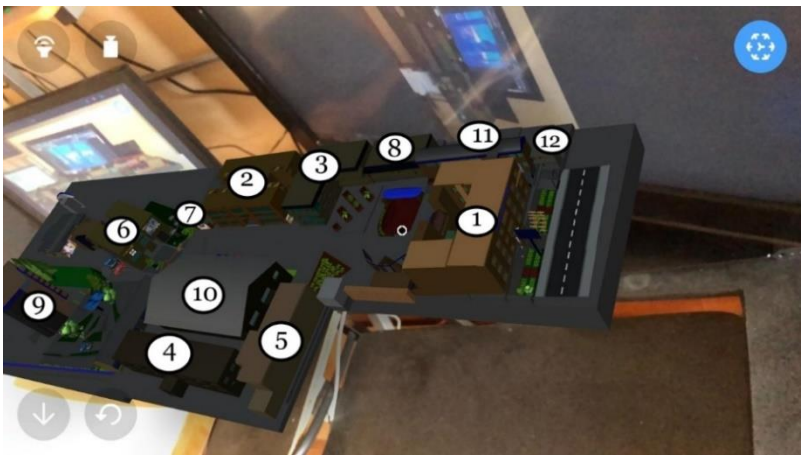

Figure 5. UNACH campus la Dolorosa in Augmented Reality detailing the location of the blocks.

3.2. Neutrosophic SERVQUAL model and results

In this subsection, we consider the Neutrosophic SERVQUAL model proposed in [1]. The model used is based on the idea that there is a staff that provides the service, which is why we must adapt it to the conditions established in the product we provide, where there is a combination of human staff that indicates to the student and workers the steps and conditions to use the Augmented Reality product and the product itself that provides a service.

Specifically, for the evaluation, a usability test is created in Google Forms. Once the test is designed, 30 users are chosen at random at UNACH, La Dolorosa campus, and the users are asked to test the product and complete the usability test. They are then asked to answer the following questions shown in Table 1.

	Dimension	Question
Q_1	Tangibility	Premises and facilities in good condition
Q_2	Tangibility	State-of-the-art equipment and product
Q_3	Tangibility	Properly uniform and impeccable staff
Q_4	Tangibility	Enough attractive and explicit advertising
Q_5	Reliability	The staff shows interest in solving requirements
Q_6	Reliability	The staff and the AR product undertake to comply in the shortest time and do so
Q_7	Reliability	The staff and the AR product offer good service at all times
Q_8	Reliability	The staff and the AR product comply with the service in the established time
Q_9	Response Capacity	The solution/response time of the request is adequate
Q_{10}	Response Capacity	Service is provided promptly
Q_{11}	Response Capacity	The staff and the AR product are always ready to help

Q_{12}	Response Capacity	Staff and the AR product are always available to help with requirements
Q_{13}	Security	The staff and the AR product transmit confidence
Q_{14}	Security	One feels secure when carrying out a procedure
Q_{15}	Security	The staff and the AR product are friendly
Q_{16}	Security	Trained personnel to give reliable answers to requirements. Also, do it with the AR product.
Q_{17}	Empathy	The staff gives you personalized attention.
Q_{18}	Empathy	Adequate staff to offer personalized attention.
Q_{19}	Empathy	Customer service hours are appropriate.
Q_{20}	Empathy	The staff cares about offering well-being.
Q_{21}	Empathy	The staff understands the requirements.

Table 1. Questions asked in the Neutrosophic SERVQUAL model, are classified according to the corresponding dimensions. Modified from [1].

The items that make up the SERVQUAL model questionnaire were measured on a Likert scale where the respondent had to indicate his/her agreement or disagreement with the statement presented to him/her [16, 17, 18]. To this end, an orderly and one-dimensional scale from 1 to 7 is used. In the case of the expectations questionnaire, it is assumed that a score of 7 corresponds to "Extremely important" and 1 to "Extremely unimportant". For the questionnaire on perceptions, a score of 7 represents "that the evaluation is always fulfilled" and 1 that "it is never fulfilled."

The numerical results in the Likert scale[19,20] were using the single-valued triangular neutrosophic numbers scale in Table 2.

Likert scale	Linguistic terms for expectations	Linguistic terms for perceptions	SVTNN
1	Extremely unimportant (EU)	Never fulfilled (NF)	$\langle(0,0,1);0.00,1.00,1.00\rangle$
2	Not very important (NVI)	Few times fulfilled (FTF)	$\langle(0,1,3);0.17,0.85,0.83\rangle$
3	Not important (NI)	Sometimes fulfilled (STF)	$\langle(1,3,5);0.33,0.75,0.67\rangle$
4	Medium (M)	Medium (M)	$\langle(3,5,7);0.50,0.50,0.50\rangle$
5	Important (I)	More fulfilled than not (MF)	$\langle(5,7,9);0.67,0.25,0.33\rangle$
6	Very important (VI)	Most of the time fulfilled (MTF)	$\langle(7,9,10);0.83,0.15,0.17\rangle$
7	Extremely important (EI)	Always fulfilled (AF)	$\langle(9,10,10);1.00,0.00,0.00\rangle$

Table 2. Linguistic variables and their associated SVTNNs for expectations and perceptions. Source: [1].

For processing the data we proceed with collecting the responses to the survey. Let us suppose for the questions Q_i (i =1,2,…, 21) in Table 1, we have both, $v_{ij} = \frac{\#\{Respondents\ having\ response\ Likert=j\}}{30}$ and $\omega_{ij} = \frac{\#\{Respondents\ having\ response\ Likert=j\}}{30}$, for j = 1,2,…,7 the answers in the Likert scale concerning i dimension, where v_{ij} is expectation and ω_{ij} is perception.

The results obtained from the survey were as follows as summarized in Table 3:

Question	Dimension	Expectation	Perception	Gap
Q_1	Tangibility	8.1129	9.9864	1.8735
Q_2	Tangibility	8.7823	9.8493	1.067
Q_3	Tangibility	8.2602	9.3978	1.1376
Q_4	Tangibility	8.8268	9.5435	0.7167
Q_5	Reliability	8.0223	9.7935	1.7712
Q_6	Reliability	8.8561	9.8893	1.0332
Q_7	Reliability	8.7942	9.8064	1.0122
Q_8	Reliability	8.1815	9.0854	0.9039
Q_9	Response Capacity	8.0443	9.4716	1.4273
Q_{10}	Response Capacity	8.5306	9.5598	1.0292
Q_{11}	Response Capacity	8.9116	8.5943	-0.3173
Q_{12}	Response Capacity	8.1399	8.3401	0.2002

Q_{13}	Security	8.9996	7.6565	-1.3431
Q_{14}	Security	8.1226	7.3309	-0.7917
Q_{15}	Security	8.1591	7.6691	-0.49
Q_{16}	Security	8.0154	7.8279	-0.1875
Q_{17}	Empathy	8.1221	8.6111	0.489
Q_{18}	Empathy	8.2798	8.3606	0.0808
Q_{19}	Empathy	8.5181	8.9459	0.4278
Q_{20}	Empathy	8.2234	8.8669	0.6435
Q_{21}	Empathy	8.1256	8.1298	0.0042

Table 3. Results of processing the de- neutrosofied data from the survey.

Note that the results in Table 3 are de-neutrosophied. The average of all gaps is 0.5089, which is a positive result. Although some aspects still need improvement that yielded negative results. In others, the expectation was exceeded[21].

Conclusion

This article consisted of the evaluation of the quality of the Augmented Reality product implemented at the La Dolorosa campus of the National University of Chimborazo in Ecuador. This is a new experience that allows students, workers, and visitors to locate themselves within the different places on campus and learn about the functions and history of each place. It is also an experience that can be enriched with more resourses and extended to other campuses of the university and other universities in the country. A group of 30 randomly selected students were asked to answer a set of 21 questions about the quality of this product and the associated service after living the experience. The Neutrosophic SERVQUAL model proposed in [1] was used. This model was adapted to the conditions of the study carried out. In general, a positive acceptance result was obtained. However, must be working on improving the aspects that turned out negative such as: “The staff and the AR product are always ready to help”, “The staff and the AR product transmit confidence”, “One feels secure when carrying out a procedure”, “The staff and the AR product are friendly” and “Trained personnel to give reliable answers to requirements. Also, do it the AR product". The use of a neutrosophic tool made it possible to take into account indeterminacy and uncertainty in the evaluation, thus achieving greater accuracy.

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Comprehensive Analysis of Application of Multi-Criteria Decision-Making Methods with TreeSoft Set for Teaching Quality Improvement of College English Learning Process

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Abstract: The assessment of teaching quality in College English is a critical component of educational improvement, ensuring that language instruction meets the evolving needs of students in a globalized world. With the increasing integration of digital tools, interactive learning strategies, and student-centered pedagogies, assessing teaching effectiveness has become more complex and multidimensional. This study explores a structured evaluation framework that incorporates pedagogical innovation, student engagement, technological integration, and outcome-based assessment. By applying multi-criteria decision-making (MCDM) techniques, this research provides a systematic method to analyze the effectiveness of College English instruction. TreeSoft set is used to divide the main and sub criteria as a tree to show the relationships between the criteria and alternatives. Then in each sub criterion we compute the criteria weights and in main criteria. The SIWEC method is used to compute the criteria weights and the ARAS method is used to rank the alternatives.

Keywords: TreeSoft Set; MCDM Methods; Application of MCDM Issue; Teaching Quality; College English.

1. Introduction

The role of College English in higher education extends beyond linguistic proficiency; it serves as a bridge for students to engage in international communication, academic research, and professional development. As English continues to dominate as the global lingua franca, universities must ensure that their English programs equip students with essential language skills tailored to real-world applications. However, assessing the quality of College English teaching is not a straightforward process, as it involves multiple variables, including curriculum design, teaching methodology, technological integration, and student learning outcomes. This necessitates a robust and dynamic evaluation framework to measure teaching effectiveness

accurately[1], [2]. With the advancement of digital technology, traditional English teaching methods are undergoing a paradigm shift. Conventional lecture-based instruction is being supplemented—or even replaced—by more interactive and student-centered approaches such as task-based learning, flipped classrooms, and AI-assisted language instruction. While these methodologies promise enhanced learning experiences, their effectiveness varies depending on institutional resources, student demographics, and pedagogical execution. Evaluating the impact of these instructional models requires a systematic analysis that considers both qualitative and quantitative indicators[3], [4].

Student engagement is another key determinant of teaching quality. In College English courses, engagement is not limited to classroom participation but extends to independent learning, collaborative projects, and language immersion opportunities. Factors such as motivation, interest in English learning, and the ability to apply language skills in real-life scenarios play a crucial role in determining the success of a course. A comprehensive evaluation model should, therefore, incorporate both student-centered and teacher-centered factors to provide a holistic picture of teaching effectiveness[2], [5].

Furthermore, assessment and feedback mechanisms are fundamental to improving teaching quality. Effective evaluation strategies include formative assessments, timely feedback, peer reviews, and self-assessments, all of which contribute to a more responsive and adaptive learning environment. In this regard, technology-enhanced assessments, such as AI-driven analytics and learning management systems, have emerged as valuable tools for tracking student progress and refining instructional strategies[6], [7]. These technological advancements need to be integrated into evaluation frameworks to ensure a data-driven approach to improving College English teaching.

Given the multidimensional nature of teaching quality evaluation, the use of multi-criteria decision-making (MCDM) methodologies offers a structured and objective approach to assessment. By assigning appropriate weights to various evaluation criteria, institutions can systematically compare different teaching models and identify best practices. Techniques ARAS and SIWEC models provide a solid foundation for analyzing diverse teaching methods and optimizing instructional strategies accordingly[8], [9].

Assessment the quality of College English teaching requires a comprehensive, data-driven approach that incorporates pedagogy, technology, student engagement, and assessment mechanisms. By leveraging MCDM techniques, universities can develop a structured evaluation framework that facilitates continuous improvement in English language education. This study aims to establish a reliable model for assessing teaching effectiveness and offers recommendations for enhancing College English instruction in higher education institutions.

The main contributions of this study are organized as follows:

We use the TreeSoft set to divide the main and sub criteria as tree, then we obtain the relation between the criteria.

We obtain the weights of the main and sub criteria by the SIWEC method. In each sub criterion we obtain the weights.

Then we obtain the global weights of all criteria.

We use the ARAS method to rank alternatives.

2. MCDM Methodology

This section displays the steps of the two MCDM methods under the TreeSoft set. We compute the criteria weights by the SIWEC method by dividing the criteria and sub criteria into tree, then we compute the criteria weights in each branch, and final we compute the global weights of each criterion. Then we rank the alternatives under these criteria weights.

TreeSoft Set

Let U be a universe of discourse and H is a non-empty subset of U, the powerset of H is a P(H)[10], [11]. then we let the set of attributes as Z and

$$Z = \{Z_1, Z_2, \dots Z_n\} \; and \; n \geq 1 \tag{1}$$

We can form the sub attribute of Z as:

Attribute	Sub-attribute
Z_1	$\{Z_{1,1}, Z_{1,2}, \dots \dots . .\}$
Z_2	$\{Z_{2,1}, Z_{2,2}, \dots \dots . .\}$
Z_3	$\{Z_{3,1}, Z_{3,2}, \dots \dots . .\}$
..	..
Z_n	$\{Z_{n,1}, Z_{n,2}, \dots \dots . .\}$

The attributes can be presented at the first level, and the second level refers to the sub-attributes[12], [13]. Fig 1 shows an example of the TreeSoft set.

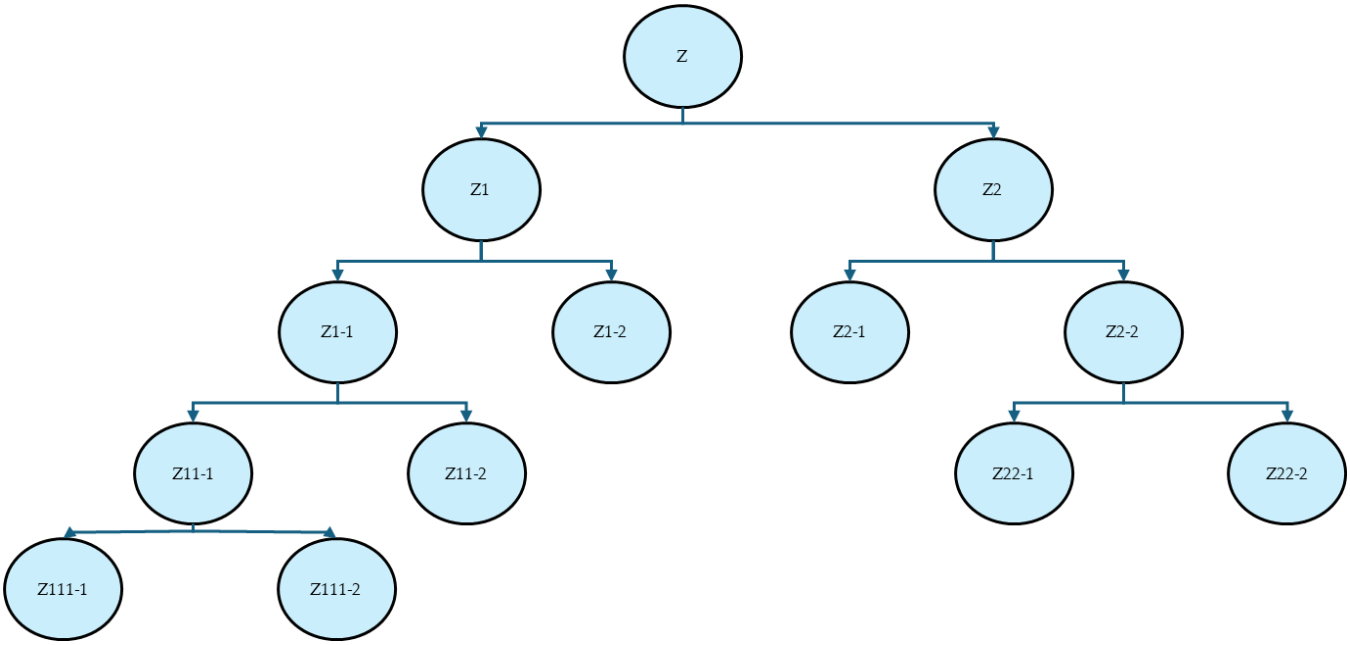

Fig 1. Example of TreeSoft set.

SIWEC Approach

The SIWEC method is used to compute the criteria weights. The steps of this method are organized as follows:

Create the assessment matrix.

Let experts and decision makers evaluate the criteria and alternatives to build the decision matrix.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \tag{2}$$

Normalize the decision matrix.

$$x_{ij} = \frac{a_{ij}}{\max a_{ij}}; i = 1, \dots, m; j = 1, \dots, n \tag{3}$$

Compute the standard deviation.

The aim of this step is to obtain the standard deviation δ_j.

Multiply the standard deviation by the normalization values.

$$r_{ij} = x_{ij}\delta_j \tag{4}$$

Compute the sum results of the previous multiplication

$$d_j = \sum_{j=1}^{n} r_{ij} \tag{5}$$

Compute the criteria weights.

The criteria weights are computed as follows:

$$w_j = \frac{d_j}{\sum_{j=1}^{n} d_j} \tag{6}$$

ARAS Method

The ARAS Method is used to rank the alternatives[14], [15].

The ARAS Method starts with the decision matrix between the criteria and the alternatives.

Normalize the decision matrix

$$k_{ij} = \frac{a_{ij}}{\sum_{i=1}^{m} a_{ij}} \tag{7}$$

Compute the weighted decision matrix.

$$q_{ij} = k_{ij} w_j \tag{8}$$

Compute the optimality function

$$E_i = \sum_{j=1}^{n} q_{ij} \tag{9}$$

Compute the utility degree

$$U_i = \frac{E_i}{V_o} \tag{10}$$

V_o refers to the optimality value of E_i

Final rank of the alternatives.

3. Application

This section shows an application of the proposed approach to show its validation. We obtained eight criteria, and eight alternatives present as eight colleges. Fig 2 shows the eight criteria their sub criteria.

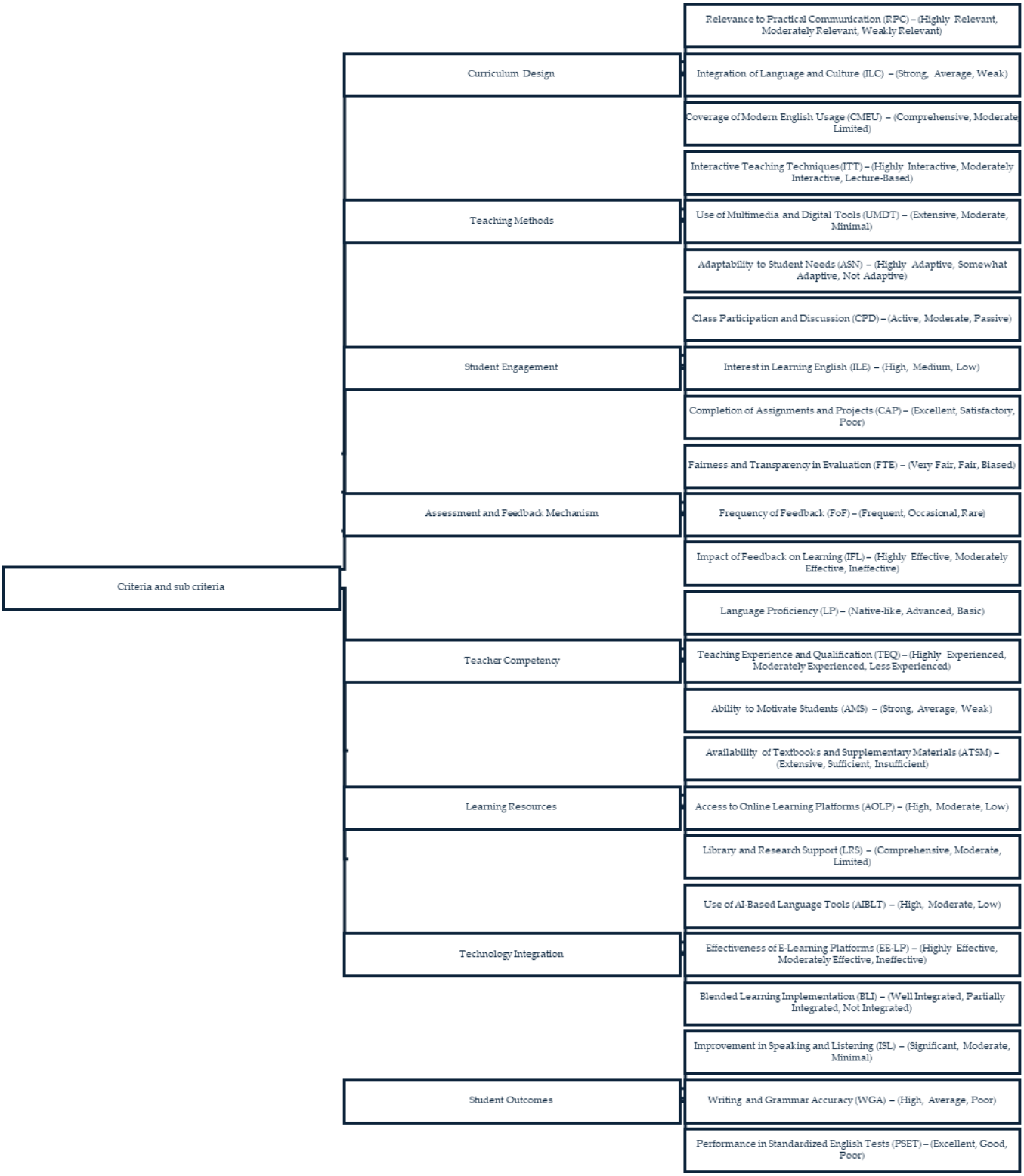
Criteria and sub criteria
Curriculum Design
Relevance to Practical Communication (RPC) – (Highly Relevant, Moderately Relevant, Weakly Relevant)
Integration of Language and Culture (ILC) – (Strong, Average, Weak)
Coverage of Modern English Usage (CMEU) – (Comprehensive, Moderate, Limited)
Teaching Methods
Interactive Teaching Techniques (ITT) – (Highly Interactive, Moderately Interactive, Lecture-Based)
Use of Multimedia and Digital Tools (UMDT) – (Extensive, Moderate, Minimal)
Adaptability to Student Needs (ASN) – (Highly Adaptive, Somewhat Adaptive, Not Adaptive)
Student Engagement
Class Participation and Discussion (CPD) – (Active, Moderate, Passive)
Interest in Learning English (ILE) – (High, Medium, Low)
Completion of Assignments and Projects (CAP) – (Excellent, Satisfactory, Poor)
Assessment and Feedback Mechanism
Fairness and Transparency in Evaluation (FTE) – (Very Fair, Fair, Biased)
Frequency of Feedback (FoF) – (Frequent, Occasional, Rare)
Impact of Feedback on Learning (IFL) – (Highly Effective, Moderately Effective, Ineffective)
Teacher Competency
Language Proficiency (LP) – (Native-like, Advanced, Basic)
Teaching Experience and Qualification (TEQ) – (Highly Experienced, Moderately Experienced, Less Experienced)
Learning Resources
Ability to Motivate Students (AMS) – (Strong, Average, Weak)
Availability of Textbooks and Supplementary Materials (ATSM) – (Extensive, Sufficient, Insufficient)
Access to Online Learning Platforms (AOLP) – (High, Moderate, Low)
Technology Integration
Library and Research Support (LRS) – (Comprehensive, Moderate, Limited)
Use of AI-Based Language Tools (AIBLT) – (High, Moderate, Low)
Effectiveness of E-Learning Platforms (EE-LP) – (Highly Effective, Moderately Effective, Ineffective)
Student Outcomes
Blended Learning Implementation (BLI) – (Well Integrated, Partially Integrated, Not Integrated)
Improvement in Speaking and Listening (ISL) – (Significant, Moderate, Minimal)
Writing and Grammar Accuracy (WGA) – (High, Average, Poor)
Performance in Standardized English Tests (PSET) – (Excellent, Good, Poor)

Fig 2. Eight criteria with sub criteria.

SIWEC Approach

In this section, we compute the weights of the main criteria, then we compute the weights of each sub criterion. Finally, we compute the criteria of global weights. We have eight main criteria, and each criterion has three sub criteria.

We created the decision matrix for the main criteria, sub criteria and global criteria. Three experts evaluated the criteria and alternatives based on scale between 0.1 to 0.9.

Eq. (3) is used to normalize the decision matrix as shown in Table 1.

Then we compute the standard deviation.

Then we multiply the standard deviation by the normalization values using Eq. (4) as shown in Table 2.

Then we compute the sum results of the previous multiplication using Eq. (5).

Then we compute the criteria weights using Eq. (6).

Table 1. Normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.461748	0.401213	1	0.611037	0.391797	0.460139	0.478554	0.45571
A_2	0.589916	0.465819	0.234856	0.463154	1	0.529318	0.507641	0.326497
A_3	0.40579	0.722511	1	0.611037	0.358934	0.722511	1	0.465819
A_4	0.753821	0.337185	0.341632	0.531177	0.267682	0.492153	0.384782	0.78601
A_5	1	0.546072	0.604905	0.537735	0.411619	0.401791	0.589916	1
A_6	0.384782	0.609571	0.530955	0.734856	0.633804	0.711294	0.478554	0.369199
A_7	0.589916	1	0.292357	0.450817	0.53918	0.743308	0.471998	0.481465
A_8	0.39554	0.700607	0.325627	1	0.666667	1	0.671915	0.391103

Table 2. The multiplication of the standard deviation by the normalization values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.099165	0.085068	0.308736	0.109691	0.091468	0.09119	0.092254	0.106851
A_2	0.12669	0.098766	0.072509	0.083144	0.233457	0.1049	0.097861	0.076554
A_3	0.087147	0.153192	0.308736	0.109691	0.083796	0.143186	0.192776	0.109221
A_4	0.16189	0.071492	0.105474	0.095355	0.062492	0.097534	0.074177	0.184297
A_5	0.214759	0.115782	0.186756	0.096532	0.096095	0.079626	0.113722	0.234471
A_6	0.082636	0.129245	0.163925	0.131918	0.147966	0.140963	0.092254	0.086567
A_7	0.12669	0.212027	0.090261	0.080929	0.125875	0.147308	0.09099	0.11289
A_8	0.084946	0.148547	0.100533	0.179516	0.155638	0.198179	0.129529	0.091703

Then we apply the previous steps into each sub criterion. In the first sub criterion, we obtain the normalization values and multiplication of normalization and standard deviation as shown in Table 3.

Table 3. Values of first sub-criterion.

	C_1	C_2	C_3
A_1	0.611037	0.391797	0.460139
A_2	0.463154	1	0.529318
A_3	0.611037	0.358934	0.722511
A_4	0.531177	0.267682	0.492153
A_5	0.537735	0.411619	0.401791
A_6	0.734856	0.633804	0.711294
A_7	0.450817	0.53918	0.743308
A_8	1	0.666667	1
	C_1	C_2	C_3
A_1	0.109691	0.091468	0.09119
A_2	0.083144	0.233457	0.1049
A_3	0.109691	0.083796	0.143186
A_4	0.095355	0.062492	0.097534
A_5	0.096532	0.096095	0.079626
A_6	0.131918	0.147966	0.140963
A_7	0.080929	0.125875	0.147308
A_8	0.179516	0.155638	0.198179

Then we apply the previous steps into each sub criterion. In the second sub criterion, we obtain the normalization values and multiplication of normalization and standard deviation as shown in Table 4.

Table 4. Values of second sub-criterion.

	C_1	C_2	C_3
A_1	0.478554	0.45571	1
A_2	0.507641	0.326497	0.234856
A_3	1	0.465819	1
A_4	0.384782	0.78601	0.341632
A_5	0.589916	1	0.604905
A_6	0.478554	0.369199	0.530955
A_7	0.471998	0.481465	0.292357
A_8	0.671915	0.391103	0.325627
	C_1	C_2	C_3
A_1	0.092254	0.106851	0.308736
A_2	0.097861	0.076554	0.072509
A_3	0.192776	0.109221	0.308736
A_4	0.074177	0.184297	0.105474
A_5	0.113722	0.234471	0.186756
A_6	0.092254	0.086567	0.163925
A_7	0.09099	0.11289	0.090261
A_8	0.129529	0.091703	0.100533

Then we apply the previous steps into each sub criterion. In the third sub criterion, we obtain the normalization values and multiplication of normalization and standard deviation as shown in Table 5.

Table 5. Values of third sub-criterion.

	C_1	C_2	C_3
A_1	0.394745	0.401213	1
A_2	0.351582	0.423118	0.326497
A_3	0.383942	0.379838	0.390526
A_4	0.340827	0.347824	0.347824
A_5	1	0.347824	0.42254
A_6	0.394745	0.454602	0.326497
A_7	0.465109	1	0.379886
A_8	0.416886	0.700607	0.423118
	C_1	C_2	C_3
A_1	0.086119	0.092036	0.224671
A_2	0.076702	0.09706	0.073355
A_3	0.083762	0.087132	0.08774
A_4	0.074356	0.079789	0.078146
A_5	0.218164	0.079789	0.094933
A_6	0.086119	0.104283	0.073355
A_7	0.10147	0.229393	0.08535
A_8	0.090949	0.160714	0.095062

Then we apply the previous steps into each sub criterion. In the fourth sub criterion, we obtain the normalization values and multiplication of normalization and standard deviation as shown in Table 6.

Table 6. Values of fourth sub-criterion.

	C_1	C_2	C_3
A_1	0.555796	0.748162	1
A_2	0.888963	0.970178	0.409757
A_3	0.734689	0.908404	0.72387
A_4	0.487774	0.908404	0.576267
A_5	1	0.908404	0.495284
A_6	0.605091	0.908404	0.614379
A_7	0.734689	0.984745	0.338292
A_8	0.476103	1	0.37679
	C_1	C_2	C_3
A_1	0.105325	0.058712	0.218038
A_2	0.168461	0.076135	0.089343
A_3	0.139225	0.071287	0.157831
A_4	0.092434	0.071287	0.125648
A_5	0.189502	0.071287	0.107991
A_6	0.114666	0.071287	0.133958

A_7	0.139225	0.077278	0.07376
A_8	0.090223	0.078475	0.082154

Then we apply the previous steps into each sub criterion. In the fifth sub criterion, we obtain the normalization values and multiplication of normalization and standard deviation as shown in Table 7.

Table 7. Values of fifth sub-criterion.

	C_1	C_2	C_3
A_1	0.864214	0.30877	1
A_2	0.354118	1	0.489904
A_3	1	0.234856	0.744952
A_4	0.284132	1	0.514542
A_5	0.802453	0.53918	0.572043
A_6	0.30877	0.411619	0.53918
A_7	0.47338	0.76959	0.292357
A_8	0.317402	0.53918	0.325627
	C_1	C_2	C_3
A_1	0.251362	0.091133	0.227385
A_2	0.102997	0.295149	0.111397
A_3	0.290856	0.069318	0.169391
A_4	0.082641	0.295149	0.116999
A_5	0.233398	0.159138	0.130074
A_6	0.089807	0.121489	0.122602
A_7	0.137685	0.227144	0.066478
A_8	0.092318	0.159138	0.074043

Then we apply the previous steps into each sub criterion. In the sixth sub criterion, we obtain the normalization values and multiplication of normalization and standard deviation as shown in Table 8.

Table 8. Values of sixth sub-criterion.

	C_1	C_2	C_3
A_1	0.373139	0.390526	1
A_2	0.373139	0.369151	0.330024
A_3	0.330024	0.347824	0.373139
A_4	0.351582	0.326497	0.373139
A_5	1	0.347824	0.330024
A_6	0.405547	0.475929	0.373139
A_7	0.621752	1	0.308467
A_8	0.416886	0.700607	0.427689
	C_1	C_2	C_3
A_1	0.084821	0.092901	0.229487
A_2	0.084821	0.087816	0.075736
A_3	0.07502	0.082743	0.085631
A_4	0.079921	0.077669	0.085631

A_5	0.227318	0.082743	0.075736
A_6	0.092188	0.113217	0.085631
A_7	0.141335	0.237887	0.070789
A_8	0.094766	0.166665	0.098149

Then we apply the previous steps into each sub criterion. In the seventh sub criterion, we obtain the normalization values and multiplication of normalization and standard deviation as shown in Table 9.

Table 9. Values of seventh sub-criterion.

	C_1	C_2	C_3
A_1	0.326497	0.369199	1
A_2	0.636578	0.657905	0.234856
A_3	1	0.668592	0.744952
A_4	0.443915	0.743308	0.53918
A_5	0.78601	0.476507	0.374495
A_6	0.401213	0.609571	0.563818
A_7	0.615107	1	0.292357
A_8	0.41243	0.700607	0.325627
	C_1	C_2	C_3
A_1	0.074663	0.069025	0.26035
A_2	0.145572	0.123002	0.061145
A_3	0.228679	0.125	0.193948
A_4	0.101514	0.138968	0.140375
A_5	0.179744	0.089087	0.0975
A_6	0.091749	0.113965	0.14679
A_7	0.140662	0.186959	0.076115
A_8	0.094314	0.130985	0.084777

Then we apply the previous steps into each sub criterion. In the eighth sub criterion, we obtain the normalization values and multiplication of normalization and standard deviation as shown in Table 10.

Table 10. Values of eighth sub-criterion.

	C_1	C_2	C_3
A_1	0.630377	0.401213	0.98697
A_2	0.426865	0.498989	0.393311
A_3	0.463821	0.369777	0.64311
A_4	0.988329	0.434383	0.792765
A_5	1	0.679713	1
A_6	0.463154	0.609571	0.661786
A_7	0.71007	1	0.489607
A_8	0.476103	0.700607	0.545325
	C_1	C_2	C_3
A_1	0.14886	0.083792	0.21973
A_2	0.100802	0.104213	0.087563

A_3	0.109529	0.077227	0.143176
A_4	0.233389	0.09072	0.176494
A_5	0.236145	0.141956	0.222631
A_6	0.109372	0.127307	0.147334
A_7	0.167679	0.208847	0.109002
A_8	0.112429	0.14632	0.121406

Finally, we compute the global weights of the criteria as shown in Fig 3.

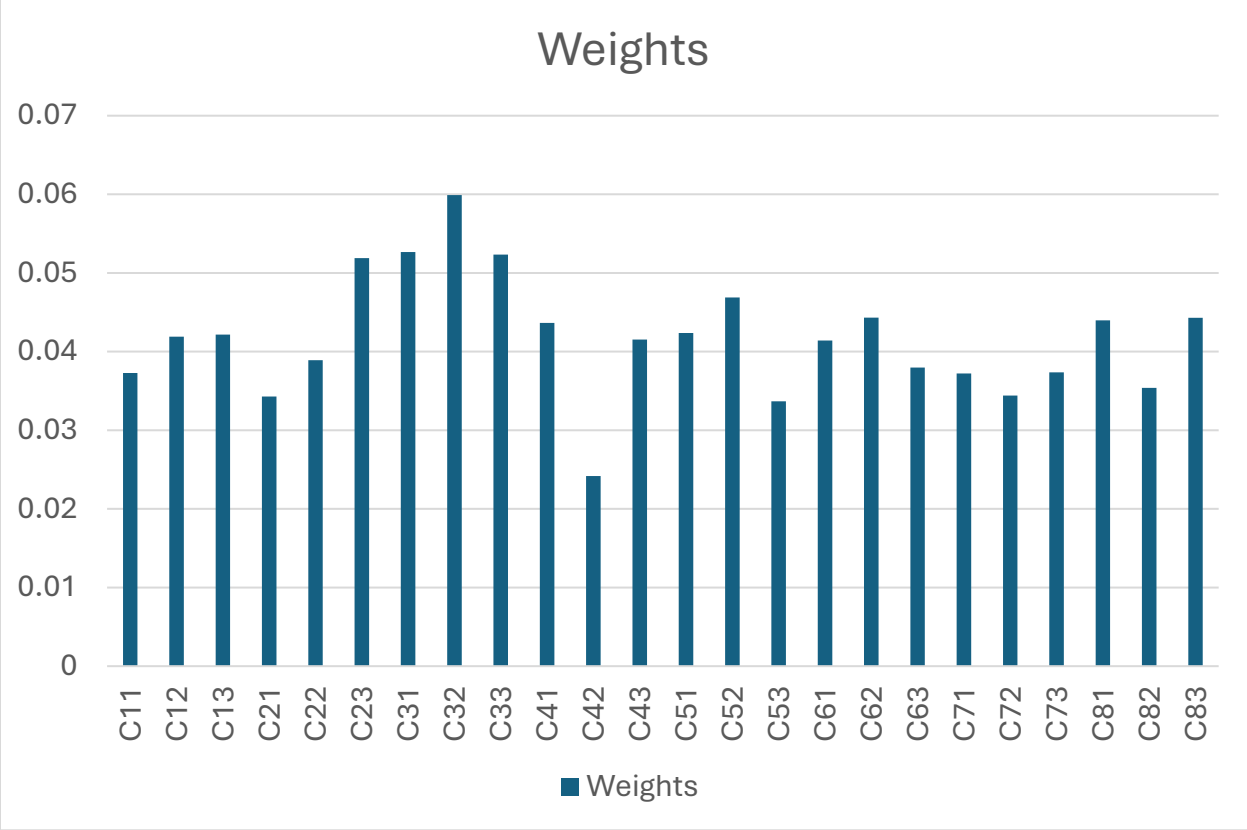

Fig 3. Global weights of the criteria.

ARAS Method

Then we build the decision matrix for all sub criteria. Then we apply the steps of the ARAS method.

Eq. (7) is used to normalize the decision matrix as shown in Table 11.

Eq. (8) is used to compute the weighted decision matrix as shown in table 12.

Eq. (9) is used to compute the optimality function

Eq. (10) is used to compute the utility degree. Final we ranked the alternatives as shown in Fig 4.

Table 11. Normalization values.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
C11	0.100785	0.12876	0.088571	0.164535	0.218268	0.083986	0.12876	0.086334
C12	0.083884	0.097391	0.151059	0.070497	0.11417	0.127446	0.209075	0.146479
C13	0.230929	0.054235	0.230929	0.078893	0.13969	0.122613	0.067514	0.075197
C21	0.123696	0.093759	0.123696	0.10753	0.108857	0.148762	0.091262	0.202437
C22	0.091763	0.234209	0.084066	0.062694	0.096405	0.148443	0.126281	0.15614
C23	0.090927	0.104598	0.142774	0.097254	0.079397	0.140558	0.146884	0.197608
C31	0.104411	0.110757	0.218181	0.083952	0.128708	0.104411	0.102981	0.146599
C32	0.106579	0.076359	0.108943	0.183827	0.233874	0.086346	0.112602	0.091469
C33	0.106182	0.106182	0.112499	0.106182	0.194856	0.115664	0.136282	0.122152
C41	0.098942	0.104344	0.093671	0.085776	0.085776	0.112108	0.246608	0.172775
C42	0.234831	0.076672	0.162015	0.08419	0.099226	0.154497	0.089209	0.099361
C43	0.135744	0.102891	0.09741	0.094669	0.08373	0.163251	0.100151	0.222154
C51	0.112616	0.221207	0.086387	0.081659	0.076941	0.074588	0.154979	0.191623
C52	0.109295	0.125728	0.174155	0.092761	0.087683	0.082618	0.090233	0.237527
C53	0.125152	0.122049	0.17304	0.095268	0.089919	0.098093	0.128787	0.167691
C61	0.126883	0.093882	0.114833	0.11171	0.117648	0.189119	0.093882	0.152043
C62	0.1075	0.111975	0.144438	0.101876	0.235096	0.083996	0.128776	0.086344
C63	0.071924	0.085421	0.133249	0.133249	0.147636	0.10162	0.201307	0.125594
C71	0.194951	0.0803	0.144908	0.144908	0.160554	0.102187	0.089704	0.082487
C72	0.101831	0.077186	0.101831	0.192381	0.085401	0.182092	0.077186	0.182092
C73	0.087637	0.22368	0.074676	0.083767	0.179492	0.120604	0.111397	0.118747
C81	0.059707	0.093728	0.177074	0.078606	0.153392	0.124059	0.114588	0.198846
C82	0.09656	0.085902	0.094165	0.201774	0.135575	0.125225	0.125225	0.135575
C83	0.099619	0.078371	0.097041	0.171823	0.218602	0.114584	0.134464	0.085496

Table 12. Weighted normalization values.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
C11	0.003758	0.004801	0.003302	0.006135	0.008138	0.003131	0.004801	0.003219
C12	0.003516	0.004082	0.006331	0.002954	0.004785	0.005341	0.008762	0.006139
C13	0.009737	0.002287	0.009737	0.003327	0.00589	0.00517	0.002847	0.003171
C21	0.004242	0.003215	0.004242	0.003687	0.003733	0.005101	0.003129	0.006942
C22	0.00357	0.009113	0.003271	0.002439	0.003751	0.005776	0.004913	0.006075
C23	0.004718	0.005427	0.007408	0.005046	0.00412	0.007293	0.007621	0.010253
C31	0.005498	0.005832	0.011489	0.004421	0.006778	0.005498	0.005423	0.00772
C32	0.006385	0.004574	0.006526	0.011013	0.014011	0.005173	0.006746	0.00548
C33	0.005557	0.005557	0.005888	0.005557	0.010198	0.006053	0.007132	0.006393
C41	0.004319	0.004555	0.004089	0.003744	0.003744	0.004894	0.010765	0.007542
C42	0.00568	0.001855	0.003919	0.002036	0.0024	0.003737	0.002158	0.002403
C43	0.005638	0.004274	0.004046	0.003932	0.003478	0.006781	0.00416	0.009228
C51	0.004772	0.009373	0.00366	0.00346	0.00326	0.00316	0.006567	0.00812
C52	0.005125	0.005895	0.008166	0.00435	0.004111	0.003874	0.004231	0.011138
C53	0.004216	0.004111	0.005829	0.003209	0.003029	0.003304	0.004338	0.005648

C61	0.005256	0.003889	0.004756	0.004627	0.004873	0.007833	0.003889	0.006298
C62	0.004764	0.004962	0.0064	0.004514	0.010418	0.003722	0.005706	0.003826
C63	0.002731	0.003243	0.005059	0.005059	0.005605	0.003858	0.007643	0.004768
C71	0.007255	0.002989	0.005393	0.005393	0.005975	0.003803	0.003338	0.00307
C72	0.003503	0.002655	0.003503	0.006618	0.002938	0.006264	0.002655	0.006264
C73	0.003274	0.008357	0.00279	0.00313	0.006706	0.004506	0.004162	0.004437
C81	0.002625	0.004121	0.007786	0.003456	0.006745	0.005455	0.005038	0.008743
C82	0.003417	0.00304	0.003332	0.00714	0.004797	0.004431	0.004431	0.004797
C83	0.004413	0.003472	0.004299	0.007612	0.009684	0.005076	0.005957	0.003787

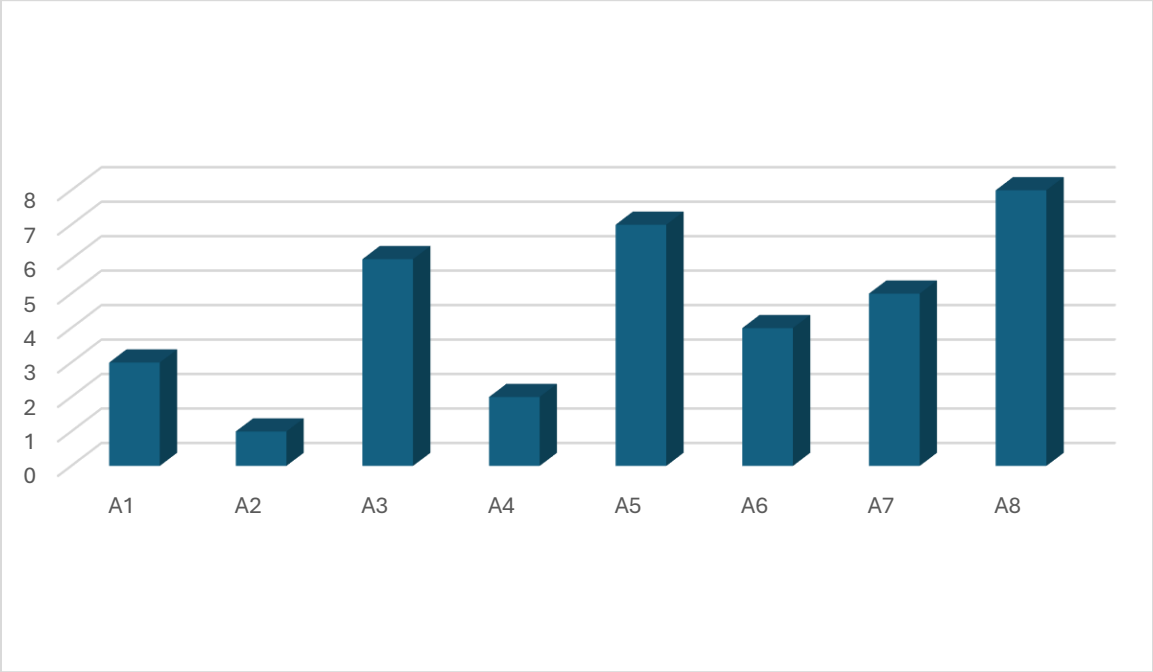

Fig 4. Ranks of alternatives.

4. Conclusions, Implications and Future Works

MCDM methods are used in this study to deal with different criteria and alternatives. Three experts are evaluated the criteria and alternatives based on their opinions. They used a scale between 0.1 to 0.9. Two MCDM methods are used in this study such as SIWEC method to compute the criteria weights and the ARAS method to rank the alternatives. We used the TreeSoft set to deal with different criteria and sub-criteria. We divided the main criteria into eight branches and each branch we computed the criteria weights. Then we obtain the global weights of this study. The results show alternative 8 is the best and alternative 2 is the worst.

The evaluation of teaching quality in College English must go beyond conventional assessment methods to embrace a holistic, multidimensional approach. This research highlights the necessity of integrating pedagogical innovation, student engagement, and technological advancements

into a structured evaluation framework. By implementing MCDM methodologies, universities can ensure a more objective and data-driven assessment of teaching effectiveness.

Findings from this study emphasize the importance of adaptive teaching methods, formative feedback, and technology-enhanced assessments in improving student outcomes. The shift toward digital learning tools, AI-driven instruction, and interactive pedagogies requires educators to continuously refine their strategies to meet the evolving needs of students. A systematic evaluation framework enables institutions to identify strengths and weaknesses, allowing for targeted improvements in College English programs.

Future research should explore the long-term impact of digital and hybrid learning environments on English language acquisition. Additionally, comparative studies across different educational contexts can provide deeper insights into effective teaching practices. As higher education continues to evolve, a well-structured evaluation mechanism will be instrumental in ensuring that College English instruction remains relevant, engaging, and effective for diverse learners.

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Estimating the prevalence of delay in comprehensive and expressive language development in preschool children in Pichincha, Ecuador based on Plithogenic Statistics

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Abstract. Language is defined as the exclusive capacity of human beings to communicate through a set of arbitrary linguistic codes materialized in speech, thanks to which cognitive, executive, social, and academic, among other skills are acquired. In the first three years of life, the child goes through the so-called critical period of language acquisition. In this period, the child develops the cognitive, neuromotor, and sensory skills necessary to generate the communicative bases, including intentionality to communicate turns of intervention, encoding, decoding, and interpretation of the message. If something happens at this stage, linguistic development could be severely compromised. This paper aims to carry out a statistical study on the prevalence of language disorders in children between one and three years of age in the Ecuadorian province of Pichincha, in 2024. Language disorders can affect both comprehensive and expressive language and are multicausal. That is why we apply Plithogenic Statistics in the study. Plithogenic Statistics extends Multivariate Statistics where, in addition to randomness, indeterminacy is included. This is the theory we selected to carry out the study because there are indeterminacies and multiple variables in the problem.

Keywords: Language, Language delay, Comprehensive language, Expressive language, Preschool children, Plithogeny, Plithogenic Statistics.

1 Introduction

Language delay is a condition in which a child does not acquire language skills at the same rate or in the same way as other children of his or her age. It can affect either the understanding of language (comprehensive language) the expression of language (expressive language) or both. Furthermore, comprehension and expressive language delay refers to a delay in the development of a child's language skills, where the acquisition of oral language occurs more slowly compared to his or her peers. This phenomenon can manifest itself in both the ability to understand (comprehension) and the ability to express thoughts and needs.

It is characterized by a chronological delay in the development of speech and language, without there being intellectual, sensory or motor deficits that justify it. Often, children with language delay can understand better than express, which means that they can follow simple instructions or recognize words, but they have difficulty formulating complex sentences or using vocabulary appropriate for their age.

Prevalence may be higher in children with other developmental disorders such as autism or learning disorders; factors that influence prevalence include biological factors (hearing problems, neurodevelopmental disorders, prematurity, and low birth weight). Environmental factors include limited exposure to language, lack of early stimulation, and disadvantaged socioeconomic conditions. This is why early detection and timely intervention are essential to improve the prognosis of children with language delay.

In Ecuador, a study was conducted on the prevalence of delay in the development of comprehensive and

expressive language in children aged 1 to 3 years 11 months, in the Caritas Alegres, Nueva Esperanza, and Cananvalle Child Development Centers in the Tabacundo parish, Pedro Moncayo canton, Pichincha province, period March-August 2018. The population was 152 children, of which 63 were female and 89 were male. The results observed that the children present a considerable delay in language development; at the comprehensive level 31.4% and expressive 50.4%, no child presented malformations of the peripheral oral mechanism and execution of praxis.

In another study conducted on children aged 1 to 3 years 11 months, at the Urku-Sisa and Quinchucajas Child Development Centers in the Cangahua parish, Cayambe canton, Pichincha province. The study population was 57 children, of which 50.8% were male and 49.1% female. The results showed that the prevalence of language delay in comprehension was 42.11% in the comprehension component and 85.96% in the expressive component.

A third study was carried out at the Child Development Centers Angel of Love, Caritas Felices and Domingo Savio in the Cayambe parish, Cayambe canton, Pichincha province. This study determined that children from 1 to 3 years 11 months have a prevalence of delay in the development of comprehensive language of 27.13% and 45.74% in expressive language.

A fourth study was conducted at the Dos Margaritas, Estrellitas Mágicas, and Cayambe Child Development Centers in the Cayambe parish, Cayambe canton, Pichincha province. The study population was 52% male and 48% female. The results obtained identified that the prevalence of language delay in children aged 1 to 3 years 11 months was 20.31% and expressive language was 50%.

In a fifth study conducted on children aged 3 to 5 years at the Hospital del Callao, it was found that 39.5% of children between 3 and 5 years had delays in expressive language, which is more common in boys (72.25%). Between 7 and 15% of preschool-aged children have some type of language disorder, including delay. Children under 3 years of age are especially vulnerable to presenting language delays, with a prevalence that can reach 22% in comprehensive language and 42% in expressive language.

As can be seen from these studies, the number of cases of language delay in children between one and four years old in Ecuador is not small. These disorders are not the same in comprehensive language and expressive language. On the other hand, the causes are different, which is why a statistical study on the subject requires multivariate statistics.

The purpose of this article is to understand the prevalence of comprehensive and expressive language disorders in preschool children from 1 to 3 years of age who attended 23 public institutions located in various areas of the Ecuadorian province of Pichincha, in 2024. These data were part of the graduation works of the students of the Speech Therapy program at the Central University of Ecuador, which are in the institutional repository.

For the calculations, we use Plithogenic Statistics [1, 2]. This is a theory based on F. Smarandache's Plithogeny, which deals with the dynamics between multiple concepts or variables and their interactions between the concepts or their opposites or their neutrals (which is neither the concept nor its opposite) [3-13]. The way they interact produces results that cannot be described with classical dialectics.

Plithogenic Statistics, in particular, generalizes Multivariate Statistics, where the uncertainty and indeterminacy that exists between the variables involved, in addition to randomness, are incorporated [14-16]. It also generalizes Neutrosophic Statistics, where the latter uses the methods of Classical Statistics applied to data or parameters in interval form, or when the size of the sample or population is not exactly known [17].

This article is divided into a Preliminaries section, where we explain the basic concepts of Plithogenic Statistics. In the Results section, we explain the methods used in the study and the results. We conclude the article with the conclusions.

2 Preliminaries

The Plithogenic Probability of an event's occurrence is composed of the probabilities of the event occurring for all the random variables or parameters that constitute it [1]. The Plithogenic Probability based on the Plithogenic Variate Analysis, is multi-dimensional. It could be said that it is a probability of sub-probabilities, where each sub-probability refers to the behavior of a variable assuming that the event is produced by one or more variables. Each variable is represented by a Probability Distribution (Density) Function (PDF).

According to F. Smarandache's classification, the Subclasses of Plithogenic Probability are the following:

(1) Classical MultiVariate Probability: Whether all PDFs are classical.

(2) Plithogenic Neutrosophic Probability is defined when the PDF is expressed in the form of (T, I, F), where T is the probability that the event occurs, I is the indeterminacy probability that the event occurs and F is the probability that the event does not occur. Such that the following is fulfilled: $T, I, F \in [0, 1]$, $0 \leq T + I + F \leq 3$.

(3) Plithogenic Indeterminate Probability: When all PDFs have indeterminate data or arguments.

(4) Plithogenic Intuitionistic Fuzzy Probability: When PDFs have the form (T, F) where $T, F \in [0, 1]$, $0 \leq T + F \leq 1$.

(5) Plithogenic Picture Fuzzy Probability: When PDFs have the form (T, N, F). $T, N, F \in [0, 1]$, $0 \leq T + N + F \leq 1$; where T is the probability that the event occurs, N is the neutral probability of the event occurring or not occurring, and F is the probability that the event does not occur.

(6) Plithogenic Spherical Fuzzy Probability: When PDFs have the form (T, H, F). $T, H, F \in [0, 1]$, $0 \leq T^2 +$

$H^2 + F^2 \leq 1$; where T is the probability that the event occurs, H is the neutral probability of it occurring or not, and F is the probability that the event does not occur.

(7) Plithogenic (fuzzy-extension) Probability: when we have that all PDFs are in the (fuzzy-extension set) style.

(8) Plithogenic Hybrid Probability: when some PDFs are in one of the above styles and others are in other styles.

Plithogenic Statistics (PS) comprises the analysis and observations of the events studied by Plithogenic Probability.

Plithogenic Statistics generalizes classical MultiVariate Statistics, and in turn allows an analysis of many output variables that are neutrosophic or indeterminate. It is also a multi-indeterminate statistic.

Various Subclasses of Plithogenic Statistics are as follows:

- Multivariate Statistics,
- Plithogenic Neutrosophic Statistics,
- Plithogenic Indeterminate Statistics,
- Plithogenic Intuitionistic Fuzzy Statistics,
- Plithogenic Picture Fuzzy Statistics,
- Plithogenic Spherical Fuzzy Statistics,
- and in general: Plithogenic (fuzzy-extension) Statistics,
- and Plithogenic Hybrid Statistics.

On the other hand, Plithogenic Refined Statistics is the most general form of statistics that studies the analysis and observations of the events described by Plithogenic Refined Probability.

In classical inference statistics, the population's average of the variable is estimated from the sample's average.

When we have a classical random variable, the exact size of the sample is known and all elements of the sample belong 100% to the population. However, this does not reflect the dynamics of a population such as the student population, which is the example illustrated by F. Smarandache, where there is fluctuation of students within courses, in addition to the fact that the membership of each student varies depending on whether he or she is studying a course full-time, part-time or over-time.

In a Neutrosophic Population, each element has a triple probability of membership equal to (T_j, I_j, F_j) such that $0 \leq T_j + I_j + F_j \leq 3$.

If we assume we have the dataset (T_j, I_j, F_j) for $j = 1, 2, \dots, n$, where n is the sample size, then the average probability for all data in the sample is calculated by Equation 1.

$$\frac{1}{n}\sum_{j=1}^{n}(T_j, I_j, F_j) = \left(\frac{\sum_{j=1}^{n} T_j}{n}, \frac{\sum_{j=1}^{n} I_j}{n}, \frac{\sum_{j=1}^{n} F_j}{n}\right) \quad (1)$$

3 Results

This research evaluated a population of 3,374 boys and girls who attended 23 public institutions located in various areas of the Ecuadorian province of Pichincha in 2024. These data were part of the graduation works of the students of the Speech Therapy program at the Central University of Ecuador.

A random sample was selected using stratified random sampling, the size of which was calculated based on Equation 2:

$$n = \frac{Z^2 Npq}{e^2(N-1)+Z^2 pq} \quad (2)$$

Where,

n = Sample size,

N = Population size (3374),

Z = deviation for the desired confidence level (1.96),

e = Maximum allowed error margin (0.05),

p = proportion that we expect to find (p= 0.5).

Thus, $n = 344.9828700401366 \approx 345$ children. This is the sample size.

23 specialists were hired, one for each institution, who had to study an average of 15 students per institution.

The aspects to be measured are the following:

First, there are the dimensions of comprehensive language and expressive language. Each of them is evaluated according to the following sub-aspects as shown in Table 1:

Language	Aspect to be measured	Definition
Expressive Language	Babbling	The quantity and variety of sounds the child produces.
	First words	The age at which the child says his or her first words and the number of words he or she knows.
	Use of phrases	The ability to combine words to form simple sentences.
	Vocabulary	The variety and quantity of words the child uses regularly.
	Clarity of speech	The ease with which the child can be understood by others.
Comprehensive Language	Response to commands	The child's ability to follow simple instructions
	Understanding questions	The ability to understand and answer simple questions.
	Recognition of words and objects	The ability to recognize and relate words to the objects they represent.
	Joint attention	The ability to maintain gaze and attention during a conversation or activity.
	Social interaction	The ability to communicate and play with other children and adults.

Table 1: Type of language and aspects to be measured and their definitions.

It is important to remember that each child is unique and can develop at his or her own pace. This is why the specialist cannot be able to determine a single probability value as occurs in classical probabilities. It may be necessary to evaluate with the support of more than one probabilistic value on the probability of occurrence, the indeterminacy of occurrence and the probability of non-occurrence.

To facilitate the assessment, a linguistic scale was associated with a numerical scale between 0 and 1 for each of the probabilities of event fulfillment T, indeterminacy I, and non-fulfillment of the event F. Therefore, for each aspect to be measured, each specialist evaluates each child in terms of the aspect with a triple of (T, I, F) probability values. The proposed scale is shown in Table 2.

Linguistic value	Numeric value
Very low	0.2
Low	0.3
More or less low	0.4
Neither low nor high	0.5
More or less high	0.6
High	0.7
Very high	0.8

Table 2: Scale of linguistic and numerical values corresponding to the probabilities of occurrence, non-occurrence, or indeterminacy of events.

The utility of the scale shown in Table 2 is that the surveyed expert can more easily determine the triple values that he or she proposes. For example, if the specialist determines that an attribute is satisfied with a "Very high" probability with a "Low" indeterminacy and a "Very low" probability of non-satisfaction, then the triple obtained is $(0.8, 0.3, 0.2)$.

Next, we will establish the procedure that we follow for the evaluations,

It is based on the set of experts as $E = \{e_1, e_2, \dots, e_{23}\}$, the children evaluated correspond to the set $U = \{u_1, u_2, \dots, u_{345}\}$.

The following aspects need to be evaluated:
A_1: Expressive Language,
A_2: Comprehensive Language.
The following sub-attributes to A_1 be evaluated are:
A_{11}: Babbling,
A_{12}: First words,
A_{13}: Use of phrases,
A_{14}: Vocabulary,
A_{15}: Clarity of speech.
The following sub-attributes to A_2 be evaluated are:
A_{21}: Response to commands,
A_{22}: Understanding questions,

A_{23}: Recognition of words and objects,
A_{24}: Joint attention,
A_{25}: Social interaction.
The study uses the Plithogenic Refined Probability (PRP) as follows:

$$PRP_1(u_i/e_j) = \left(\left(p_{A_1ij}, p_{A_2ij}\right), \left(indp_{A_1ij}, indp_{A_2ij}\right), \left(np_{A_1ij}, np_{A_2ij}\right)\right) \quad (3)$$

Where:

u_i: This is the ith child to be investigated ($i = 1, 2, \dots, 345$),
e_j: This is the jth expert who conducts the investigation ($j = 1, 2, .., 23$),

p_{A_1ij}: It is the probability assigned to the ith child by the jth expert that he or she has a disorder in expressive language.

p_{A_2ij}: It is the probability assigned to the ith child by the jth expert that he or she has a disorder in comprehensive language.

$indp_{A_1ij}$: It is the probability assigned to the ith child by the jth expert such that there is indeterminacy in establishing whether he or she has an expressive language disorder.

$indp_{A_2ij}$: It is the probability assigned to the ith child by the jth expert such that there is indeterminacy in establishing whether he or she has a disorder in comprehensive language.

np_{A_1ij}: It is the probability assigned to the ith child by the jth expert such that he or she does not have a disorder in expressive language.

np_{A_2ij}: It is the probability assigned to the ith child by the jth expert such that he or she does not have a disorder in comprehensive language.

Equation 3 can be broken down into further detail by substituting the probabilities with respect to the attributes into the probabilities with respect to the sub-attributes, as follows, see Equation 4:

$$\begin{aligned} &PRP_2(u_i/e_j) \\ &\left(\left(p_{A_{11}ij}, p_{A_{12}ij}, p_{A_{13}ij}, p_{A_{14}ij}, p_{A_{15}ij}, p_{A_{21}ij}, p_{A_{22}ij}, p_{A_{23}ij}, p_{A_{24}ij}, p_{A_{25}ij}\right)\right. \\ &= \left(indp_{A_{11}ij}, indp_{A_{12}ij}, indp_{A_{13}ij}, indp_{A_{14}ij}, indp_{A_{15}ij}, indp_{A_{21}ij}, indp_{A_{22}ij}, indp_{A_{23}ij}, indp_{A_{24}ij}, indp_{A_{25}ij}\right), \\ &\left.\left(np_{A_{11}ij}, np_{A_{12}ij}, np_{A_{13}ij}, np_{A_{14}ij}, np_{A_{15}ij}, np_{A_{21}ij}, np_{A_{22}ij}, np_{A_{23}ij}, np_{A_{24}ij}, np_{A_{25}ij}\right)\right) \end{aligned} \quad (4)$$

Where:

$p_{A_{1k}ij}$: It is the sub-probability of p_{A_1ij} to A_{1k} ($k = 1, 2, \dots, 5$).

$p_{A_{2k}ij}$: It is the sub-probability of p_{A_2ij} to A_{2k} ($k = 1, 2, \dots, 5$).

$indp_{A_{1k}ij}$: It is the sub-probability of $indp_{A_1ij}$ to A_{1k} ($k = 1, 2, \dots, 5$).

$indp_{A_{2k}ij}$: It is the sub-probability of $indp_{A_2ij}$ to A_{2k} ($k = 1, 2, \dots, 5$).

$np_{A_{1k}ij}$: It is the sub-probability of np_{A_1ij} to A_{1k} ($k = 1, 2, \dots, 5$).

$np_{A_{2k}ij}$: It is the sub-probability of np_{A_2ij} to A_{2k} ($k = 1, 2, \dots, 5$).

Equation 4 is simplified by Equation 5:

$$\begin{aligned} \overline{PRP}_2(u_i) = &\left(\left(\bar{p}_{A_{11}i}, \bar{p}_{A_{12}i}, \bar{p}_{A_{13}i}, \bar{p}_{A_{14}i}, \bar{p}_{A_{15}i}, \bar{p}_{A_{21}i}, \bar{p}_{A_{22}i}, \bar{p}_{A_{23}i}, \bar{p}_{A_{24}i}, \bar{p}_{A_{25}i}\right)\right. \\ &\left(\overline{indp}_{A_{11}i}, \overline{indp}_{A_{12}i}, \overline{indp}_{A_{13}i}, \overline{indp}_{A_{14}i}, \overline{indp}_{A_{15}i}, \overline{indp}_{A_{21}i}, \overline{indp}_{A_{22}i}, \overline{indp}_{A_{23}i}, \overline{indp}_{A_{24}i}, \overline{indp}_{A_{25}i}\right), \\ &\left.\left(\overline{np}_{A_{11}i}, \overline{np}_{A_{12}i}, \overline{np}_{A_{13}i}, \overline{np}_{A_{14}i}, \overline{np}_{A_{15}i}, \overline{np}_{A_{21}i}, \overline{np}_{A_{22}i}, \overline{np}_{A_{23}i}, \overline{np}_{A_{24}i}, \overline{np}_{A_{25}i}\right)\right) \end{aligned} \quad (5)$$

Where:

$$\bar{p}_{A_{1k}i} = \frac{\sum_{j=1}^{23} p_{A_{1k}ij}}{23},$$

$$\bar{p}_{A_{2k}i} = \frac{\sum_{j=1}^{23} p_{A_{2k}ij}}{23},$$

$$\overline{\overline{indp}}_{A_{1k}i} = \frac{\sum_{j=1}^{23} indp_{A_{1k}ij}}{23},$$

$$\overline{\overline{indp}}_{A_{2k}i} = \frac{\sum_{j=1}^{23} indp_{A_{2k}ij}}{23},$$

$$\overline{\overline{np}}_{A_{1k}i} = \frac{\sum_{j=1}^{23} np_{A_{1k}ij}}{23},$$

$$\overline{\overline{np}}_{A_{2k}i} = \frac{\sum_{j=1}^{23} np_{A_{2k}ij}}{23}.$$

Equation 5 is further reduced to obtain Equation 6 which corresponds to all children studied: This formula represents the probability that any given child will meet them.

$$\overline{\overline{PRP}}_2(\mathrm{u}) = \begin{matrix} \left(\left(\bar{\bar{p}}_{A_{11}}, \bar{\bar{p}}_{A_{12}}, \bar{\bar{p}}_{A_{13}}, \bar{\bar{p}}_{A_{14}}, \bar{\bar{p}}_{A_{15}}, \bar{\bar{p}}_{A_{21}}, \bar{\bar{p}}_{A_{22}}, \bar{\bar{p}}_{A_{23}}, \bar{\bar{p}}_{A_{24}}, \bar{\bar{p}}_{A_{25}}\right)\right. \\ \left(\overline{\overline{indp}}_{A_{11}}, \overline{\overline{indp}}_{A_{12}}, \overline{\overline{indp}}_{A_{13}}, \overline{\overline{indp}}_{A_{14}}, \overline{\overline{indp}}_{A_{15}}, \overline{\overline{indp}}_{A_{21}}, \overline{\overline{indp}}_{A_{22}}, \overline{\overline{indp}}_{A_{23}}, \overline{\overline{indp}}_{A_{24}}, \overline{\overline{indp}}_{A_{25}}\right), \\ \left.\overline{\overline{np}}_{A_{11}}, \overline{\overline{np}}_{A_{12}}, \overline{\overline{np}}_{A_{13}}, \overline{\overline{np}}_{A_{14}}, \overline{\overline{np}}_{A_{15}}, \overline{\overline{np}}_{A_{21}}, \overline{\overline{np}}_{A_{22}}, \overline{\overline{np}}_{A_{23}}, \overline{\overline{np}}_{A_{24}}, \overline{\overline{np}}_{A_{25}}\right) \end{matrix} \quad (6)$$

Where:

$$\bar{\bar{p}}_{A_{1k}} = \frac{\sum_{j=1}^{345} \bar{p}_{A_{1k}i}}{345},$$

$$\bar{\bar{p}}_{A_{2k}} = \frac{\sum_{j=1}^{345} \bar{p}_{A_{2k}i}}{345},$$

$$\overline{\overline{indp}}_{A_{1k}} = \frac{\sum_{j=1}^{345} \overline{indp}_{A_{1k}i}}{345},$$

$$\overline{\overline{indp}}_{A_{2k}} = \frac{\sum_{j=1}^{345} \overline{indp}_{A_{2k}i}}{345},$$

$$\overline{\overline{np}}_{A_{1k}} = \frac{\sum_{j=1}^{345} \overline{np}_{A_{1k}i}}{345},$$

$$\overline{\overline{np}}_{A_{2k}} = \frac{\sum_{j=1}^{345} \overline{np}_{A_{2k}i}}{345}.$$

What relates formula 6 to an aggregation formula is calculated with Equation 7:

$$\overline{\overline{PRP}}_1(\mathrm{u}) = \left(\left(min_k p_{A_{1k}}, min_k p_{A_{2k}}\right), \left(max_k indp_{A_{1k}}, max_k indp_{A_{2k}}\right), \left(max_k np_{A_{1k}}, max_k np_{A_{2k}}\right)\right) \quad (7)$$

After applying the calculations on the formulas above, i.e. Equations 3-7, the following results were obtained:

$$\overline{\overline{PRP}}_1(\mathrm{u}) = \left((0.487,0.267), (0.273,0.374), (0.681,0.776)\right).$$

This means that the plithogenic neutrosophic probability is $PNP_{A_1} = (0.487,0.273,0.681)$ for having an alteration in expressive language, whereas $PNP_{A_2} = (0.267,0.374,0.776)$ is the probability that there will be an alteration in comprehensive language.

These values can be reduced to a single numerical value, with Equation 8 called the score function [18,19,20]:

$$\mathcal{S}\left((T,I,F)\right) = \frac{2+T-I-F}{3} \quad (8)$$

Therefore, we have $\mathcal{S}\left(PNP_{A_1}\right) = 0.511$ and $\mathcal{S}\left(PNP_{A_2}\right) = 0.372$. These values are classified, according to the distance to the values shown in Table 1, as "neither low nor high" probability of suffering from expressive language disorder and "more or less low" probability of suffering from comprehensive language disorder.

Conclusion

The age between 1 and 3 years is critical for language development in children. Delays and alterations in the proper use of language may occur in children of that age than would be expected. In this article, we presented a study carried out on a random sample of 345 children from a population of 3374, who are associated with childcare institutions in the province of Pichincha in Ecuador, 2024. For the calculations, we based ourselves on Plithogenic Statistics, because each of the alterations has probabilities and sub-probabilities related to the attributes and sub-attributes being studied. As attributes, the alteration of expressive language and the alteration of comprehensive language were evaluated. Each of them was evaluated according to five sub-attributes. The results were that the

probability of suffering from alterations in expressive language in this region is “neither low nor high” and the probability of suffering from alterations in comprehensive language is “more or less low”.

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Decision Making Methodology to Assess Big Data Professional Education in Vocational and Technical Colleges: Forest HyperSoft Set Approach and Implementation

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Abstract: The rapid advancement of big data technologies has transformed industries, necessitating the development of specialized educational programs to equip students with relevant skills. Vocational and technical colleges play a crucial role in bridging the skill gap by offering big data professional education tailored to industry demands. However, assessing the quality and effectiveness of such programs remains a challenge due to the evolving nature of big data, the need for practical training, and alignment with industry requirements. This study proposes a comprehensive evaluation framework incorporating multiple criteria such as curriculum relevance, faculty expertise, infrastructure, industry collaboration, and student outcomes. By employing a Multi-Criteria Decision-Making (MCDM) approach, this research provides an in-depth analysis of big data education quality, ensuring that vocational institutions produce industry-ready graduates. We use two MCDM methods such as CRITIC method to compute the criteria weights and the VIKOR method to rank the alternatives. These methods are used with the Forest HyperSoft set to deal with criteria, sub criteria and sub-sub-criteria. We use five criteria and six alternatives in this study. These criteria are divided into Trees. Then we compute the criteria weights and rank the alternatives under each criterion. Then we obtain the rank of each criterion and combine these ranks into a final rank.

Keywords: Decision Making; Forest HyperSoft Set; Big Data; Education; Technical Colleges.

1. Introduction

In the era of digital transformation, big data has emerged as a critical component across various industries, influencing decision-making, automation, and business intelligence. As organizations increasingly rely on data-driven strategies, there is a growing demand for professionals equipped with big data skills, including data analytics, machine learning, and cloud computing. Vocational and technical colleges have recognized this trend and are offering specialized programs in big data to prepare students for industry-specific roles[1], [2]. However, the quality of these programs

varies significantly, raising concerns about their effectiveness in meeting industry expectations. A systematic evaluation framework is necessary to assess whether these educational programs equip students with the competencies required in the evolving technological landscape.

Evaluating the quality of big data professional education in vocational and technical colleges involves multiple dimensions. A well-structured curriculum that aligns with industry standards, faculty expertise in both theoretical and practical applications, access to modern technological infrastructure, and strong industry collaborations are essential factors that determine the success of these programs. Additionally, student learning outcomes, employability rates, and exposure to real-world big data projects are crucial indicators of program effectiveness[3], [4]. Traditional educational evaluation methods may not be sufficient to capture the complexities of big data education, necessitating the use of advanced evaluation methodologies such as Multi-Criteria Decision-Making (MCDM) approaches.

A robust evaluation framework should consider not only the academic aspects of big data education but also the practical training opportunities and industry involvement. Many vocational colleges struggle with outdated curricula, limited faculty expertise in emerging technologies, and inadequate access to real-world datasets, which hinder the effectiveness of their programs. Furthermore, the lack of industry certifications, limited internship opportunities, and weak industry-academic partnerships can affect students' readiness for the job market[5], [6]. Therefore, a systematic assessment that integrates qualitative and quantitative criteria is essential to ensure the continuous improvement of big data professional education.

This study proposes a comprehensive evaluation framework that incorporates key criteria such as curriculum relevance, faculty competency, infrastructure support, industry linkages, and student performance outcomes. By employing MCDM techniques, the study aims to rank and compare different educational programs, identifying best practices and areas for improvement. The findings will provide valuable insights for policymakers, educators, and industry stakeholders in refining big data curricula, enhancing teaching methodologies, and strengthening industry-academic collaborations[7], [8].

Ultimately, improving the quality of big data professional education in vocational and technical colleges is crucial for producing a highly skilled workforce capable of driving digital innovation. As the demand for data professionals continues to rise, vocational institutions must adopt evidence-based strategies to enhance their educational offerings.[9], [10] This research contributes to the ongoing discourse on higher education quality assessment, offering practical recommendations to optimize big data training programs and ensure that graduates meet the evolving needs of the industry

The main contributions of this study are organized as follows:

We propose a MCDM approach by using two methods, such as CRITIC method to compute the criteria weights and the VIKOR method to rank the alternatives.

The decision matrix is created by opinions of three experts and decision makers who have experience in the related field.

We employ the Forest HyperSoft set to divide the criteria as levels and each criterion present as TreeSoft set. Then we compute the criteria weights and ranking the alternatives in each criterion.

Finally, we combined the rank of the alternatives as a final rank. We show the rank of alternative stable under different criteria.

The rest of this study is organized as follows: Section 2 shows the steps of the CRITIC methodology to compute the criteria weights and the VIKORB methodology to rank the alternatives. Also, this section shows the ideal of the Forest HyperSoft to divide each criterion as Tree. Section 3 shows the results of the proposed approach. Section 5 shows the conclusions of this study.

2. Forest HyperSoft Set with CRITIC and VIKOR Methods

This section shows the steps of the CRITIC method to compute the criteria weights and the VIKOR method to rank the alternatives. We use the Forest HyperSoft set to classify the different criteria into different TreeSoft. Ans in each TreeSoft we rank the alternatives and compute the criteria weights. Finally, we combined the rank of the alternatives to select the best one.

Forest HyperSoft Set

Let U be a universe of discourse and H non-empty set of U, and A be a set of criteria, each of this criterion has different levels.

Level 1 refers to the sub criteria values

Level 2 refers to the sub-sub-criteria values.

Level 3 refers to the sub-sub-sub-criteria values.

Each of the criteria refers to the TreeSoft set and all these TreeSoft set can be combined to obtain the Forest HyperSoft Set[11], [12].

Fig 1 shows the example of the Forest HyperSoft set with different TreeSoft set by different level with values.

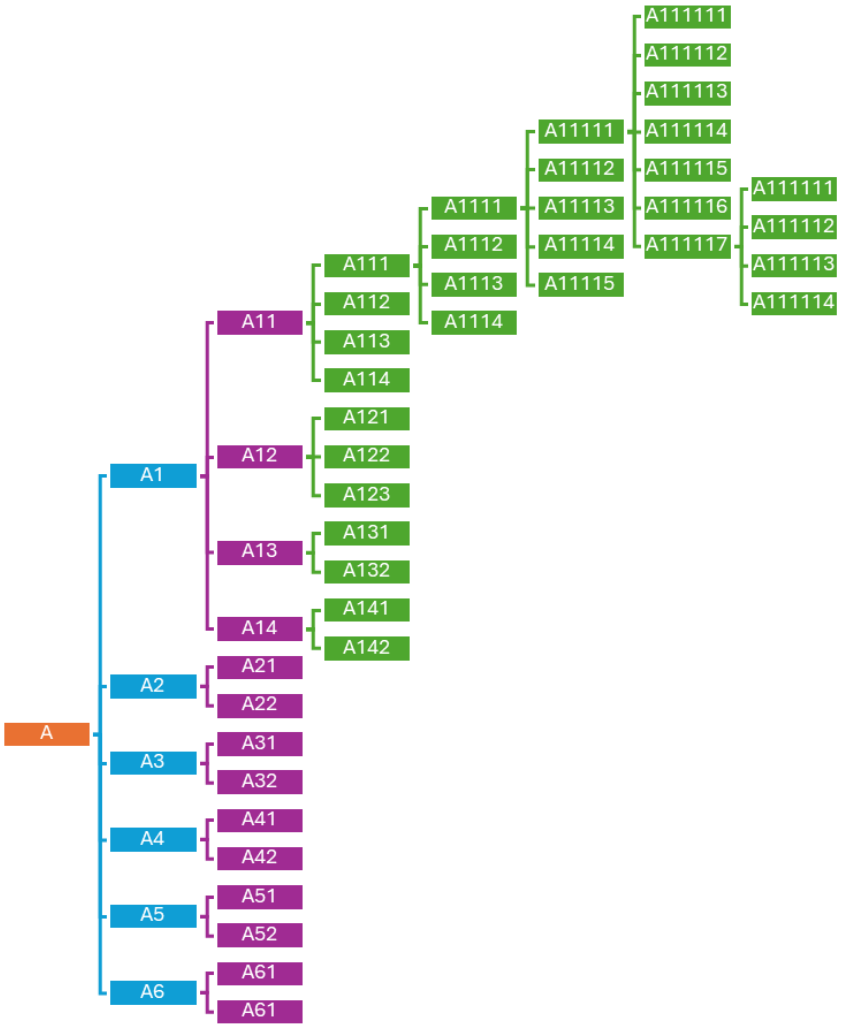

Fig 1. Example of Forest HyperSoft.

CRITIC Method

This part shows the steps of the CRITIC method to compute the criteria weights[13], [14].

Create the decision matrix.

$$F = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix}; i = 1, \dots, m; j = 1, \dots, n \quad (1)$$

Normalize the decision matrix.

We can normalize the decision matrix for positive and negative criteria such as:

$$x_{ij} = \frac{f_{ij} - \min_i f_{ij}}{\max_i f_{ij} - \min_i f_{ij}} \quad (2)$$

$$x_{ij} = \frac{f_{ij} - \max_i f_{ij}}{\min_i f_{ij} - \max_i f_{ij}} \quad (3)$$

Compute the correlation coefficient μ_{jk}

Compute the standard deviation σ_j

Compute the C index

$$C_j = \sigma_j \sum_{k=1}^{n} (1 - \mu_{jk}) \quad (4)$$

Compute the criteria weights.

$$w_j = \frac{c_j}{\sum_{j=1}^{n} c_j} \quad (5)$$

VIKOR Method

The VIKOR Method is used to rank the alternatives[15], [16].

Normalize the decision matrix.

The decision matrix is normalized for positive and negative criteria such as:

$$y_{ij} = \frac{\max_i f_{ij} - f_{ij}}{\max_i f_{ij} - \min_i f_{ij}} \quad (6)$$

$$y_{ij} = \frac{\min_i f_{ij} - f_{ij}}{\min_i f_{ij} - \max_i f_{ij}} \quad (7)$$

Compute the weighted decision matrix.

$$d_{ij} = w_j y_{ij} \quad (8)$$

Compute the S and R indexes

$$S_i = \sum_{j=1}^{n} d_{ij} \quad (9)$$

$$R_i = \max_j d_{ij} \quad (10)$$

Compute the VIKOR index

$$G_i = \theta \times \left(\frac{S_i - \min_i S_i}{\max_i S_i - \min_i S_i} \right) + (1 - \theta) * \left(\frac{R_i - \min_i R_i}{\max_i R_i - \min_i R_i} \right) \quad (11)$$

Where θ value between 0 and 1. In this study we used the θ value with 0.5.

3. Results and Discussion

This section shows the implementation of the proposed approach. We display the results of the criteria weights and ranking the alternatives. This study uses the Forest HyperSoft set to deal

with criteria and sub criteria. We divide the case study into five TreeSoft sets to deal with criteria and sub criteria with six alternatives as shown in Fig 2.

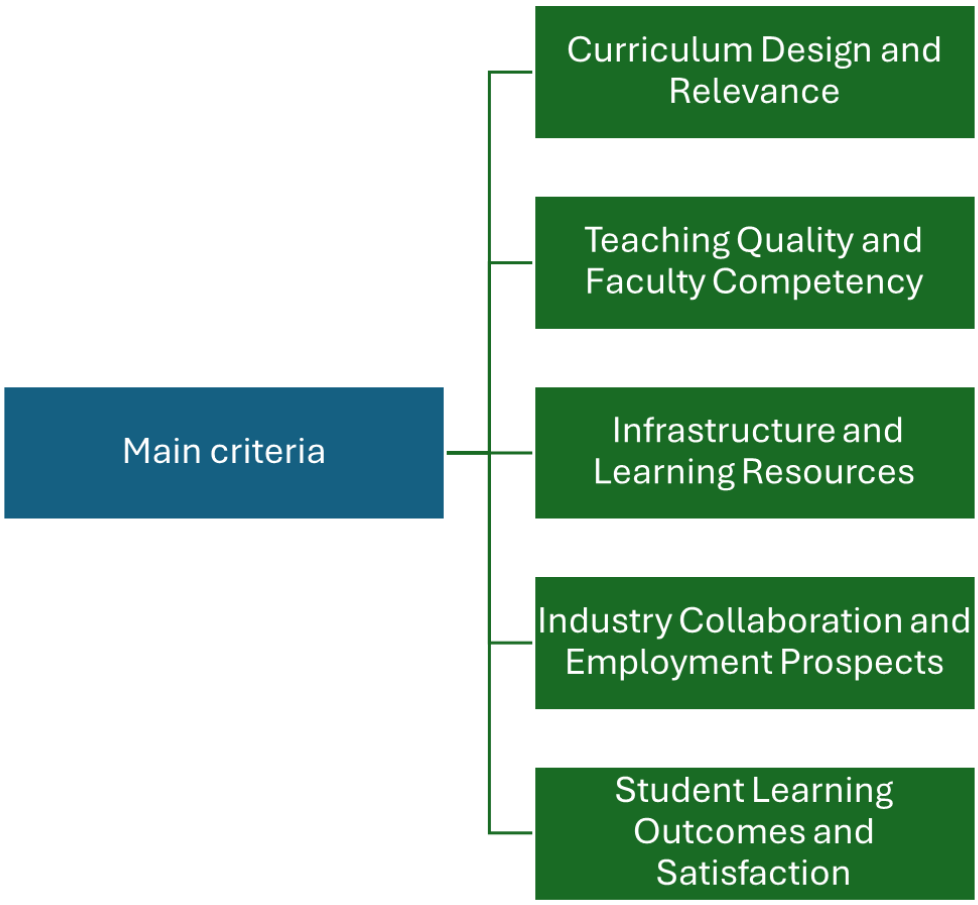
Main criteria
Curriculum Design and Relevance
Teaching Quality and Faculty Competency
Infrastructure and Learning Resources
Industry Collaboration and Employment Prospects
Student Learning Outcomes and Satisfaction

Fig 2. The main criteria.

Curriculum Design and Relevance

We use the first criterion. We divide the first criterion into TreeSoft set with criteria and sub criteria as shown in Fig 3.

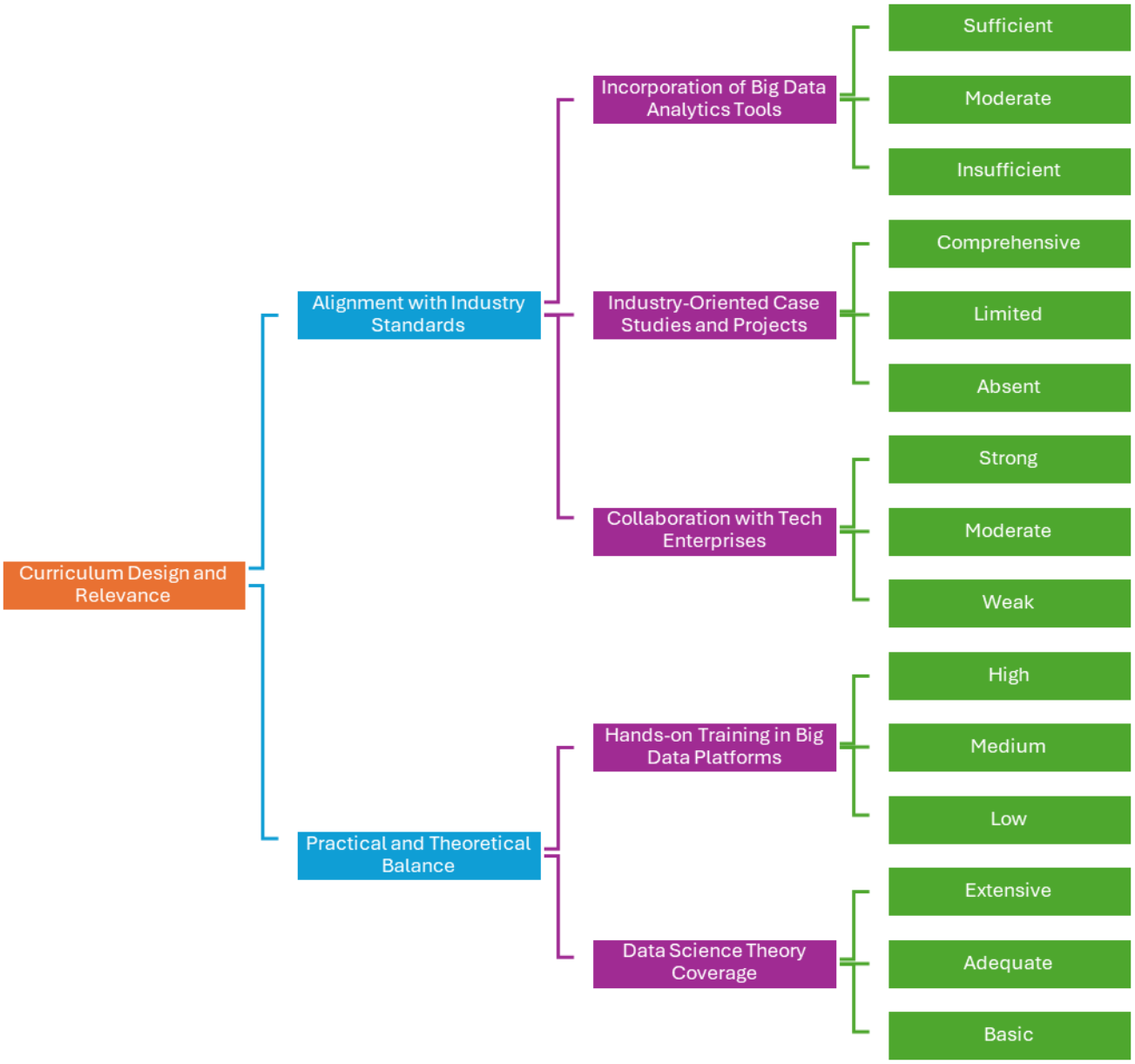

Fig 3. The sub criteria of first criterion.

This criterion has two sub criteria and each sub criteria has three and two sub criteria. Then we select the best values in this criterion as: $A_{1111} \times A_{1121} \times A_{1131} \times A_{1211} \times A_{1221}$. Then we apply the CRITIC and VIKOR method in these values.

In the CRITIC Method, we built the decision matrix between the criteria and alternatives using scale between 0.1 to 0.9. Then we combine these values into a single matrix.

Eq. (2) is used to normalize the decision matrix as shown in Table 1.

Then we compute the correlation coefficient μ_{jk}

Then we compute the standard deviation σ_j

Then we compute the C index using Eq. (4).

Then we compute the criteria weights using Eq. (5) as shown in Table 2.

Table 1. Normalized decision matrix by CRITIC Method.

	A_{1111}	A_{1121}	A_{1131}	A_{1211}	A_{1221}
R_1	-0.64552	0	-0.41557	-1	-0.73879
R_2	-0.39687	-0.58297	-0.89855	-0.83666	-1
R_3	0	-0.49227	-1	-0.6022	-0.91322
R_4	-0.91058	-0.53765	0	0	-0.25005
R_5	-1	-1	-0.4413	0	0
R_6	-0.95536	-0.91023	-0.96618	-0.51135	-0.45661

Table 2. The criteria weights.

	A_{1111}	A_{1121}	A_{1131}	A_{1211}	A_{1221}
Criteria	0.263924	0.197501	0.156383	0.194405	0.187787

In the VIKOR Method, we start with the combined decision matrix between the criteria and alternatives.

We normalize the decision matrix using Eq. (6) as shown in Table 3.

Then we compute the weighted decision matrix using Eq. (8) as shown in Table 4.

Then we compute the S and R indexes using Eqs. (9 and 10).

Then we compute the VIKOR index using Eq. (11). We use the value of θ with 0.5

Table 3. Normalized decision matrix by VIKOR Method.

	A_{1111}	A_{1121}	A_{1131}	A_{1211}	A_{1221}
R_1	0.354484	1	0.584427	0	0.261214
R_2	0.603126	0.417025	0.10145	0.163344	0
R_3	1	0.507733	0	0.397795	0.086778
R_4	0.089416	0.462345	1	1	0.749951
R_5	0	0	0.558702	1	1
R_6	0.044642	0.089766	0.033817	0.488654	0.543389

Table 4. Weighted normalized decision matrix by VIKOR Method.

	A_{1111}	A_{1121}	A_{1131}	A_{1211}	A_{1221}
R_1	0.093557	0.197501	0.091395	0	0.049053
R_2	0.159179	0.082363	0.015865	0.031755	0

R_3	0.263924	0.100278	0	0.077334	0.016296
R_4	0.023599	0.091313	0.156383	0.194405	0.140831
R_5	0	0	0.087372	0.194405	0.187787
R_6	0.011782	0.017729	0.005288	0.094997	0.102042

Teaching Quality and Faculty Competency

We divide this criterion into TreeSoft set with criteria and sub criteria as shown in Fig 4.

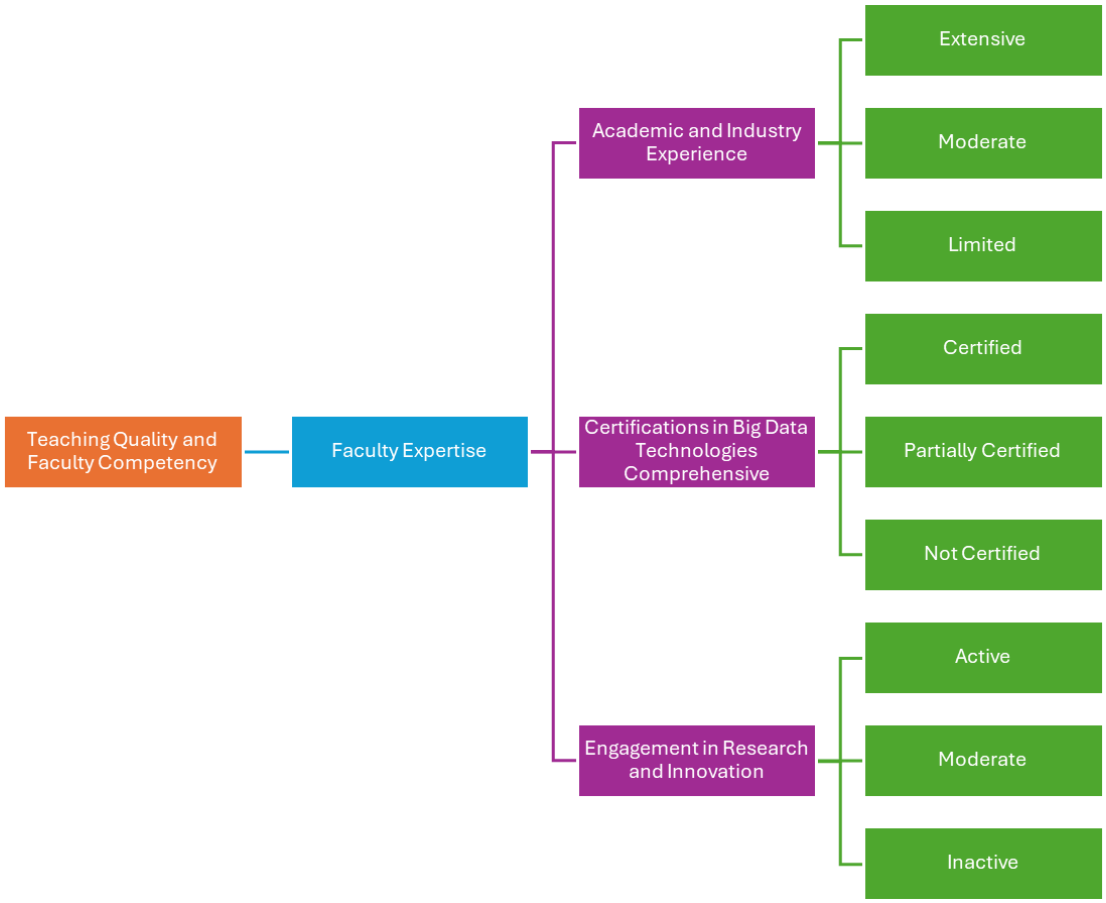

Fig 4. The sub criteria of second criterion.

This criterion has one sub criterion and each sub criteria has three sub criteria. Then we select the best values in this criterion as: $A_{2111} \times A_{2121} \times A_{2131}$. Then we apply the CRITIC and VIKOR method in these values.

In the CRITIC Method, we built the decision matrix between the criteria and alternatives using scale between 0.1 to 0.9. Then we combine these values into a single matrix.

Eq. (2) is used to normalize the decision matrix as shown in Table 5.

Then we compute the correlation coefficient μ_{jk}

Then we compute the standard deviation σ_j

Then we compute the C index using Eq. (4).

Then we compute the criteria weights using Eq. (5) as shown in Table 6.

Table 5. Normalized decision matrix by CRITIC Method.

	A_{2111}	A_{2121}	A_{2131}
R_1	0	0	-0.75015
R_2	-1	-0.95434	-0.78936
R_3	-0.63648	-0.77229	-0.84184
R_4	-0.22736	-0.63653	0
R_5	-0.54713	-1	-0.38242
R_6	-0.47823	-0.86332	-1

Table 6. The criteria weights.

	A_{2111}	A_{2121}	A_{2131}
Criteria	0.220868	0.322615	0.456517

In the VIKOR Method, we start with the combined decision matrix between the criteria and alternatives.

We normalize the decision matrix using Eq. (6) as shown in Table 7.

Then we compute the weighted decision matrix using Eq. (8) as shown in Table 8.

Then we compute the S and R indexes using Eqs. (9 and 10).

Then we compute the VIKOR index using Eq. (11). We use the value of θ with 0.5

Table 7. Normalized decision matrix by VIKOR Method.

	A_{2111}	A_{2121}	A_{2131}
R_1	1	1	0.249852
R_2	0	0.0457	0.210639
R_3	0.3635	0.2277	0.158157
R_4	0.7726	0.3635	1
R_5	0.4529	0	0.617581
R_6	0.5218	0.1367	0

Table 8. Weighted normalized decision matrix by VIKOR Method.

	A_{2111}	A_{2121}	A_{2131}

R_1	0.2209	0.3226	0.114062
R_2	0	0.0147	0.09616
R_3	0.0803	0.0735	0.072201
R_4	0.1707	0.1173	0.456517
R_5	0.1	0	0.281936
R_6	0.1152	0.0441	0

Infrastructure and Learning Resources

We divide this criterion into TreeSoft set with criteria and sub criteria as shown in Fig 5.

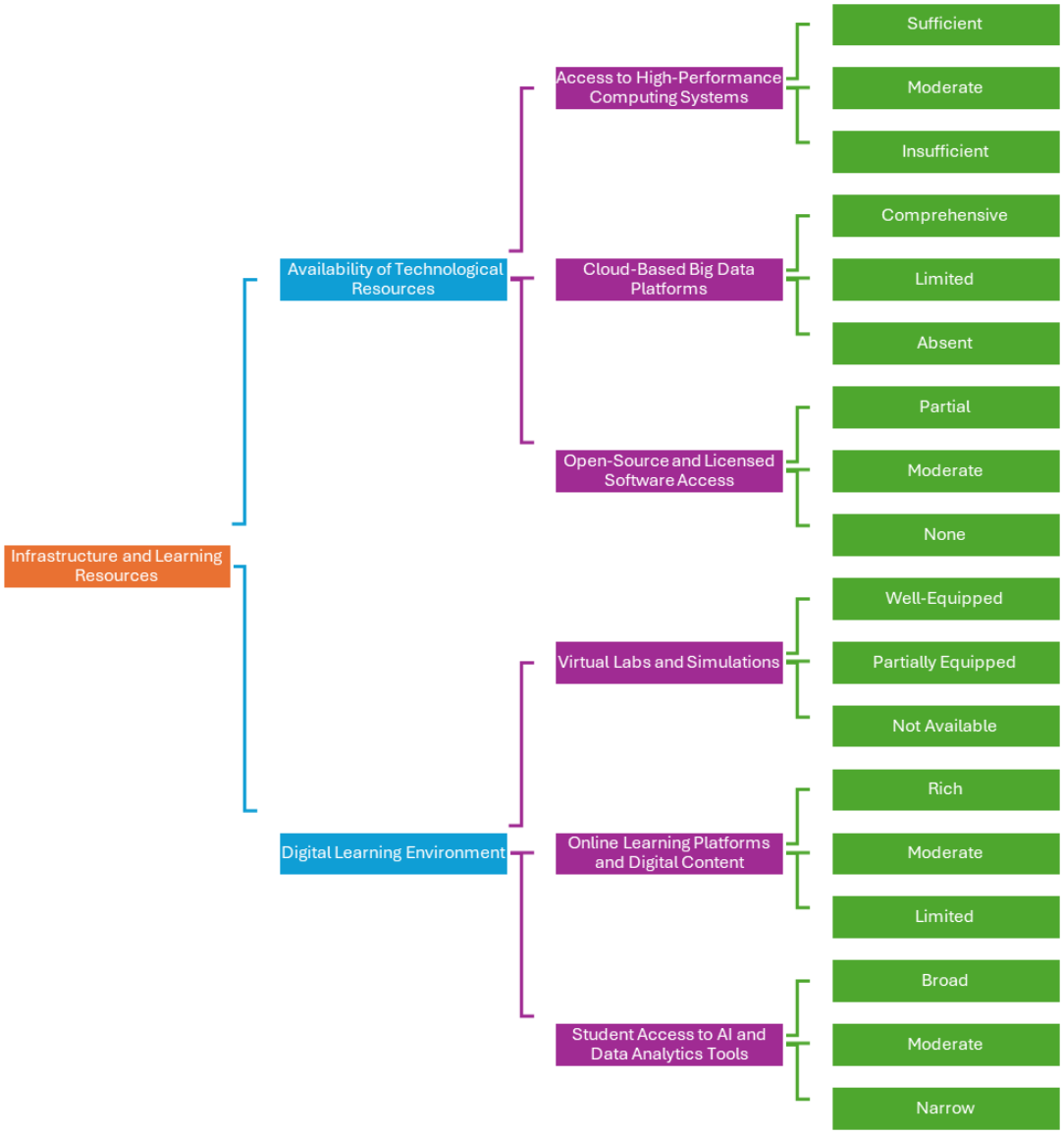

Fig 5. The sub criteria of third criterion.

This criterion has two sub criteria and each sub criteria has three and two sub criteria. Then we select the best values in this criterion as: $A_{3111} \times A_{3121} \times A_{3131} \times A_{3211} \times A_{3221} \times A_{3231}$. Then we apply the CRITIC and VIKOR method in these values.

In the CRITIC Method, we built the decision matrix between the criteria and alternatives using scale between 0.1 to 0.9. Then we combine these values into a single matrix.

Eq. (2) is used to normalize the decision matrix as shown in Table 9.

Then we compute the correlation coefficient μ_{jk}

Then we compute the standard deviation σ_j

Then we compute the C index using Eq. (4).

Then we compute the criteria weights using Eq. (5) as shown in Table 10.

Table 9. Normalized decision matrix by CRITIC Method.

	A_{3111}	A_{3121}	A_{3131}	A_{3211}	A_{3221}	A_{3231}
R_1	0	-1	0	0	-0.18494	-1
R_2	-0.28175	0	-0.09972	-0.26309	0	-0.34886
R_3	-0.68096	-0.74358	-0.42709	-0.52617	-0.45426	0
R_4	-0.62025	-0.6515	-0.37731	-0.7279	-0.39801	-0.53556
R_5	-0.62007	-0.60533	-1	-0.69722	-0.42614	-0.39396
R_6	-1	-0.97077	-1	-1	-1	-0.60625

Table 10. The criteria weights.

	A_{3111}	A_{3121}	A_{3131}	A_{3211}	A_{3221}	A_{3231}
Criteria	0.137941	0.181304	0.163361	0.134714	0.099207	0.283472

In the VIKOR Method, we start with the combined decision matrix between the criteria and alternatives.

We normalize the decision matrix using Eq. (6) as shown in Table 11.

Then we compute the weighted decision matrix using Eq. (8) as shown in Table 12.

Then we compute the S and R indexes using Eqs. (9 and 10).

Then we compute the VIKOR index using Eq. (11). We use the value of θ with 0.5

Table 11. Normalized decision matrix by VIKOR Method.

	A_{3111}	A_{3121}	A_{3131}	A_{3211}	A_{3221}	A_{3231}
R_1	1	0	1	1	0.815056	0

R_2	0.718249	1	0.900281	0.736914	1	0.651143
R_3	0.319042	0.256421	0.572906	0.473828	0.545735	1
R_4	0.379751	0.348497	0.622692	0.272098	0.601986	0.464443
R_5	0.379932	0.394672	0	0.302777	0.573861	0.606042
R_6	0	0.029235	0	0	0	0.393749

Table 12. Weighted normalized decision matrix by VIKOR Method.

	A_{3111}	A_{3121}	A_{3131}	A_{3211}	A_{3221}	A_{3231}
R_1	0.137941	0	0.163361	0.134714	0.080859	0
R_2	0.099076	0.181304	0.147071	0.099272	0.099207	0.184581
R_3	0.044009	0.04649	0.093591	0.063831	0.054141	0.283472
R_4	0.052383	0.063184	0.101724	0.036655	0.059721	0.131657
R_5	0.052408	0.071556	0	0.040788	0.056931	0.171796
R_6	0	0.0053	0	0	0	0.111617

Industry Collaboration and Employment Prospects

We divide this criterion into TreeSoft set with criteria and sub criteria as shown in Fig 6.

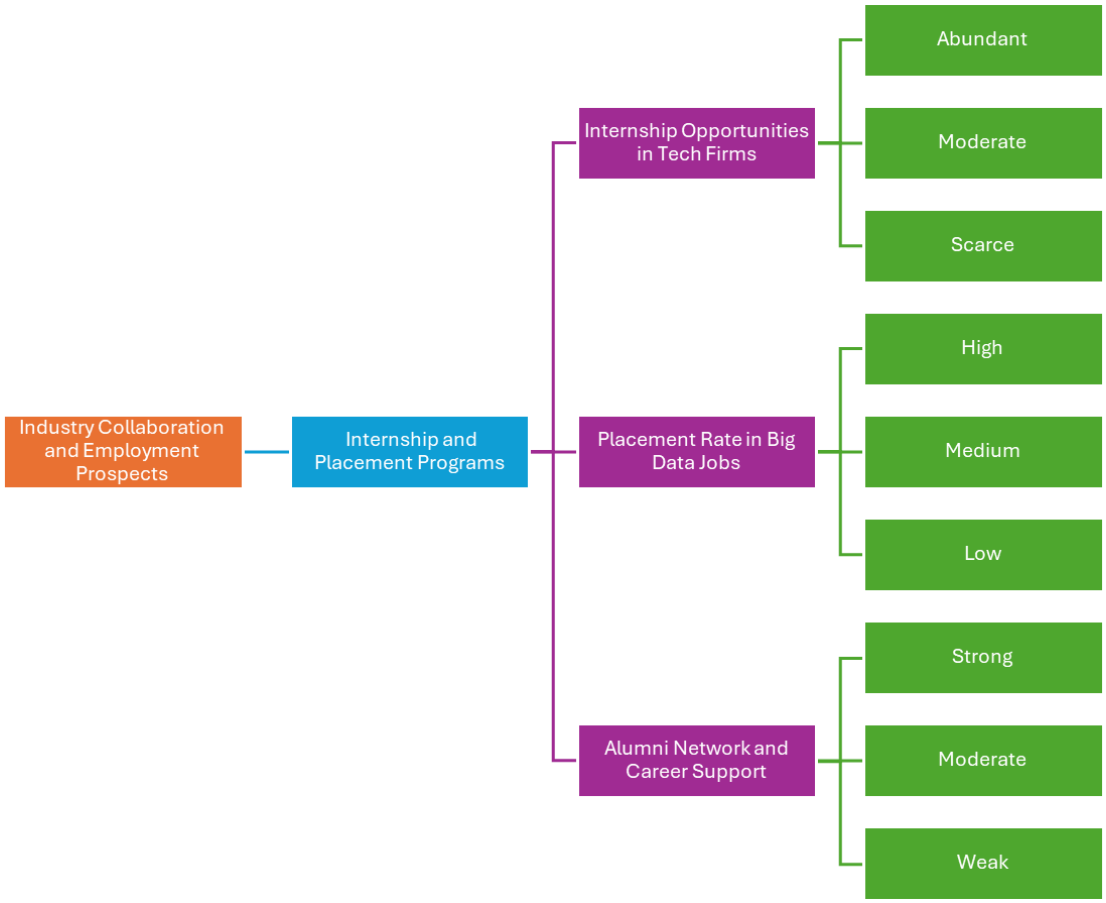

Fig 6. The sub criteria of the fourth criterion.

This criterion has two sub criteria and each sub criteria has three and two sub criteria. Then we select the best values in this criterion as: $A_{4111} \times A_{4121} \times A_{4131}$. Then we apply the CRITIC and VIKOR method in these values.

In the CRITIC Method, we built the decision matrix between the criteria and alternatives using scale between 0.1 to 0.9. Then we combine these values into a single matrix.

Eq. (2) is used to normalize the decision matrix as shown in Table 13.

Then we compute the correlation coefficient μ_{jk}

Then we compute the standard deviation σ_j

Then we compute the C index using Eq. (4).

Then we compute the criteria weights using Eq. (5) as shown in Table 14.

Table 13. Normalized decision matrix by CRITIC Method.

	A_{4111}	A_{4121}	A_{4131}
R_1	-1	0	-0.83337
R_2	0	-0.80409	-0.19333
R_3	-0.00014	-0.56571	-0.63175
R_4	-0.52992	-0.37719	0
R_5	-0.97077	-0.9101	-0.45674
R_6	-0.92473	-1	-1

Table 14. The criteria weights.

	A_{4111}	A_{4121}	A_{4131}
Criteria	0.364783	0.36631	0.268908

In the VIKOR Method, we start with the combined decision matrix between the criteria and alternatives.

We normalize the decision matrix using Eq. (6) as shown in Table 15.

Then we compute the weighted decision matrix using Eq. (8) as shown in Table 16.

Then we compute the S and R indexes using Eqs. (9 and 10).

Then we compute the VIKOR index using Eq. (11). We use the value of θ with 0.5

Table 15. Normalized decision matrix by VIKOR Method.

	A_{4111}	A_{4121}	A_{4131}
R_1	0	1	0.166627
R_2	1	0.195908	0.806668
R_3	0.999863	0.434291	0.368255
R_4	0.470082	0.622812	1
R_5	0.029235	0.089902	0.543257
R_6	0.075273	0	0

Table 16. Weighted normalized decision matrix by VIKOR Method.

	A_{4111}	A_{4121}	A_{4131}
R_1	0	0.36631	0.044807
R_2	0.364783	0.071763	0.216919
R_3	0.364733	0.159085	0.099026
R_4	0.171478	0.228142	0.268908
R_5	0.010664	0.032932	0.146086
R_6	0.027458	0	0

Student Learning Outcomes and Satisfaction

We divide this criterion into TreeSoft set with criteria and sub criteria as shown in Fig 7.

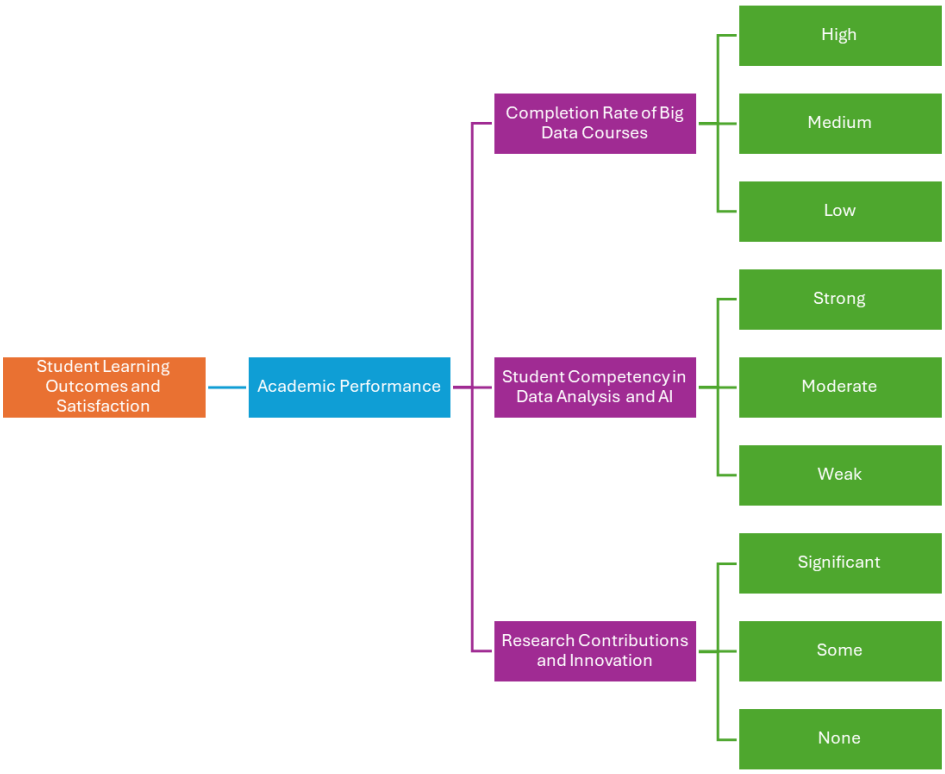

Fig 7. The sub criteria of the fifth criterion.

This criterion has two sub criteria and each sub criteria has three and two sub criteria. Then we select the best values in this criterion as: $A_{5111} \times A_{5121} \times A_{5131}$. Then we apply the CRITIC and VIKOR method in these values.

In the CRITIC Method, we built the decision matrix between the criteria and alternatives using scale between 0.1 to 0.9. Then we combine these values into a single matrix.

Eq. (2) is used to normalize the decision matrix as shown in Table 17.

Then we compute the correlation coefficient μ_{jk}

Then we compute the standard deviation σ_j

Then we compute the C index using Eq. (4).

Then we compute the criteria weights using Eq. (5) as shown in Table 18.

Table 17. Normalized decision matrix by CRITIC Method.

	A_{5111}	A_{5121}	A_{5131}
R_1	-0.52278	0	-0.28758
R_2	-0.69336	-0.5001	-1
R_3	-1	-0.91322	-0.57494
R_4	0	-1	-0.47506
R_5	-0.43166	0	-0.86229
R_6	-0.69305	-0.8839	0

Table 18. The criteria weights.

	A_{5111}	A_{5121}	A_{5131}
Criteria	0.256628	0.410091	0.33328

In the VIKOR Method, we start with the combined decision matrix between the criteria and alternatives.

We normalize the decision matrix using Eq. (6) as shown in Table 19.

Then we compute the weighted decision matrix using Eq. (8) as shown in Table 20.

Then we compute the S and R indexes using Eqs. (9 and 10).

Then we compute the VIKOR index using Eq. (11). We use the value of θ with 0.5

Table 19. Normalized decision matrix by VIKOR Method.

	A_{5111}	A_{5121}	A_{5131}
R_1	0.477219	1	0.71242

R_2	0.306645	0.499902	0
R_3	0	0.086778	0.425065
R_4	1	0	0.524941
R_5	0.568342	1	0.13771
R_6	0.306952	0.116095	1

Table 20. Weighted normalized decision matrix by VIKOR Method.

	A_{5111}	A_{5121}	A_{5131}
R_1	0.122468	0.410091	0.237435
R_2	0.078694	0.205005	0
R_3	0	0.035587	0.141666
R_4	0.256628	0	0.174952
R_5	0.145853	0.410091	0.045896
R_6	0.078773	0.04761	0.33328

In the final, we obtain the ranks of the alternative in each criterion as shown in Table 21. Then we combine these ranks into final ranks.

Table 21. Ranks of alternatives.

	A_1	A_2	A_3	A_4	A_5	Final Rank
R_1	4	2	3	4	1	4
R_2	5	5	1	2	5	5
R_3	3	4	2	3	6	3
R_4	1	1	4	1	4	1
R_5	2	3	5	5	2	2
R_6	6	6	6	6	3	6

5. Conclusions

The evaluation of big data professional education in vocational and technical colleges is a critical step in ensuring that graduates possess the necessary skills to thrive in a data-driven economy. This study highlights the importance of a multi-dimensional assessment framework that considers curriculum relevance, faculty expertise, technological infrastructure, industry collaborations, and student outcomes. By leveraging MCDM methodologies, the research provides a structured approach to ranking and improving educational programs, allowing institutions to align their training with industry requirements.

This study used two MCDM methods such as CRITIC method to compute the criteria weights and the VIKOR method to rank the alternatives. We used the Forest HyperSoft set to divide each criterion as a Tree, then in each tree we compute the criteria weights and ranking the alternatives.

Then we combined these ranks into a final rank. The results show alternative 6 is the best and alternative 4 is the worst.

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An Overview of Fermatean Neutrosophic Graphs

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Abstract: Fermatean Neutrosophic graph theory plays a significant role in the modeling and organization of optimization problems. In real-world scenarios, uncertainty is often present due to undetermined or incomplete information. Consequently, experts face challenges in designing optimization problems using fuzzy graphs. To address concerns about inconsistency, unpredictability, and vagueness in graphical optimization problems, various expansions of graph mathematical concepts have been developed. One such expansion is the concept of Fermatean neutrosophic graphs, which effectively manage uncertainties associated with unspecified and inadequate data in optimization problems. This article explores different types of Fermatean neutrosophic graphs and their operations. Additionally, it provides a numerical example showcasing the application of Fermatean Neutrosophic Sets in decision-making. These operations serve as valuable tools for analyzing and manipulating neutrosophic graphs in optimization problems.

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Keywords: Fuzzy set, Fermatean, fuzzy set, neutrosophic set, single valued neutrosophic set, Fermatean neutrosophic set, Fermatean neutrosophic graph

1. Introduction

Graph theory possesses numerous applications including computer applications, networking, transportation, and system analysis. It acts as a relational model, representing relationships between real-world objects. A graph's vertices and edges represent the objects and their associations in a given problem. Factors such as insufficient data, lack of evidence, or insufficient information frequently lead to inaccurate information in optimization problems. Zadeh [1] initially introduced the idea of the Fuzzy Set (FS) to address uncertainty.

A FS assigns membership degrees ranging from 0 to 1 to objects. Sitara et al. [2] conducted a comprehensive study on fuzzy graphs, showcasing some of their essential characteristics. Fuzzy graphs are significant because they can better handle ambiguity. However, Atanassov [3] noted that FSs are only capable of managing uncertainties by introducing degree of membership. Following this, Atanassov proposed intuitionistic FSs, a more generalized form of FSs. Every element in the intuitionistic FSs has a membership and non-membership function. Smarandache [4] introduced the novel concept known as Neutrosophic Set (NS), considered an effective method for handling vague, indistinguishable, and indetermined information in practical scenarios. Wang et al. [5] suggested the idea of single-valued NS (SVNS) to make NSs more usable in real-life applications. Real-life applications along with theoretical development of NSs were depicted in the studies [6, 7, 8, 9, 10, 11, 12, 13]. Broumi et al. [14] presented the single-valued neutrosophic graphs. Overview of neutrosophic graphs was presented in [15]. NSs have been used for dealing with shortest-path problems [16, 17, 18, 19, 20, 21].

Graph theories have widely studied in fuzzy and bipolar FS environment [22, 23, 24, 25, 26] . Fermatean FS (FFS) [27] was developed to effectively handle indeterminate evidence by expanding the domain of influence for membership and non-membership evaluations. Thamizhendhi et al. [28] introduced the Fermatean Fuzzy Hyper Graphs (FFHGs) and provided definitions and properties for this concept. Jaikum et al. [29] and Anirudh et al. [30] investigated vulnerability parameters in the NS environment. Broumi et al. [31] recently proposed the complex FNS and its application to decision-making. There are numerous applications of Fermatean Neutrosophic Graph (FNG)s in various fields, including shortest-path problems [32, 33].

The following are the primary contributions to the study:

- This paper presents several types of operations on FNGs, including lexicographic product, Cartesian product, union, composition, and join, as well as their properties.
- This study evaluates the significance of a novel class of graphs and looks into their potential applications in decision-making problems.

Fermatean neutrosophic graphs make significant contributions to real-world applications by capturing and representing uncertainty. Neutrosophic graphs can handle uncertain data, while traditional graphs can only represent clear and precise information. This capability is especially useful in fields like social networks, decision-making, and pattern recognition, where uncertainty and ambiguity are common.

Organization of the manuscripts

The subsequent sections of the research are titled as follows: Section-2 highlights some definitions. Section-3 highlights different types of FNGs. Section-4 highlights the different operations of FNGs. Section-5 highlights a numerical example, i.e., the application of Fermatean Neutrosophic Sets (FNS) in decision-making. Section-6 summarizes the conclusions of this article.

2.. **Preliminaries:**

2.1 FNS [27]

A FNS $\hat{A}$ in $\hat{X}$ is defined as: $\widehat{A} = \{\langle \hat{x}, T_{\widehat{A}}(\hat{x}), I_{\widehat{A}}(\hat{x}), F_{\widehat{A}}(\hat{x})\rangle | \hat{x} \in \hat{X}\}$

where $T_{\widehat{A}}(\hat{x}), I_{\widehat{A}}(\hat{x}), F_{\widehat{A}}(\hat{x}): \widehat{X} \to [0,1]$ represents respectively the membership, indeterminacy, and falsity degrees such that

$0 \leq (T_{\widehat{A}}(\hat{x}))^3 + (F_{\widehat{A}}(\hat{x}))^3 \leq 1$

$0 \leq (I_{\widehat{A}}(\hat{x}))^3 \leq 1$

then $0 \leq (T_{\widehat{A}}(\hat{x}))^3 + (F_{\widehat{A}}(\hat{x}))^3 + (I_{\widehat{A}}(\hat{x}))^3 \leq 2 \; \forall \hat{x} \in \widehat{X}$.

2.2 Intervalued FNS [32]

An IFNS $\hat{A}$ in $\hat{X}$ is defined as: $\widehat{A} = \{\langle \hat{x}, T_{\widehat{A}}(\hat{x}), I_{\widehat{A}}(\hat{x}), F_{\widehat{A}}(\hat{x})\rangle | \hat{x} \in \hat{X}\}$

where $T_{\widehat{A}}(\hat{x}) = [T_{\widehat{A}}^{-}(\hat{x}), T_{\widehat{A}}^{+}(\hat{x})]$, $I_{\widehat{A}}(\hat{x}) = [I_{\widehat{A}}^{-}(\hat{x}), I_{\widehat{A}}^{+}(\hat{x})]$, and $F_{\widehat{A}}(\hat{x} = [F_{\widehat{A}}^{-}(\hat{x}), F_{\widehat{A}}^{+}(\hat{x})]$ indicates the truth, indeterminacy and falsity membership degree.

$0 \leq (T_{\widehat{A}}(\hat{x}))^3 + (F_{\widehat{A}}(\hat{x}))^3 \leq 1$ and $0 \leq (I_{\widehat{A}}(\hat{x}))^3 \leq 1$

$$0 \leq (T_{\widehat{A}}(\hat{x}))^3 + (F_{\widehat{A}}(\hat{x}))^3 + (I_{\widehat{A}}(\hat{x}))^3 \leq 2, \forall \hat{x} \in \widehat{X}$$

Means

$$0 \leq (T_{\widehat{A}}^{+}(\hat{x}))^3 + \left(F_{\widehat{A}}^{+}(\hat{x})\right)^3 + \left(I_{\widehat{A}}^{+}(\hat{x})\right)^3 \leq 2, \forall \hat{x} \in \widehat{X}$$

2.3 Fermatean neutrosophic graphs [28]

FNGs combine the ideas of Fermatean graphs and neutrosophic graphs. The vertices and edges are assigned with Fermatean neutrosophic values, which can have true, false, and indeterminate components

A graph $G = (V, E)$ is called FNG, if the following condition holds

Let $G = (\widehat{P}, \widehat{R})$ where $\widehat{P}$ is set on $\hat{X}$ and $\widehat{R}$ is a relation on $\hat{X}$ then:

$$T_{\widehat{R}}(\hat{u}, \hat{v}) \leq \min\{T_{\widehat{P}}(\hat{u}), T_{\widehat{P}}(\hat{v})\}$$

$$I_{\widehat{R}}(\hat{u}, \hat{v}) \geq \max\{I_{\widehat{P}}(\hat{u}), I_{\widehat{P}}(\hat{v})\}$$

$$F_{\widehat{R}}(\hat{u}, \hat{v}) \geq \max\{F_{\widehat{P}}(\hat{u}), F_{\widehat{P}}(\hat{v})\}$$

and $0 \leq (T_{\widehat{R}}(\hat{u}, \hat{v}))^3 + \left(F_{\widehat{R}}(\hat{u}, \hat{v})\right)^3 + \left(I_{\widehat{R}}(\hat{u}, \hat{v})\right)^3 \leq 2, \forall \hat{u}, \hat{v} \in \hat{X}$

Where, $T_{\widehat{R}}(\hat{u}, \hat{v}): \hat{X} \times \hat{X} \to [0,1], I_{\widehat{R}}(\hat{u}, \hat{v}): \hat{X} \times \hat{X} \to [0,1], F_{\widehat{R}}(\hat{u}, \hat{v}): \hat{X} \times \hat{X} \to [0,1]$ represents three membership degree such as truth, indeterminacy and falsity of $\hat{R}$.

3. Different Types of FNGs

We provide regular FNGs, strong FNGs, and uniform FNGs in this section.

3.1 Regular FNGs

A regular FNG is a specialized type of graph in the domain of FNSs, characterized by nodes and edges that carry degrees of truth, indeterminacy, and falsity membership, all defined within the Fermatean neutrosophic framework.

Formally, a FNG $G = (V, E, T, I, F)$ satisfies the following:

1. Fermatean Neutrosophic Degrees:
   For each vertex $v \in V$ and edge $e \in E$, there are functions:
   - $T(v), T(e): V \cup E \to [0,1]$
   - $I(v), I(e): V \cup E \to [0,1]$

- $F(v), F(e): V \cup E \rightarrow [0,1]$ These degrees satisfy the Fermatean neutrosophic condition:

$(T(x))^3 + (F(x))^3 \leq 1 and (T(x))^3 + (F(x))^3 + (I(x))^3 \leq 2, \forall x \in \text{V} \cup \text{E}.$

A Regular Fermatean Neutrosophic Graph (RFNG) is a type of graph in which every vertex has the same degree under the Fermatean neutrosophic environment, meaning that each vertex has an equal number of adjacent vertices, while considering membership, non-membership, and indeterminacy in a Fermatean neutrosophic setting.

Example of a Regular Fermatean Neutrosophic Graph:

Consider a 3-regular Fermatean neutrosophic graph with 4 vertices $V = \{v_1, v_2, v_3, v_4\}$ and 4 edges $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$ forming a cycle.Each vertex has exactly 2 adjacent vertices, making it 2-regular.

Fermatean Neutrosophic Representation:

For each edge (v_i, v_j), assign Fermatean neutrosophic values:

- Membership M_{ij}: The degree of connectivity between $v_i \, and \, v_j$
- Non-membership N_{ij}: The degree of non-connection.
- Indeterminacy I_{ij}: The uncertainty in the connection.

Consider three membership value

- $M_{ij} = 0.8$, meaning the confidence in edge existence is high.
- $N_{ij} = 0.2$, meaning the confidence in non-existence is low.
- $I_{ij} = 0.1$, representing a small amount of uncertainty.

These values satisfy the Fermatean neutrosophic condition:

$$(M_{ij})^2 + (N_{ij})^2 + (I_{ij})^2 \leq 1$$
$$(0.8)^2 + (0.2)^2 + (0.1)^2 \leq 1$$
$$0.64 + 0.04 + 0.01 \leq 1$$
$$0.69 \leq 1$$

Thus, this is a valid 2-regular Fermatean neutrosophic graph**.**

3.2 Strong FNGs

A regular FNG is a specialized type of graph in the domain of FNSs, characterized by nodes and edges that carry degrees of truth, indeterminacy, and falsity membership, all defined within the Fermatean neutrosophic framework.

Formally, a FNG $G = (V, E, T, I, F)$ satisfies the following:

1. Strong Relationship Condition:

For any edge $e = (v_i, v_j)$

- The truth-membership degree $T(e)$ dominates, meaning it is significantly higher than the indeterminacy $I(e)$ and falsity $F(e)$ degrees: $T(e) > I(e)$and $T(e) > F(e)$.

Example of a Strong Fermatean Neutrosophic Graph:

Consider a 3-regular Strong Fermatean neutrosophic graph with 4 vertices $V = \{v_1, v_2, v_3, v_4\}$ and 5 edges $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1), (v_1, v_3)\}$

Fermatean Neutrosophic Representation:

For each edge (v_i, v_j), assign Fermatean neutrosophic values:

- High Membership Degree $M_{ij} \geq 0.9$
- Low membership Degree $N_{ij} \leq 0.1$
- Low Indeterminacy Degree $I_{ij} \leq 0.1$

Edge (v_i, v_j)	Membership(M_{ij})	Non-Membership(N_{ij})	Indeterminacy(I_{ij})
(v_1, v_2)	0.95	0.05	0.05
(v_2, v_3)	0.92	0.07	0.06
(v_3, v_4)	0.93	0.06	0.05
(v_4, v_1)	0.94	0.05	0.06
(v_1, v_3)	0.91	0.08	0.07

These values satisfy the Fermatean neutrosophic condition:

$$(M_{ij})^2 + (N_{ij})^2 + (I_{ij})^2 \leq 1$$
$$(0.95)^2 + (0.5)^2 + (0.5)^2 \leq 1$$
$$0.9025 + 0.0025 + 0.0025 \leq 1$$
$$0.9075 \leq 1$$

Thus, this is a valid example of strong neutrosophic graph**.**

3.3 Uniform FNGs

A uniform FNG (RFNG) is a FNG where the Fermatean neutrosophic degrees (truth, indeterminacy, and falsity) are consistent across all vertices and edges, ensuring uniformity in the representation of uncertainty and vagueness throughout the graph.

Formally, a FNG $G = (V, E, T, I, F)$ satisfies the following:

1. uniformity conditions

- All vertices $v \in V$ have identical Fermatean neutrosophic degrees $T(v), I(v), F(v)$.
- All edges $e \in E$ have identical Fermatean neutrosophic degrees $T(e), I(e), F(e)$.

i.e. $T(v_1) = T(v_2) = \cdots = T(v_n)$, $I(v_1) = I(v_2) = \cdots = I(v_n)$, $F(v_1) = F(v_2) = \cdots = F(v_n)$,

for all $v \in V$, and similarly for $e \in E$.

4. Operations on FNGs

This section introduces several significant graph-theoretic operations applied to FNGs, accompanied by key results and illustrative examples

Assume that $G_1 = (P_1, R_1)$ and $G_2 = (P_2, R_2)$ are two FNGs, which relate to the classical graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Here, P_1 & P_2 represent the Fermatean neutrosophic vertex sets corresponding to V_1 & V_2 , while R_1 & R_2 denote the Fermatean neutrosophic edge sets corresponding to E_1 & E_2 .

The subsequent section explores some of these operations within the framework of FNS theory and examines their properties in detail.

4.1 Cartesian Product of FNGs

The Cartesian product of two FNGs $G_1$and G_2 is denoted by $G_1 \times G_2$ and defined as : $G_1 \times G_2 = (P_1 \times P_2 , R_1 \times R_2)$, where

$$T_{P_1 \times P_2}(u_1, u_2) = min\{T_{P_1}(u_1), T_{P_2}(u_2)\}$$
$$I_{P_1 \times P_2}(u_1, u_2) = max\{I_{P_1}(u_1), I_{P_2}(u_2)\}$$
$$F_{P_1 \times P_2}(u_1, u_2) = max\{F_{P_1}(u_1), F_{P_2}(u_2)\}$$
$$\forall (u_1, u_2) \in (V_1, V_2)$$

The membership value of the edges in $G_1 \times G_2$ can be computed as

$$T_{R_1 \times R_2}((u, u_2), (u, v_2)) = min\{T_{P_1}(u), T_{R_2}(u_2, v_2)\}$$
$$I_{R_1 \times R_2}((u, u_2), (u, v_2)) = max\{I_{P_1}(u), I_{R_2}(u_2, v_2)\}$$
$$F_{R_1 \times R_2}((u, u_2), (u, v_2)) = max\{F_{P_1}(u), F_{R_2}(u_2, v_2)\}$$
$$\forall u \in V_1, (u_2, v_2) \in E_2$$
$$T_{R_1 \times R_2}((u_1, \beta), (v_1, \beta)) = min\{T_{R_1}(u_1, v_1), T_{P_2}(\beta)\}$$
$$I_{R_1 \times R_2}((u_1, \beta), (v_1, \beta)) = max\{I_{R_1}(u_1, v_1), I_{P_2}(\beta)\}$$
$$F_{R_1 \times R_2}((u_1, \beta), (v_1, \beta)) = max\{F_{R_1}(u_1, v_1), F_{P_2}(\beta)\}$$
$$\forall \beta \in V_2, (u_1, v_1) \in E_1$$

4.2 Composition of FNGs

The composition of two FNGs G_1 and G_2, denoted by $G_1 \circ G_2$, and defined as: $G_1 \circ G_2 = (P_1 \circ P_2 , R_1 \circ R_2)$ where,

$$T_{P_1 \circ P_2}(u_1, u_2) = min\{T_{P_1}(u_1), T_{P_2}(u_2)\}$$
$$I_{P_1 \circ P_2}(u_1, u_2) = max\{I_{P_1}(u_1), I_{P_2}(u_2)\}$$
$$F_{P_1 \circ P_2}(u_1, u_2) = min\{F_{P_1}(u_1), F_{P_2}(u_2)\}$$
$$\forall (u_1, u_2) \in (V_1, V_2)$$
$$T_{R_1 \circ R_2}((\beta, u_2), (\beta, v_2)) = min\{T_{P_1}(\beta), T_{R_2}(u_2, v_2)\}$$
$$I_{R_1 \circ R_2}((\beta, u_2), (\beta, v_2)) = max\{I_{P_1}(\beta), I_{R_2}(u_2, v_2)\}$$
$$F_{R_1 \circ R_2}((\beta, u_2), (\beta, v_2)) = max\{F_{P_1}(\beta), F_{R_2}(u_2, v_2)\}$$
$$\forall \beta \in V_1, (u_2, v_2) \in E_2$$
$$T_{R_1 \circ R_2}((u_1, \gamma), (v_1, \gamma)) = min\{T_{R_1}(u_1, v_1), T_{P_2}(\gamma)\}$$
$$I_{R_1 \circ R_2}((u_1, \gamma), (v_1, \gamma)) = min\{I_{R_1}(u_1, v_1), I_{P_2}(\gamma)\}$$

$$F_{R_1\circ R_2}((u_1,\gamma),(v_1,\gamma)) = min\{F_{R_1}(u_1,v_1), F_{P_2}(\gamma)\}$$
$$\forall\, \gamma \in V_2, (u_1,v_1) \in E_1$$
$$T_{R_1\circ R_2}((u_1,u_2),(v_1,v_2)) = min\{T_{P_2}(u_2), T_{P_2}(v_2), T_{R_1}(u_1,v_1)\}$$
$$I_{R_1\circ R_2}((u_1,u_2),(v_1,v_2)) = max\{I_{P_2}(u_2), I_{P_2}(v_2), I_{R_1}(u_1,v_1)\}$$
$$F_{R_1\circ R_2}((u_1,u_2),(v_1,v_2)) = max\{F_{P_2}(u_2), F_{P_2}(v_2), F_{R_1}(u_1,v_1)\}$$
$$\forall\, (u_1,u_2),(v_1,v_2) \in E_1$$

Example 4.2.1

Consider two FNGs G_1 , and G_2, as presented below. Then the composition of two graph $G_1 \circ G_2$, are shown graphically in Figure below.

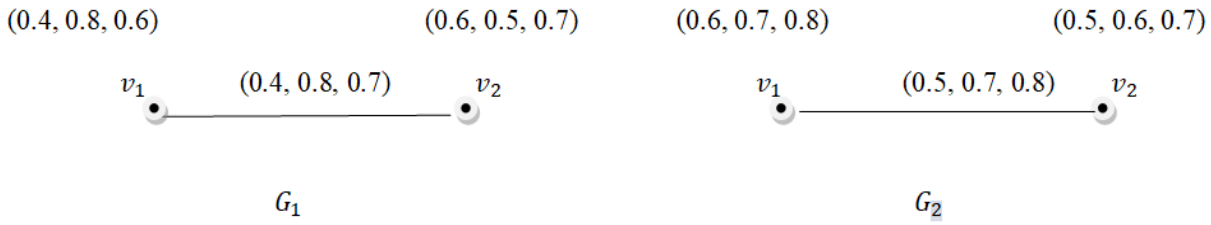

Fig. 1: Two FNGs

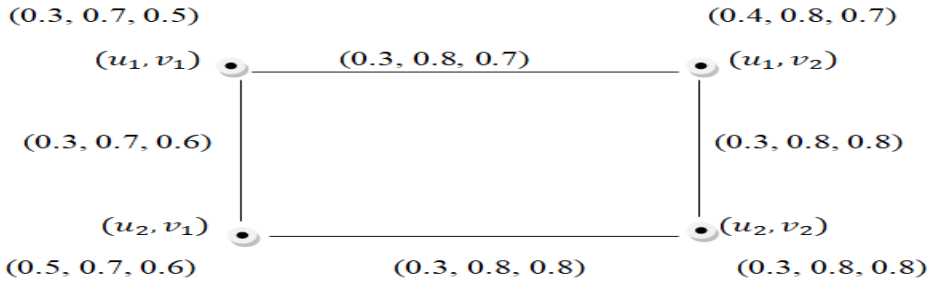

Fig. 2: Composition of two FNGs

4.3 Lexicographic product of FNGs

The lexicographic product of two FNGs G_1 and G_2 , denoted by $G_1 \cdot G_2$, and defined as: $G_1 \cdot G_2 = (P_1 \cdot P_2, R_1 \cdot R_2)$ where

$T_{P_1\cdot P_2}(u_1,u_2) = min\{T_{P_1}(u_1), T_{P_2}(u_2)\}$

$I_{P_1\cdot P_2}(u_1,u_2) = max\{I_{P_1}(u_1), I_{P_2}(u_2)\}$

$F_{P_1\cdot P_2}(u_1,u_2) = max\{F_{P_1}(u_1), F_{P_2}(u_2)\}$

$\forall\, (u_1,u_2) \in (V_1,V_2)$

$T_{R_1\cdot R_2}((\beta,u_2),(\beta,v_2)) = min\{T_{P_1}(\beta), T_{R_2}(u_2,v_2)\}$

$I_{R_1\cdot R_2}((\beta,u_2),(\beta,v_2)) = max\{I_{P_1}(\beta), I_{R_2}(u_2,v_2)\}$

$F_{R_1\cdot R_2}((\beta,u_2),(\beta,v_2)) = max\{F_{P_1}(\beta), F_{R_2}(u_2,v_2)\}$

$\forall\, \beta \in V_1, (u_2,v_2) \in E_2$

$T_{R_1\cdot R_2}((u_1,u_2),(v_1,v_2)) = min\{T_{R_1}(u_1,v_1), T_{R_2}(u_2,v_2)\}$

$I_{R_1 \cdot R_2}((u_1, u_2), (v_1, v_2)) = max\{I_{R_1}(u_1, v_1), I_{R_2}(u_2, v_2)\}$

$F_{R_1 \cdot R_2}((u_1, u_2), (v_1, v_2)) = max\{F_{R_1}(u_1, v_1), F_{R_2}(u_2, v_2)\}$

$\forall (u_1, v_1) \in E_1, (u_2, v_2) \in E_2$

Example 4.3.1

Consider two FNGs G_1 G_2, as presented below. Then the lexicographic product of two graph $G_1 \cdot G_2$, are shown graphically in Figure below.

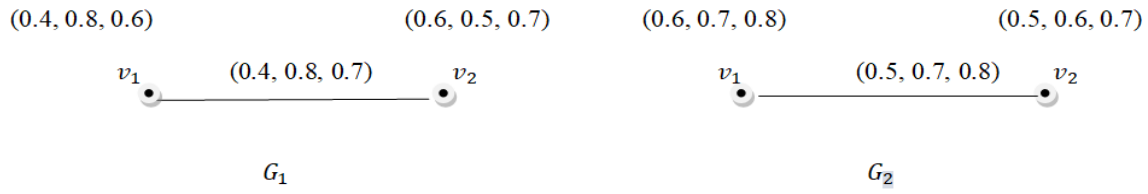

Fig. 3: Two FNGs

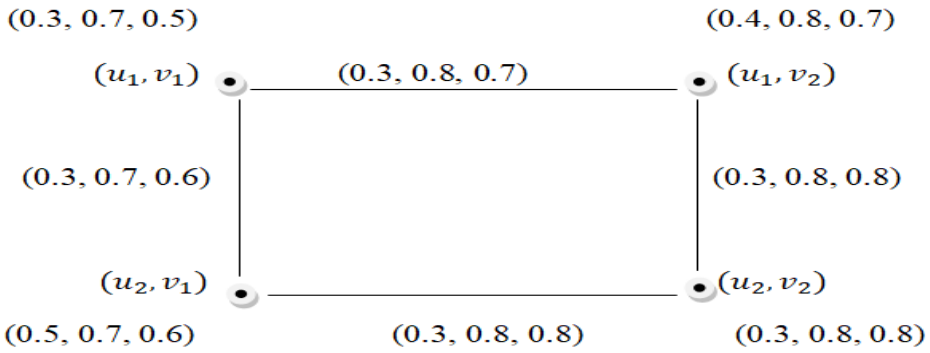

Fig. 4: Lexicographic product of two FNGs

4.4 Union of FNGs

The union of two FNGs G_1 and G_2, denoted by $G_1 \cup G_2$, and defined as: $G_1 \cup G_2 = (P_1 \cup P_2, R_1 \cup R_2)$ where

$$T_{P_1 \cup P_2}(u) = \begin{cases} T_{P_1}(u) \text{ if } u \in V_1 - V_2 \\ T_{P_2}(u) \text{ if } u \in V_2 - V_1 \\ max\{T_{P_1}(u), T_{P_2}(v)\} \text{ if } u \in V_1 \cup V_2 \end{cases}$$

$$I_{P_1 \cup P_2}(u) = \begin{cases} I_{P_1}(u) \text{ if } u \in V_1 - V_2 \\ I_{P_2}(u) \text{ if } u \in V_2 - V_1 \\ min\{I_{P_1}(u), I_{P_2}(v)\} \text{ if } u \in V_1 \cup V_2 \end{cases}$$

$$F_{P_1 \cup P_2}(u) = \begin{cases} F_{P_1}(u) \text{ if } u \in V_1 - V_2 \\ F_{P_2}(u) \text{ if } u \in V_2 - V_1 \\ min\{F_{P_1}(u), F_{P_2}(v)\} \text{ if } u \in V_1 \cup V_2 \end{cases}$$

$$T_{P_1 \cup P_2}(u,v) = \begin{cases} T_{R_1}(u,v) \; if \; (u,v) \in E_1 - E_2 \\ T_{R_2}(u,v) \; if \; (u,v) \in E_2 - E_1 \\ max\{T_{R_1}(u,v), T_{R_2}(u,v)\} \; if \; (u,v) \in E_1 \cup E_2 \end{cases}$$

$$I_{P_1 \cup P_2}(u,v) = \begin{cases} I_{R_1}(u,v) \; if \; (u,v) \in E_1 - E_2 \\ I_{R_2}(u,v) \; if \; (u,v) \in E_2 - E_1 \\ min\{I_{R_1}(u,v), I_{R_2}(u,v)\} \; if \; (u,v) \in E_1 \cup E_2 \end{cases}$$

$$F_{P_1 \cup P_2}(u,v) = \begin{cases} F_{R_1}(u,v) \; if \; (u,v) \in E_1 - E_2 \\ F_{R_2}(u,v) \; if \; (u,v) \in E_2 - E_1 \\ min\{F_{R_1}(u,v), F_{R_2}(u,v)\} \; if \; (u,v) \in E_1 \cup E_2 \end{cases}$$

Example 4.4.1

Consider two FNGs G_1 *and* G_2, as presented below. Then the union of $G_1 \cup G_2$, are shown graphically in Figure 5.

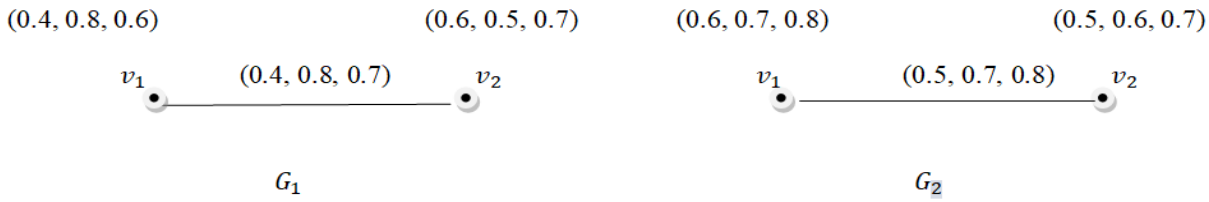

Fig. 5: Fermatean Neutrosophic Graphs

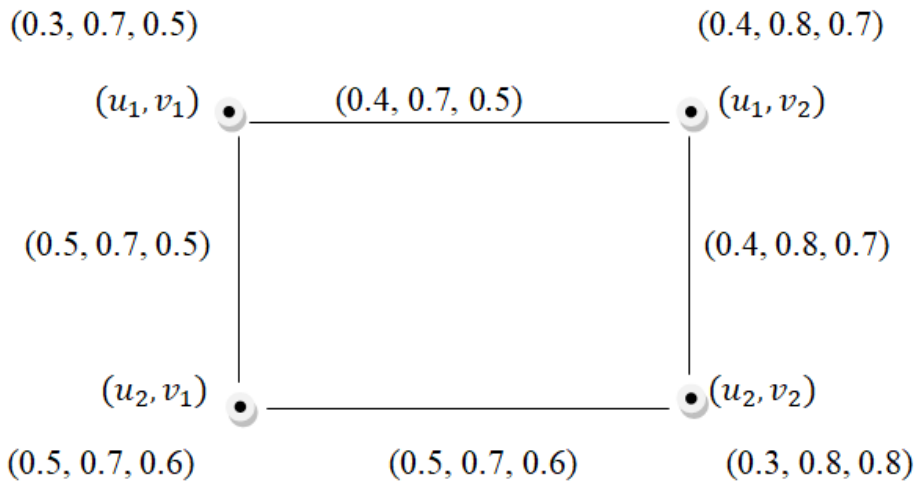

Fig. 6: Union of two FNGs

4.5 Join of FNGs

The Join of two FNGs G_1 and G_2, denoted by $G_1 + G_2$, is defined as: $G_1 + G_2 = (P_1 + P_2, R_1 + R_2)$ where,

$$T_{P_1+P_2}(u) = \begin{cases} T_{P_1}(u)\ if\ u \in V_1 - V_2 \\ T_{P_2}(u)\ if\ u \in V_2 - V_1 \\ min\{T_{P_1}(u), T_{P_2}(v)\}\ if\ u \in V_1 \cup V_2 \end{cases}$$

$$I_{P_1+P_2}(u) = \begin{cases} I_{P_1}(u)\ if\ u \in V_1 - V_2 \\ I_{P_2}(u)\ if\ u \in V_2 - V_1 \\ max\{I_{P_1}(u), I_{P_2}(v)\}\ if\ u \in V_1 \cup V_2 \end{cases}$$

$$F_{P_1+P_2}(u) = \begin{cases} F_{P_1}(u)\ if\ u \in V_1 - V_2 \\ F_{P_2}(u)\ if\ u \in V_2 - V_1 \\ max\{F_{P_1}(u), F_{P_2}(v)\}\ if\ u \in V_1 \cup V_2 \end{cases}$$

$$T_{P_1+P_2}(u,v) = \begin{cases} T_{R_1}(u,v)\ if\ (u,v) \in E_1 - E_2 \\ T_{R_2}(u,v)\ if\ (u,v) \in E_2 - E_1 \\ min\{T_{R_1}(u,v), T_{R_2}(u,v)\}\ if\ (u,v) \in E_1 \cup E_2 \end{cases}$$

$$I_{P_1+P_2}(u,v) = \begin{cases} I_{R_1}(u,v)\ if\ (u,v) \in E_1 - E_2 \\ I_{R_2}(u,v)\ if\ (u,v) \in E_2 - E_1 \\ max\{I_{R_1}(u,v), I_{R_2}(u,v)\}\ if\ (u,v) \in E_1 \cup E_2 \end{cases}$$

$$F_{P_1+P_2}(u,v) = \begin{cases} F_{R_1}(u,v)\ if\ (u,v) \in E_1 - E_2 \\ F_{R_2}(u,v)\ if\ (u,v) \in E_2 - E_1 \\ max\{F_{R_1}(u,v), F_{R_2}(u,v)\}\ if\ (u,v) \in E_1 \cup E_2 \end{cases}$$

Example 4.5.1

Consider two FNGs G_1 *and* G_2, as presented below. Then the join of two graph $G_1 + G_2$, are shown graphically in Figure below.

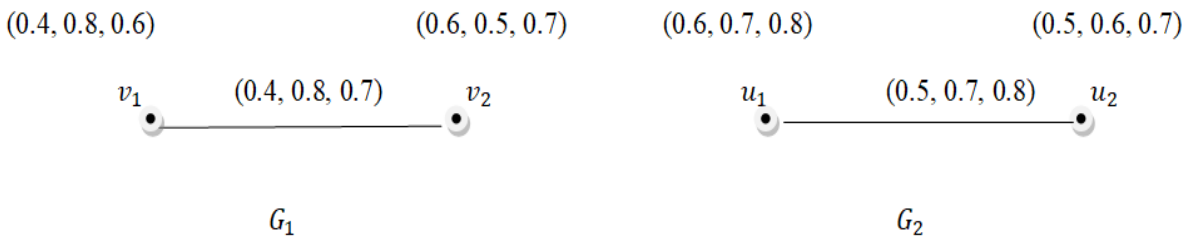

Fig. 7: Two FNGs

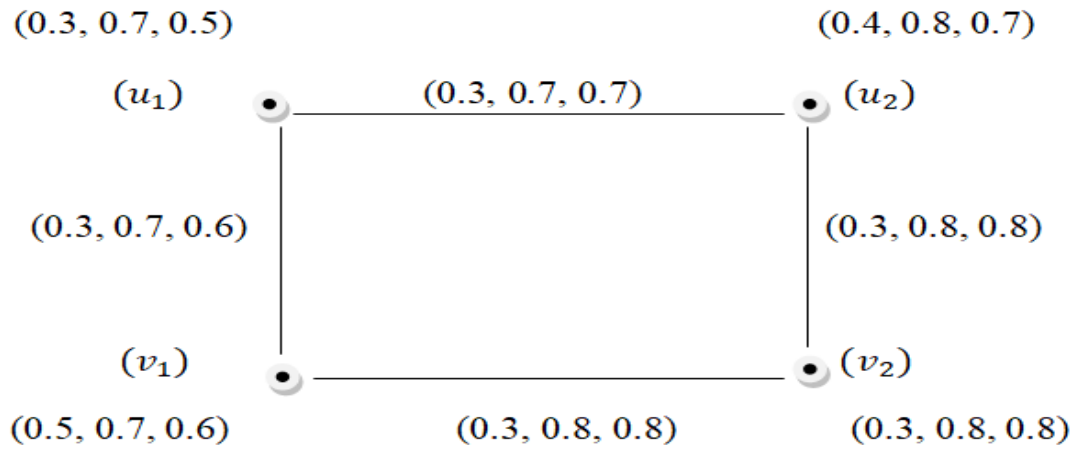

Fig. 8 Join of two FNGS

5. Numerical example (Application of FNSs) in decision-making)

NSs and their extensions have widely used in decision making such as identification of key factors affecting soil liquefaction under seismic risk [34], Choosing the most critical water quality parameter for an aquaponic system [35], green supplier selection [36], E-commerce site selection [37], career selection problem [38], vaccine selection [39], clay-brick selection [40], and so on. Different extension of NSs have been proposed in the literature such as quadripartition NS (QNS) [41], interval QNS [42], Pentapartitioned NS (PNS) [43] , Interval PNS [44], Pythagorean/ Fermatean NSs [45]. Several researchers have recently used Fermatean fuzzy sets, for multicriteria decision making such as WASPAS [46, 47], TOPSIS [48, 49], ELECTRE method [50] and so on. To demonstrate the application of Fermatean in decision making, let us consider a scenario in which a company must decide which supplier to use for a specific product. The decision is made based on three factors: price, quality, and delivery time.

We consider that there are three potential suppliers: A, B, and C. The decision-making team assigns Fermatean Neutrosophic membership values to each supplier for each criterion, ranging from 0 to 1.

Supplier A:

Table-1: Supplier A's membership values

	Belief	Indeterminacy	Unbelief
Price	0.8	0.3	0.7
Quality	0.7	0.2	0.6
Delivery Time	0.8	0.4	0.7

Supplier B:

Table-2: Supplier B's membership values

	Belief	Indeterminacy	Unbelief
Price	0.7	0.3	0.7
Quality	0.8	0.3	0.7

Delivery Time	0.9	0.4	0.6

Supplier C:

Table-3: Supplier C's membership values

	Belief	Indeterminacy	Unbelief
Price	0.7	0.2	0.6
Quality	0.8	0.3	0.7
Delivery Time	0.7	0.4	0.6

To calculate the overall Fermatean neutrosophic score for each supplier, we can use the weighted average method. Let's assume equal weights for all criteria.

Supplier A:

Price: $(0.8 \times 1/3) + (0.3 \times 1/3) + (0.7 \times 1/3) = 0.59$

Quality: $(0.7 \times 1/3) + (0.2 \times 1/3) + (0.6 \times 1/3) = 0.49$

Delivery Time: $(0.8 \times 1/3) + (0.4 \times 1/3) + (0.7 \times 1/3) = 0.62$

Overall Score for Supplier A: $\frac{(0.59 + 0.49 + 0.62)}{3} = 0.56$

Similarly, we can compute the total scores for Suppliers B and C.

Supplier B: Overall Score is 0.59 ,Supplier C: Overall score is 0.54.

Based on the Fermatean neutrosophic scores, Supplier B has the highest overall score and should be considered the company's preferred choice.

This example demonstrates how FNSs can be used in decision making by allowing experts to consider the belief, indeterminacy, and unbelief values associated with each decision alternative and criteria.

6. Conclusions

Fermatean Neutrosophic graphs offer a unique and efficient framework for representing and examining intricate systems. These graphs utilize the Fermatean Neutrosophic set to effectively capture and assess the varying degrees of membership, non-membership, and uncertainty present in a system. This facilitates a more comprehensive comprehension and interpretation of the information, making Fermatean Neutrosophic graphs an invaluable tool in areas like decision-making, pattern recognition, and data analysis. The development and application of Fermatean Neutrosophic graphs have the potential to significantly enhance our understanding of complex systems, enabling more precise and well-informed decision-making. In the future the concept of Fermentinean neutrosophic graph can be extended to Fermentinean pentapartitioned neutrosophic graph [51].

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Improving and Boosting Higher Education Competitiveness: Assessment, Ranking, and Analysis using TreeSoft Set

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Abstract: Higher education competitiveness has become a crucial factor in determining the global standing of universities and colleges. With rapid advancements in technology, globalization, and the evolving job market, institutions must continuously adapt to enhance their academic offerings, research output, and industry collaborations. This paper explores the key factors influencing higher education competitiveness, including academic excellence, infrastructure, internationalization, financial stability, industry partnerships, and student satisfaction. By employing a multi-criteria decision-making (MCDM) approach, this study evaluates and ranks higher education institutions based on these essential criteria. Two MCDM methods are used in this study such as CIMAS methodology to compute the criteria weights and the WASPAS methodology to rank the alternatives. We used the TreeSoft set to deal with different criteria and sub-criteria by dividing the criteria and sub-criteria into a Tree to obtain the relations.

Keywords: TreeSoft Set; Higher Education Competitiveness; Assessment; Analysis; Ranking.

1. Introduction

Higher education institutions play a pivotal role in shaping knowledge economies and driving societal progress. The growing demand for quality education, coupled with the rapid integration of digital technologies, has intensified competition among universities worldwide. As students, employers, and policymakers seek institutions that offer exceptional academic programs, robust research opportunities, and strong industry links, the ability to remain competitive has become a critical challenge[1], [2]. Universities must continuously evolve to meet these expectations while maintaining high standards of education, research, and employability outcomes. One of the fundamental aspects of higher education competitiveness is academic excellence, which encompasses faculty qualifications, curriculum innovation, and research contributions. Institutions with distinguished faculty members and cutting-edge research facilities tend to attract top students and funding opportunities. Moreover, the quality of education is heavily influenced by an institution's ability to integrate interdisciplinary learning, real-world

applications, and digital advancements into its academic framework. Universities that foster a dynamic learning environment gain a strategic advantage in the global education sector[3], [4]. Beyond academic performance, infrastructure and technological resources significantly impact an institution's competitiveness. State-of-the-art laboratories, digital learning platforms, and well-equipped libraries enhance students' learning experiences and facilitate groundbreaking research. Additionally, the implementation of smart campus initiatives and IT-enabled classrooms contributes to a more engaging and interactive educational environment. As universities embrace digital transformation, those with superior technological integration stand out in the highly competitive higher education landscape[5], [6].

Industry collaboration and employability outcomes are also key indicators of institutional competitiveness. Universities that establish strong partnerships with industries, provide extensive internship programs, and incorporate skill-based training in their curricula produce graduates who are better prepared for the workforce. The growing emphasis on entrepreneurship, innovation hubs, and startup ecosystems within academic institutions further enhances their reputation and attractiveness to prospective students and employers[7], [8].

Furthermore, internationalization and global engagement have become essential components of higher education competitiveness. Universities that foster student exchange programs, research collaborations, and international faculty recruitment create a diverse and globally connected academic community. Institutions that establish strong networks with international organizations and maintain high global rankings gain a competitive edge by attracting students from different cultural and academic backgrounds[9], [10].

As the higher education sector continues to evolve, institutions must adopt data-driven decision-making processes to assess and enhance their competitive position. Multi-criteria decision-making (MCDM) methodologies provide an effective framework for evaluating universities based on various performance metrics. By analyzing factors such as academic excellence, infrastructure, financial sustainability, and global outreach, MCDM approaches enable a systematic assessment of higher education competitiveness. This study applies such methodologies to rank institutions and identify areas for strategic improvement, ultimately contributing to the advancement of higher education standards worldwide.

This study proposes MCDM methods to compute the criteria weights by the CIMAS methodology and ranking the alternatives by the WASPAS methodology. We use the TreeSoft set to deal with criteria and sub-criteria to divide it into a Tree. We compute the criteria weights in each criterion and in the main criteria.

2. Methodological framework

The methodological framework of this study is built to compute the criteria weights and ranking of the alternatives of MCDM problem. Three main steps of this study are to evaluate the MCDM problem. In the first step, we use the TreeSoft set to deal with the different criteria and

alternatives. in the second step, we apply the CIMAS method to compute the criteria weights. In the third step, we apply the WASPAS method to rank the alternatives.

Dividing the criteria and sub-criteria into a Tree

The TreeSoft set is a hierarchical soft set with multi-level criteria presented in Tree to show the relations between criteria and sub criteria[11], [12].

Let U be a universe discourse and H non-empty subset of U and the powerset of U is P(H). Let the set of criteria $D = \{D_1, D_2, \dots D_n\}; n \geq 1$. These criteria can be divided into sub-criteria such as:

$$D_1 = \{D_{1-1}, D_{1-2}\ , D_{1-3}, \dots\} \quad (1)$$

$$D_2 = \{D_{2-1}, D_{2-2}\ , D_{2-3}, \dots\} \quad (2)$$

$$D_3 = \{D_{3-1}, D_{3-2}\ , D_{3-3}, \dots\} \quad (3)$$

$$D_4 = \{D_{4-1}, D_{4-2}\ , D_{4-3}, \dots\} \quad (4)$$

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$$D_n = \{D_{n-1}, D_{n-2}\ , D_{n-3}, \dots\} \quad (5)$$

A TreeSoft set is a mapping defined as:

$$F: P(Tree(D)) \rightarrow P(H) \quad (6)$$

Tree(A) presents a set of nodes from level 1 to level m.

Calculating the criteria weights

We can compute the criteria weights by the CIMAS methodology[13], [14].

Create the decision matrix.

Experts create the decision matrix between the criteria and alternatives.

Normalize the decision matrix

The decision matrix is normalized such as:

$$y_{ij} = \frac{x_{ij}}{r_j}; i = 1, \dots m; j = 1, \dots, n \quad (7)$$

Where r_j refers to the standard deviation.

Computation of the weighted decision matrix

$$h_{ij} = y_{ij} v_{ij} \quad (8)$$

Where v_{ij} refers to the weight of experts.

Compute the maximum and minimum values in the weighted decision matrix.

$$h_{ij}^{max} = \max_i h_{ij} \tag{9}$$

$$h_{ij}^{min} = \min_i h_{ij} \tag{10}$$

Compute the difference between the maximum and minimum values.

$$l_j = h_{ij}^{max} - h_{ij}^{min} \tag{11}$$

Compute the criteria weights.

$$w_j = \frac{l_j}{\sum_{j=1}^{n} l_j} \tag{12}$$

Ranking the alternatives

This step is applied to rank the alternatives by the WASPAS method[15], [16].

Normalize the decision matrix for the beneficial and non-beneficial criteria

$$u_{ij} = \frac{x_{ij}}{\max_i x_{ij}} \tag{13}$$

$$u_{ij} = \frac{x_{ij}}{\min_i x_{ij}} \tag{14}$$

Compute the additive relative importance

$$D_i^{(1)} = \sum_{j=1}^{n} u_{ij} w_j \tag{15}$$

Compute the multiplicative relative importance

$$D_i^{(2)} = \prod_{j=1}^{n} (u_{ij})^{w_j} \tag{16}$$

Compute the joint generalized criterion

$$D_i = \frac{1}{2}\left(D_i^{(1)} + D_i^{(2)}\right) \tag{17}$$

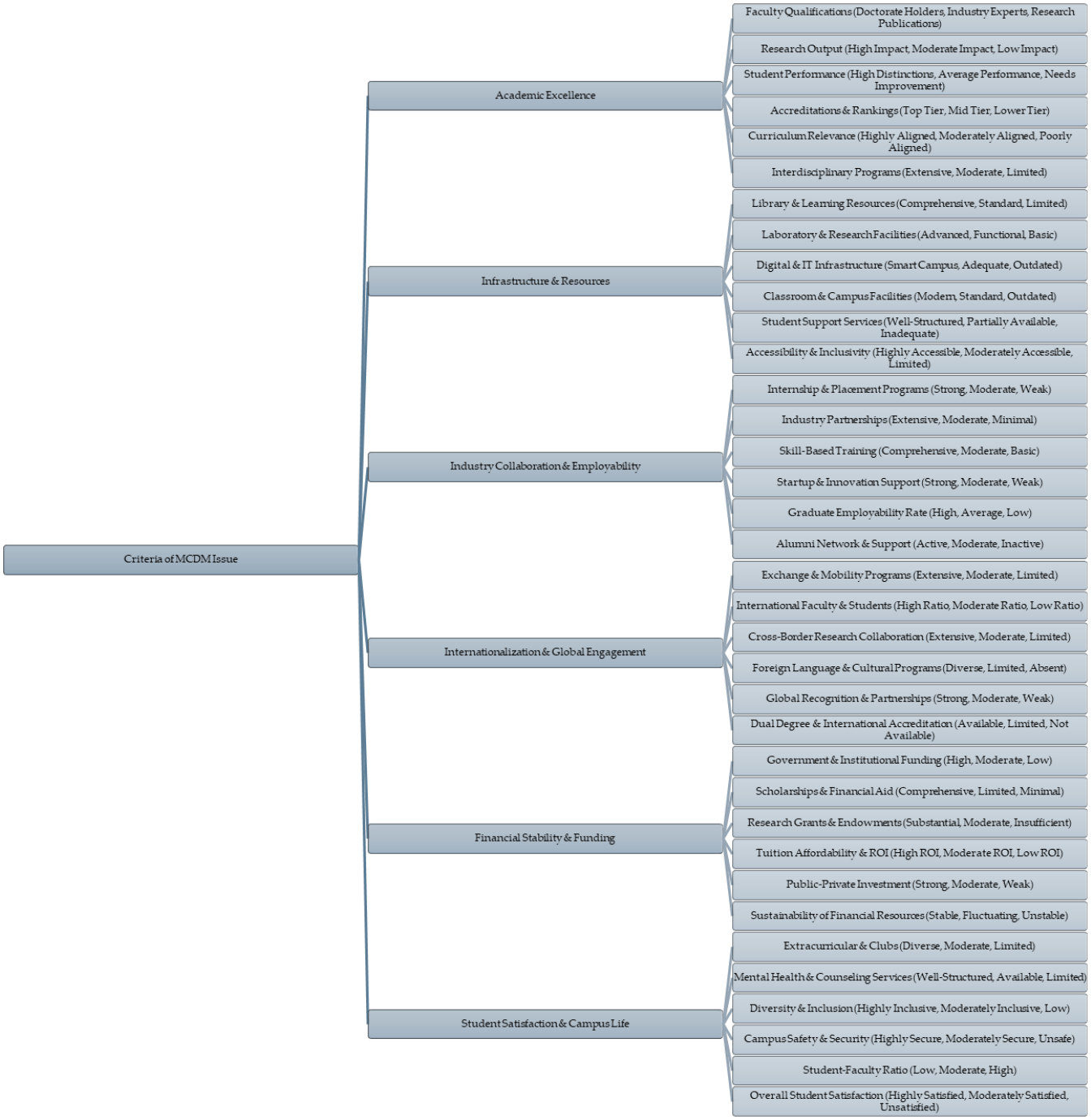
Criteria of MCDM Issue
Academic Excellence
Faculty Qualifications (Doctorate Holders, Industry Experts, Research Publications)
Research Output (High Impact, Moderate Impact, Low Impact)
Student Performance (High Distinctions, Average Performance, Needs Improvement)
Accreditations & Rankings (Top Tier, Mid Tier, Lower Tier)
Curriculum Relevance (Highly Aligned, Moderately Aligned, Poorly Aligned)
Interdisciplinary Programs (Extensive, Moderate, Limited)
Infrastructure & Resources
Library & Learning Resources (Comprehensive, Standard, Limited)
Laboratory & Research Facilities (Advanced, Functional, Basic)
Digital & IT Infrastructure (Smart Campus, Adequate, Outdated)
Classroom & Campus Facilities (Modern, Standard, Outdated)
Student Support Services (Well-Structured, Partially Available, Inadequate)
Accessibility & Inclusivity (Highly Accessible, Moderately Accessible, Limited)
Industry Collaboration & Employability
Internship & Placement Programs (Strong, Moderate, Weak)
Industry Partnerships (Extensive, Moderate, Minimal)
Skill-Based Training (Comprehensive, Moderate, Basic)
Startup & Innovation Support (Strong, Moderate, Weak)
Graduate Employability Rate (High, Average, Low)
Alumni Network & Support (Active, Moderate, Inactive)
Internationalization & Global Engagement
Exchange & Mobility Programs (Extensive, Moderate, Limited)
International Faculty & Students (High Ratio, Moderate Ratio, Low Ratio)
Cross-Border Research Collaboration (Extensive, Moderate, Limited)
Foreign Language & Cultural Programs (Diverse, Limited, Absent)
Global Recognition & Partnerships (Strong, Moderate, Weak)
Dual Degree & International Accreditation (Available, Limited, Not Available)
Financial Stability & Funding
Government & Institutional Funding (High, Moderate, Low)
Scholarships & Financial Aid (Comprehensive, Limited, Minimal)
Research Grants & Endowments (Substantial, Moderate, Insufficient)
Tuition Affordability & ROI (High ROI, Moderate ROI, Low ROI)
Public-Private Investment (Strong, Moderate, Weak)
Sustainability of Financial Resources (Stable, Fluctuating, Unstable)
Student Satisfaction & Campus Life
Extracurricular & Clubs (Diverse, Moderate, Limited)
Mental Health & Counseling Services (Well-Structured, Available, Limited)
Diversity & Inclusion (Highly Inclusive, Moderately Inclusive, Low)
Campus Safety & Security (Highly Secure, Moderately Secure, Unsafe)
Student-Faculty Ratio (Low, Moderate, High)
Overall Student Satisfaction (Highly Satisfied, Moderately Satisfied, Unsatisfied)

Fig 1. The criteria of Higher Education Competitiveness.

3. Case Study

This section shows the case study of the Higher Education Competitiveness Evaluation. We collect six criteria and eleven universities to be evaluated. Three experts evaluated these criteria and eleven universities. We used the TreeSoft set to divide the criteria into different levels. We

divided it into two levels. In the first level, we put the six main criteria. In the second level, we put the sub-criteria. Then we put the values of each sub-criteria to select the best one.

Calculating the criteria weights

In this part, we apply the steps of the CIMAS methodology to calculate the criteria weights. This step is divided into different stages. First, we compute the criteria weights of the main criteria, then we compute the weights in each sub-criterion. We have six main criteria and each criterion has six sub-criteria.

Create the decision matrix.

We create the decision matrix using a scale between 0.1 to 0.9 to evaluate the criteria and alternatives by the experts and decision makers.

Eq. (7) is used to normalize the decision matrix as shown in Table 1.

Then we obtain the weighted decision matrix using Eq. (2) as shown in Table 2.

Then we compute the maximum and minimum values in the weighted decision matrix using Eqs. (9 and 10).

Then we compute the difference between the maximum and minimum values using Eq. (11).

Then we compute the criteria weights using Eq. (12) as shown in Fig 2.

Table 1. The normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	1.969431	3.190429	4.296314	2.724285	3.988006	1.692277
A_2	4.647021	5.774561	1.569217	3.16582	3.208086	4.633238
A_3	3.131694	4.020019	2.25007	3.991653	2.560925	2.724461
A_4	2.839604	5.541966	2.003268	1.792156	4.669888	1.032183
A_5	4.549464	2.554928	4.388404	1.792156	3.828696	3.631511
A_6	4.418807	3.302075	2.023119	4.819717	3.373816	3.016659
A_7	3.968768	3.416305	3.796571	1.899938	3.144479	3.113984
A_8	3.357295	3.391237	2.003268	3.360571	4.703151	2.061056
A_9	4.777678	4.252614	2.162277	2.425468	1.979122	3.532199
A_{10}	3.809948	4.680589	3.116332	3.993883	5.28437	2.257694
A_{11}	2.100088	3.935509	3.024241	3.16582	4.440552	2.477725

Table 2. The weighted decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.309207	0.626616	0.740883	0.419662	0.648435	0.266541
A_2	0.729597	1.134153	0.270605	0.487678	0.521623	0.729755
A_3	0.491686	0.789552	0.388016	0.614894	0.416397	0.429114
A_4	0.445827	1.08847	0.345456	0.276072	0.759307	0.162573
A_5	0.714281	0.501801	0.756764	0.276072	0.622532	0.571979

A_6	0.693767	0.648544	0.348879	0.742452	0.54857	0.475137
A_7	0.623109	0.670979	0.654704	0.292676	0.511281	0.490466
A_8	0.527106	0.666056	0.345456	0.517679	0.764715	0.324625
A_9	0.750111	0.835235	0.372877	0.373631	0.321798	0.556337
A_{10}	0.598174	0.919291	0.5374	0.615237	0.859219	0.355597
A_{11}	0.329721	0.772954	0.521519	0.487678	0.722017	0.390253

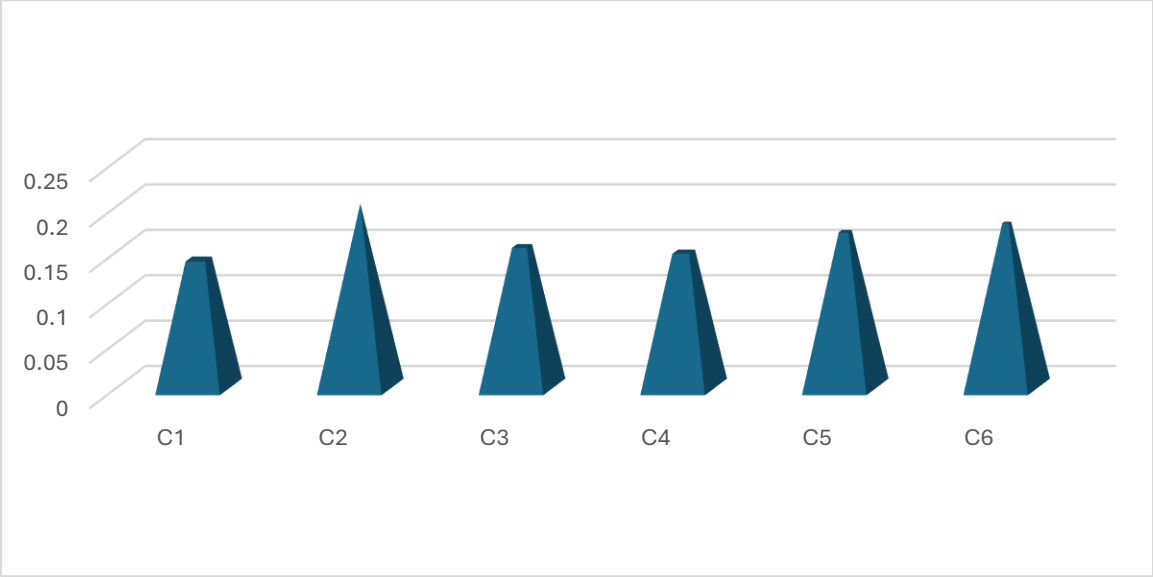

Fig 2. The weights of the main criteria.

Then we apply the previous steps in each sub-criteria to compute the weights of each sub-criterion.

In the First sub-criterion

Fig 3 shows the normalized decision matrix. Fig 4 shows the weighted decision matrix. Fig 5 shows the weights of first sub-criterion.

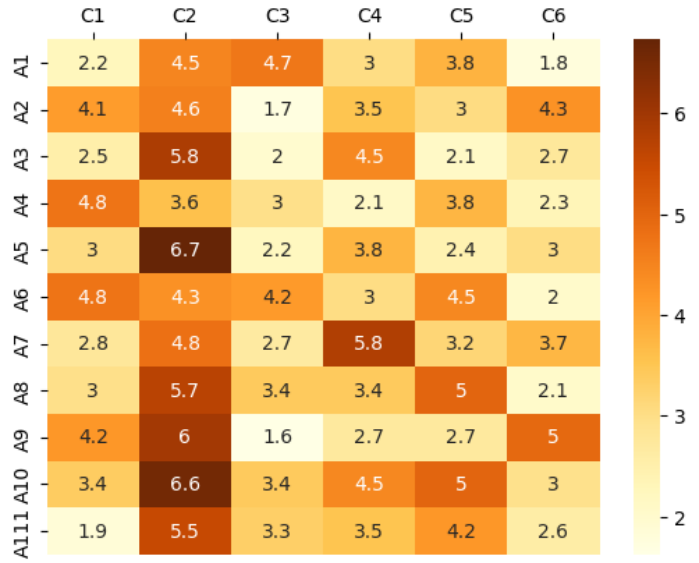

Fig 3. Normalized values.

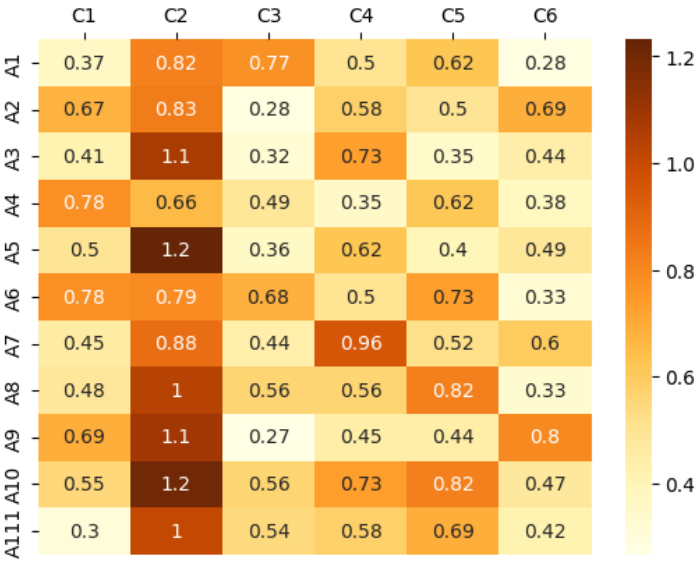

Fig 4. Weighted decision values.

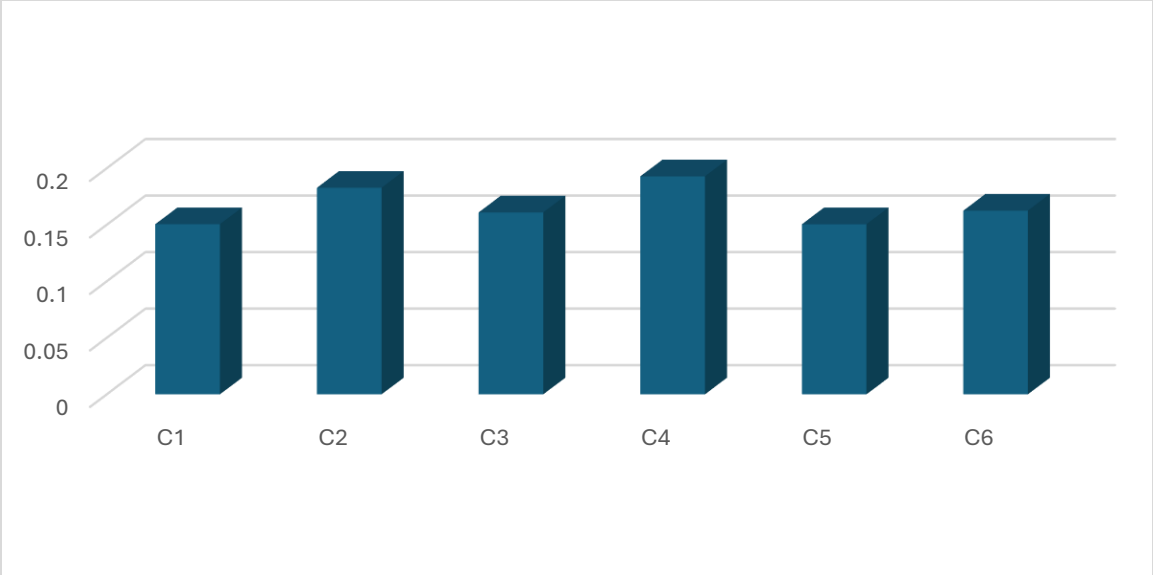

Fig 5. Weights of first sub-criterion.

In the second sub-criterion

Fig 6 shows the normalized decision matrix. Fig 7 shows the weighted decision matrix. Fig 8 shows the weights of first sub-criterion.

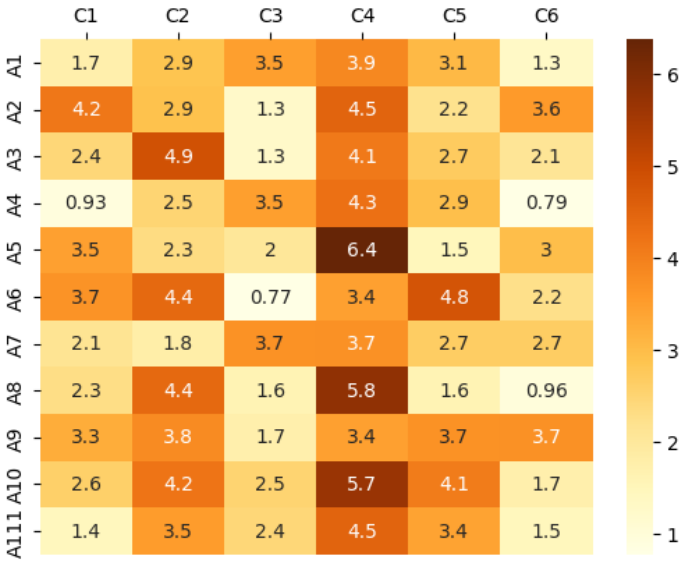

Fig 6. Normalized values.

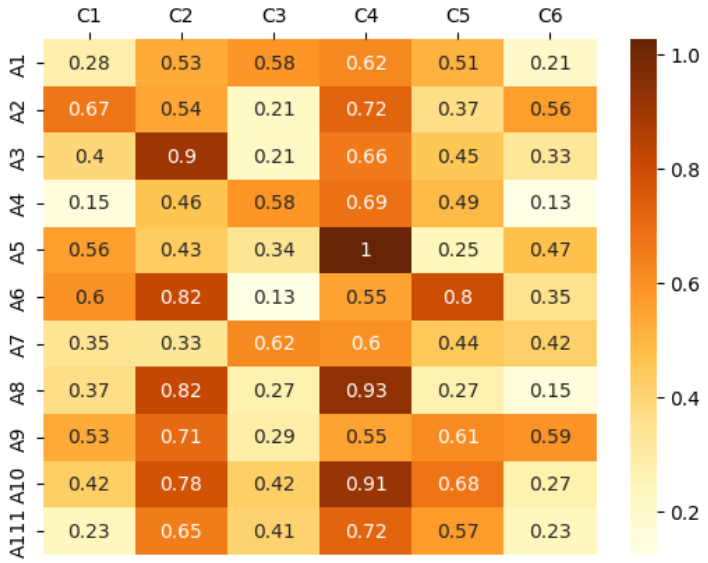

Fig 7. Weighted decision values.

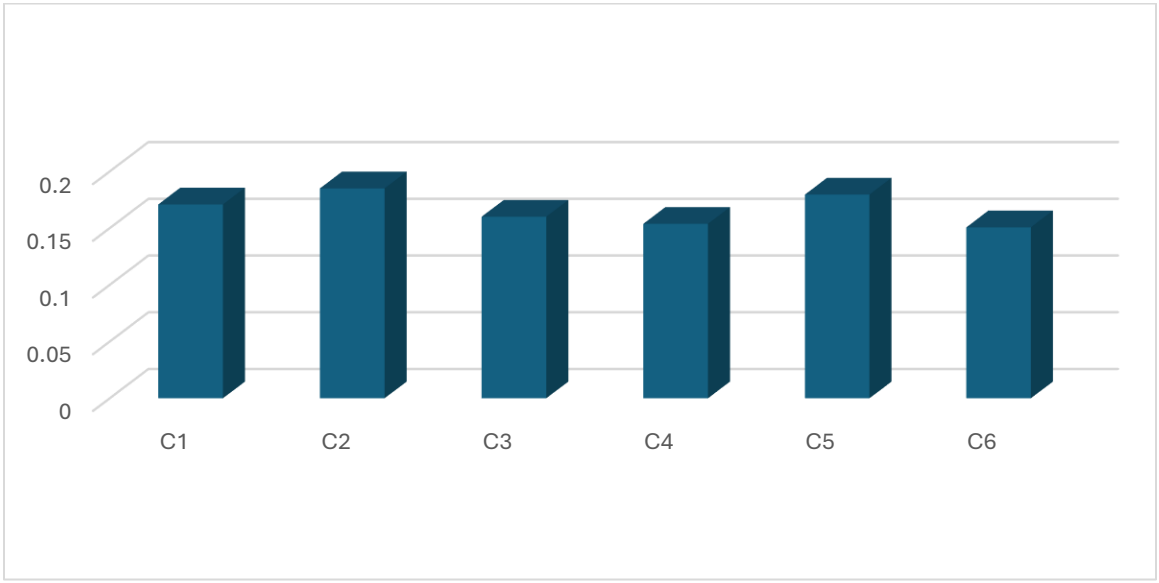

Fig 8. Weights of second sub-criterion.

In the third sub-criterion

Fig 9 shows the normalized decision matrix. Fig 10 shows the weighted decision matrix. Fig 11 shows the weights of the third sub-criterion.

Fig 9. Normalized values.

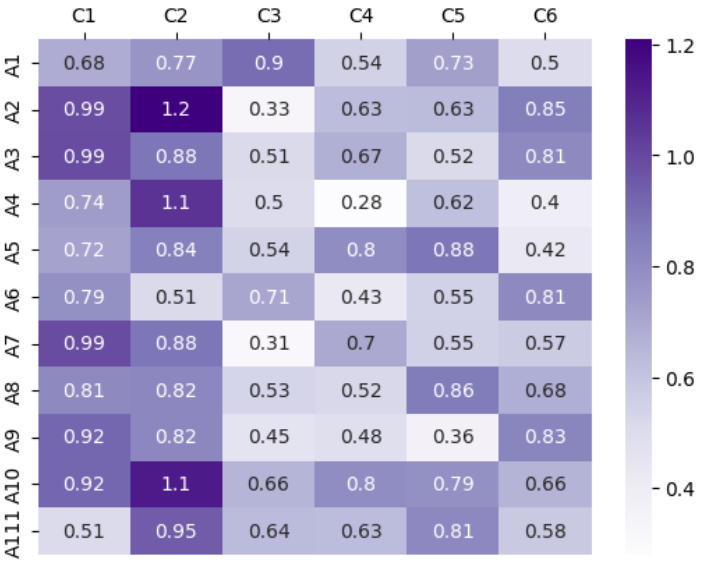

Fig 10. Weighted decision values.

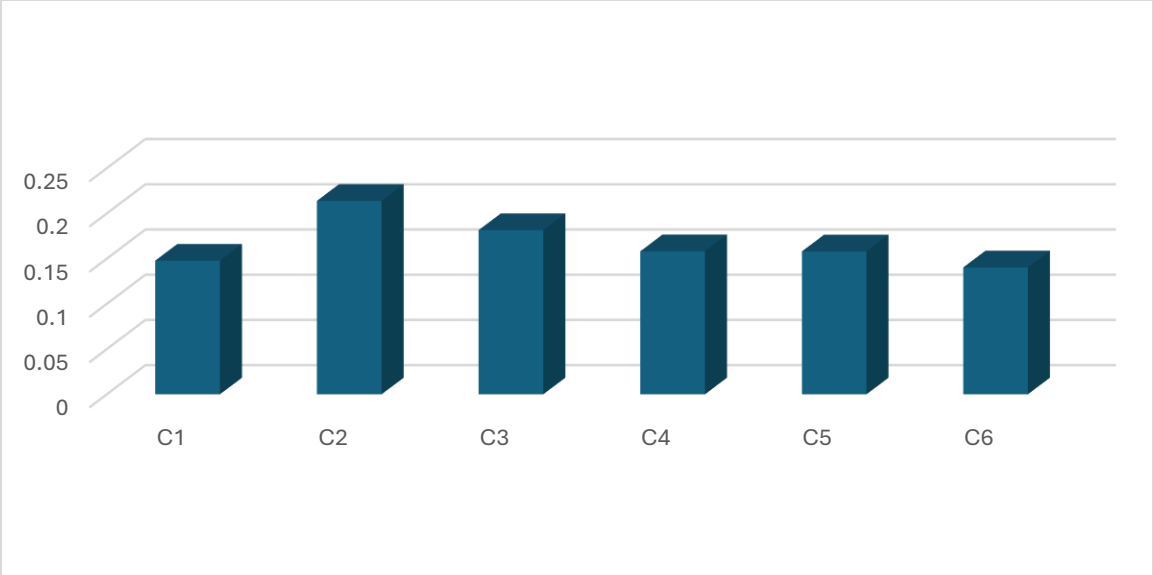

Fig 11. Weights of third sub-criterion.

In the fourth sub-criterion

Fig 12 shows the normalized decision matrix. Fig 13 shows the weighted decision matrix. Fig 14 shows the weights of the fourth criterion.

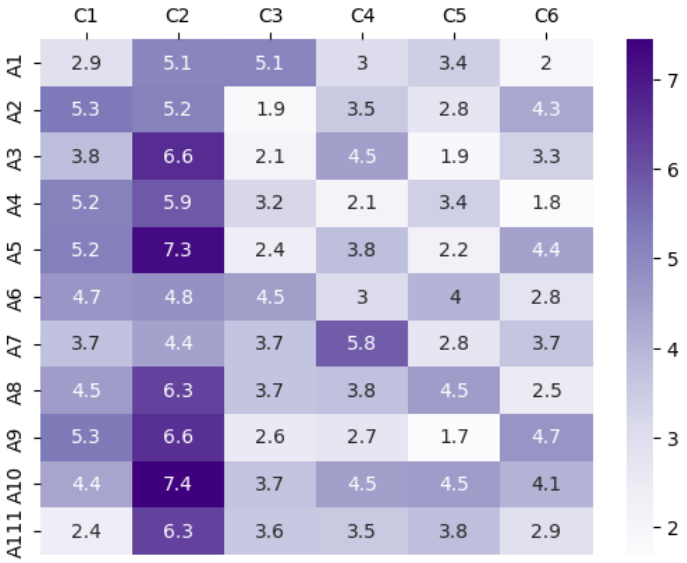

Fig 12. Normalized values.

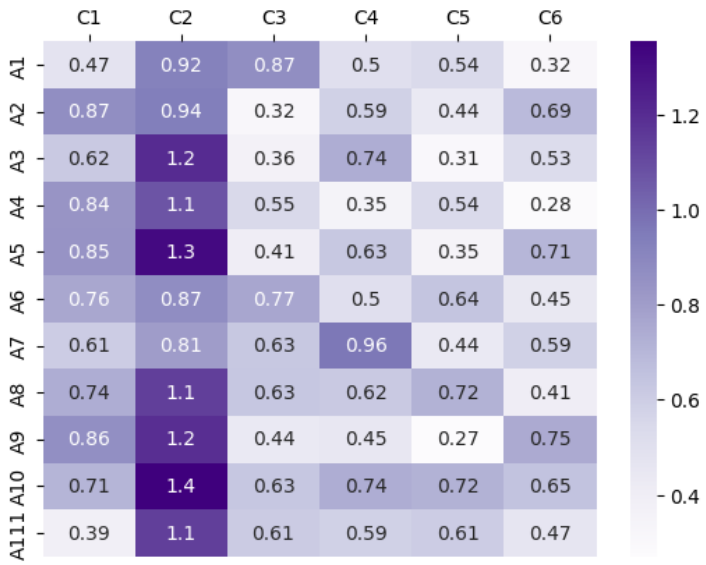

Fig 13. Weighted decision values.

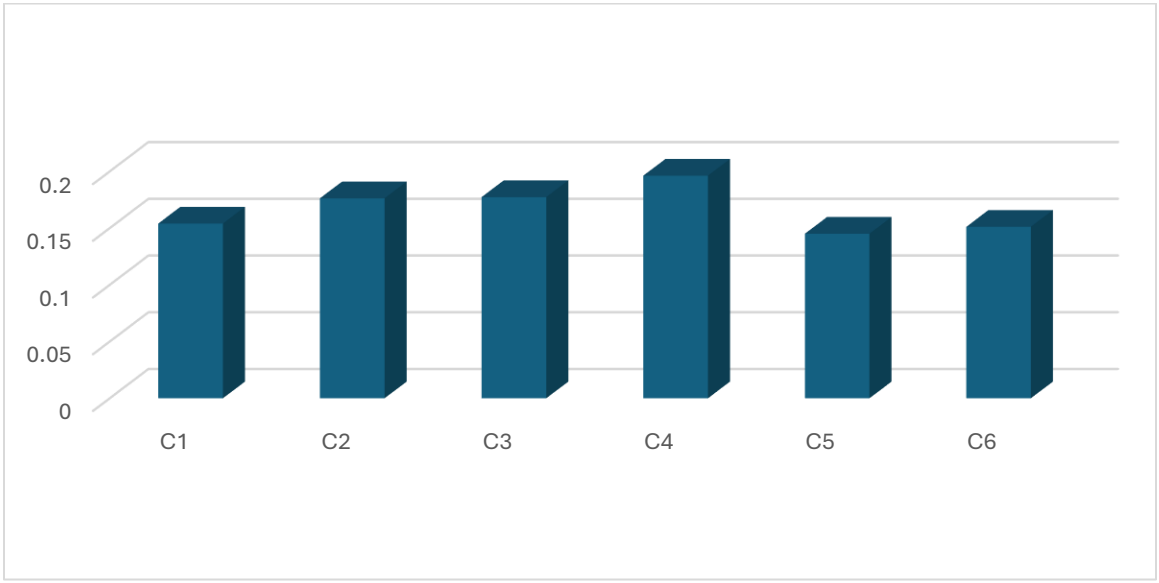

Fig 14. Weights of fourth sub-criterion.

In the fifth sub-criterion

Fig 15 shows the normalized decision matrix. Fig 16 shows the weighted decision matrix. Fig 17 shows the weights of fifth sub-criterion.

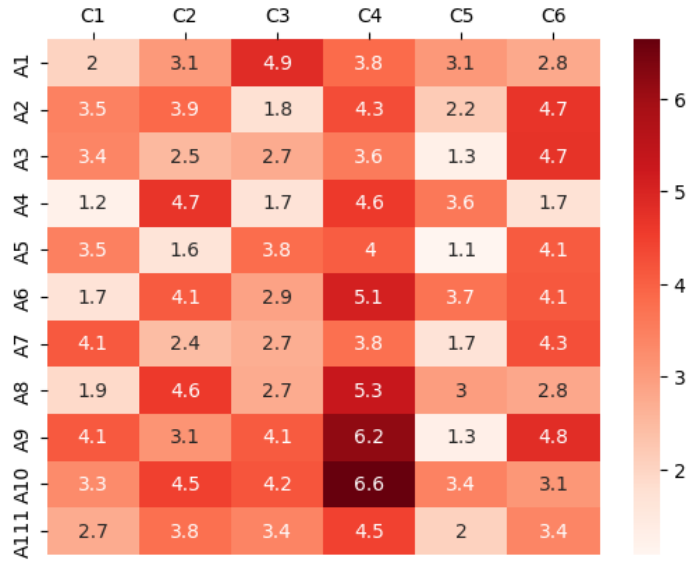

Fig 15. Normalized values.

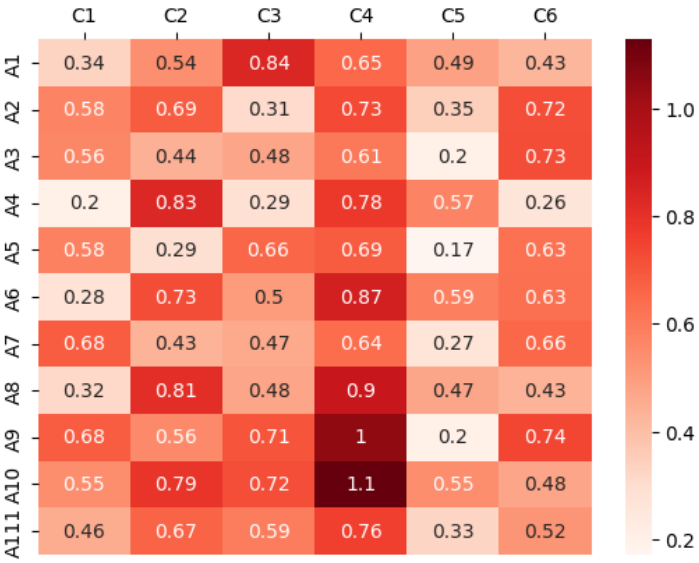

Fig 16. Weighted decision values.

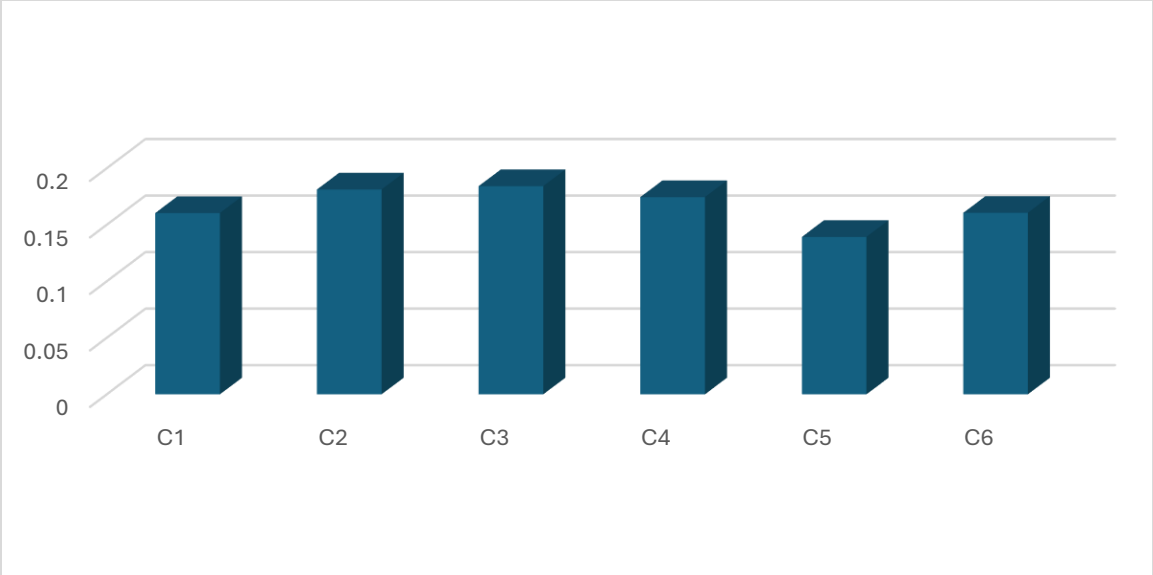

Fig 17. Weights of fifth sub-criterion.

In the sixth sub-criterion

Fig 18 shows the normalized decision matrix. Fig 19 shows the weighted decision matrix. Fig 20 shows the weights of sixth sub-criterion.

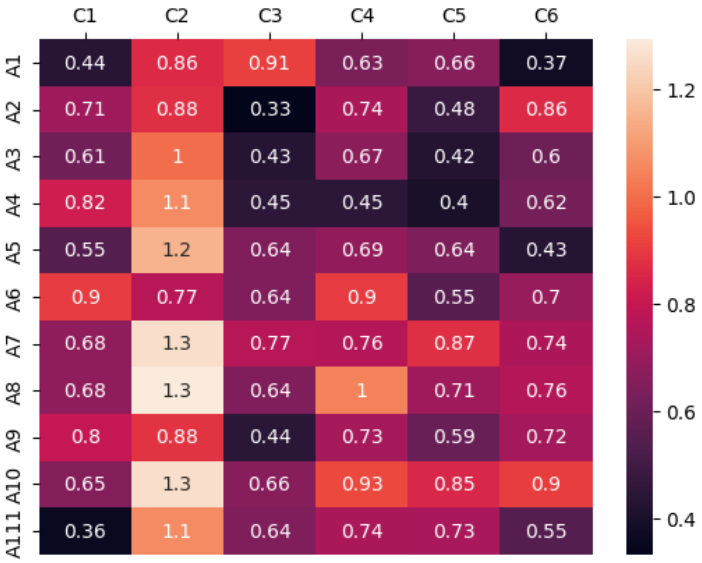

Fig 18. Normalized values.

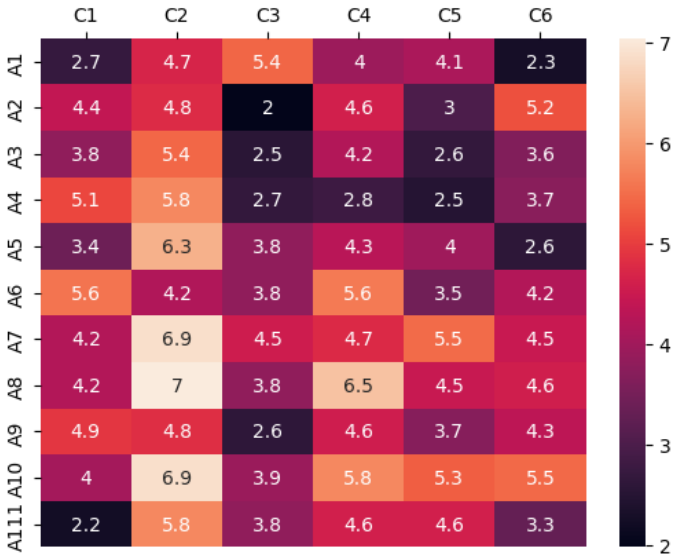

Fig 19. Weighted decision values.

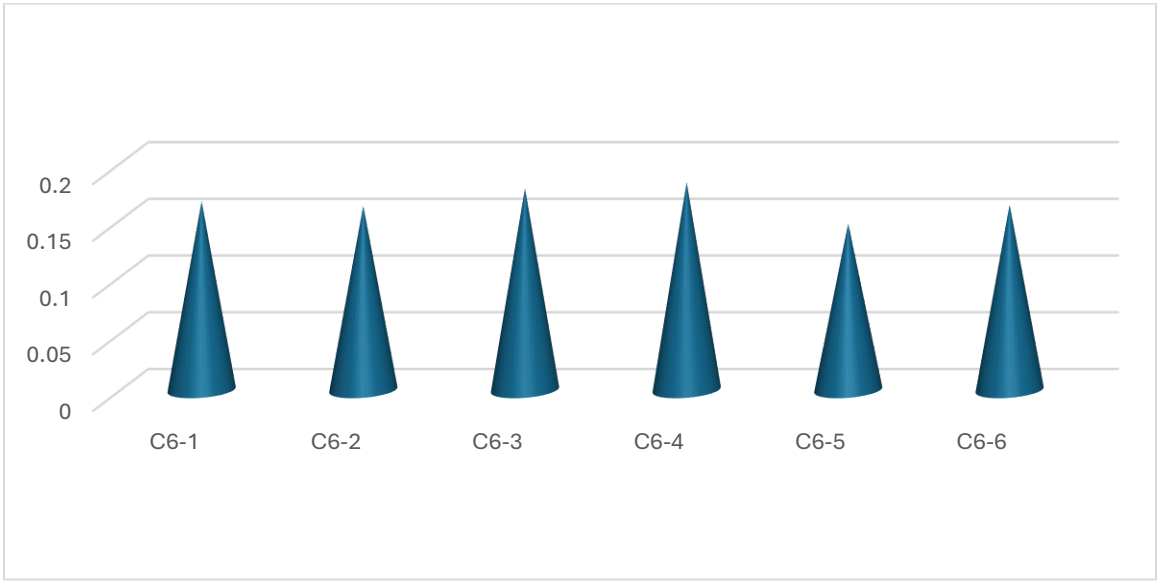

Fig 20. Weights of sixth sub-criterion.

Then we compute the global weights of the criteria by multiplications the weights of sub criteria by the weights of the main criteria. Then we select the best values of each sub-criterion.

Ranking the alternatives

Then we apply the steps of the WASPAS method to rank the alternatives.

Eq. (13) is used to normalize the decision matrix for the beneficial and non-beneficial criteria as shown in Table 3.

Eq. (15) is used to compute the additive relative importance as shown in Table 4.

Eq. (16) is used to compute the multiplicative relative importance as shown in Table 5.

Eq. (17) is used to compute the joint generalized criterion. Then we rank the alternatives as shown in Fig 21.

Table 3. Normalizations by the WASPAS method.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}
$C_{1\text{-}1}$	0.412215	0.972653	0.655485	0.594348	0.952233	0.924886	0.83069	0.702704	1	0.797448	0.439562
$C_{1\text{-}2}$	0.552497	1	0.69616	0.959721	0.442445	0.571831	0.591613	0.587272	0.736439	0.810553	0.681525
$C_{1\text{-}3}$	0.979015	0.357583	0.512731	0.456491	1	0.461015	0.865137	0.456491	0.492725	0.710129	0.689144
$C_{1\text{-}4}$	0.565238	0.656848	0.828192	0.371838	0.371838	1	0.394201	0.697255	0.503239	0.828655	0.656848
$C_{1\text{-}5}$	0.754679	0.60709	0.484623	0.883717	0.724532	0.638452	0.595053	0.890012	0.374524	1	0.840318
$C_{1\text{-}6}$	0.365247	1	0.588025	0.222778	0.783795	0.651091	0.672097	0.444841	0.762361	0.487282	0.534772
$C_{2\text{-}1}$	0.473103	0.86837	0.532881	1	0.641171	1	0.581393	0.623275	0.886966	0.707309	0.389877
$C_{2\text{-}2}$	0.665427	0.677016	0.86837	0.532881	1	0.641171	0.712538	0.845084	0.886966	0.976229	0.820828
$C_{2\text{-}3}$	1	0.365247	0.418072	0.630037	0.470896	0.883681	0.56659	0.72535	0.344527	0.72535	0.703915
$C_{2\text{-}4}$	0.523721	0.608602	0.767362	0.365247	0.651043	0.523721	1	0.588025	0.466276	0.767791	0.608602
$C_{2\text{-}5}$	0.754679	0.60709	0.423389	0.748882	0.484623	0.889515	0.631936	1	0.5405	1	0.840318
$C_{2\text{-}6}$	0.357583	0.865137	0.554701	0.47174	0.611919	0.409299	0.751259	0.414522	1	0.596251	0.52355
$C_{3\text{-}1}$	0.418072	1	0.588025	0.222778	0.831285	0.883681	0.513766	0.550776	0.783795	0.625036	0.344527
$C_{3\text{-}2}$	0.588025	0.598266	1	0.513766	0.481852	0.905544	0.370582	0.905544	0.783795	0.862675	0.72535
$C_{3\text{-}3}$	0.939581	0.343179	0.343179	0.939581	0.552497	0.209318	1	0.438104	0.472879	0.681525	0.661386
$C_{3\text{-}4}$	0.60709	0.705483	0.644249	0.668654	1	0.5405	0.583402	0.908564	0.5405	0.890012	0.705483
$C_{3\text{-}5}$	0.637381	0.461015	0.564727	0.611919	0.311089	1	0.554701	0.337297	0.767348	0.844572	0.709709
$C_{3\text{-}6}$	0.350233	0.958893	0.563853	0.21362	0.797113	0.603773	0.715676	0.259158	1	0.467251	0.396045
$C_{4\text{-}1}$	0.692684	0.999363	1	0.748353	0.724556	0.79517	1	0.818967	0.929386	0.929386	0.512288
$C_{4\text{-}2}$	0.634639	1	0.725375	0.874049	0.697255	0.423093	0.725838	0.674584	0.674584	0.931061	0.78285
$C_{4\text{-}3}$	1	0.365247	0.566162	0.555778	0.598266	0.783795	0.344527	0.588025	0.503287	0.72535	0.703915
$C_{4\text{-}4}$	0.682114	0.792667	0.847943	0.352007	1	0.537937	0.875923	0.6555	0.607296	1	0.792667
$C_{4\text{-}5}$	0.830629	0.722334	0.587542	0.702704	1	0.621696	0.628137	0.979581	0.412215	0.898086	0.924886
$C_{4\text{-}6}$	0.584099	1	0.947197	0.46363	0.497221	0.946653	0.671399	0.792774	0.966953	0.772475	0.678287
$C_{5\text{-}1}$	0.544817	1	0.710428	0.965549	0.972626	0.875854	0.696896	0.848479	0.993482	0.814525	0.448976
$C_{5\text{-}2}$	0.68163	0.693501	0.889515	0.791121	0.975153	0.644249	0.595053	0.840815	0.883717	1	0.840815
$C_{5\text{-}3}$	1	0.365247	0.418072	0.630037	0.470896	0.883681	0.724921	0.72535	0.503287	0.72535	0.703915
$C_{5\text{-}4}$	0.523721	0.608602	0.767362	0.365247	0.651043	0.523721	1	0.646042	0.466276	0.767791	0.608602
$C_{5\text{-}5}$	0.754679	0.60709	0.423389	0.748882	0.484623	0.889515	0.60709	1	0.374524	1	0.840318
$C_{5\text{-}6}$	0.423389	0.91392	0.70598	0.374524	0.938766	0.595053	0.779085	0.540003	1	0.865165	0.6199
$C_{6\text{-}1}$	0.491475	0.84228	0.825071	0.2986	0.848279	0.409431	1	0.461899	1	0.801389	0.666877
$C_{6\text{-}2}$	0.649361	0.82468	0.532113	1	0.350902	0.871075	0.520015	0.975856	0.666561	0.952659	0.80101
$C_{6\text{-}3}$	1	0.365247	0.56659	0.344527	0.783795	0.598266	0.555778	0.566162	0.84124	0.862675	0.703915
$C_{6\text{-}4}$	0.578623	0.649195	0.538364	0.690559	0.608515	0.766235	0.567151	0.800916	0.929428	1	0.672403
$C_{6\text{-}5}$	0.828192	0.588371	0.342947	0.977329	0.291692	1	0.451213	0.805521	0.348705	0.931061	0.55403
$C_{6\text{-}6}$	0.576507	0.971884	0.979569	0.354514	0.847423	0.854046	0.8816	0.583505	1	0.639175	0.701468

Table 4. Values of additive relative importance.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}
$C_{1\text{-}1}$	0.008738	0.020619	0.013895	0.012599	0.020186	0.019606	0.017609	0.014896	0.021198	0.016905	0.009318
$C_{1\text{-}2}$	0.014221	0.02574	0.017919	0.024703	0.011389	0.014719	0.015228	0.015116	0.018956	0.020864	0.017542
$C_{1\text{-}3}$	0.022191	0.008105	0.011622	0.010347	0.022667	0.01045	0.01961	0.010347	0.011169	0.016097	0.015621
$C_{1\text{-}4}$	0.015351	0.01784	0.022493	0.010099	0.010099	0.027159	0.010706	0.018937	0.013668	0.022506	0.01784
$C_{1\text{-}5}$	0.015995	0.012867	0.010271	0.01873	0.015356	0.013532	0.012612	0.018863	0.007938	0.021194	0.01781
$C_{1\text{-}6}$	0.008359	0.022887	0.013458	0.005099	0.017939	0.014901	0.015382	0.010181	0.017448	0.011152	0.012239
$C_{2\text{-}1}$	0.016331	0.029976	0.018395	0.034519	0.022133	0.034519	0.020069	0.021515	0.030617	0.024416	0.013458
$C_{2\text{-}2}$	0.024871	0.025304	0.032456	0.019917	0.037376	0.023964	0.026632	0.031586	0.033151	0.036487	0.030679
$C_{2\text{-}3}$	0.032328	0.011808	0.013515	0.020368	0.015223	0.028567	0.018317	0.023449	0.011138	0.023449	0.022756
$C_{2\text{-}4}$	0.01627	0.018907	0.023839	0.011347	0.020225	0.01627	0.031066	0.018268	0.014485	0.023852	0.018907
$C_{2\text{-}5}$	0.027376	0.022022	0.015359	0.027166	0.01758	0.032267	0.022924	0.036275	0.019607	0.036275	0.030483
$C_{2\text{-}6}$	0.010885	0.026334	0.016885	0.01436	0.018627	0.012459	0.022868	0.012618	0.03044	0.01815	0.015937
$C_{3\text{-}1}$	0.0096	0.022962	0.013502	0.005115	0.019088	0.020291	0.011797	0.012647	0.017997	0.014352	0.007911

$C_{3\text{-}2}$	0.019538	0.019879	0.033227	0.017071	0.01601	0.030088	0.012313	0.030088	0.026043	0.028664	0.024101
$C_{3\text{-}3}$	0.026493	0.009677	0.009677	0.026493	0.015579	0.005902	0.028197	0.012353	0.013334	0.019217	0.018649
$C_{3\text{-}4}$	0.014913	0.01733	0.015826	0.016426	0.024565	0.013278	0.014331	0.022319	0.013278	0.021863	0.01733
$C_{3\text{-}5}$	0.015648	0.011318	0.013864	0.015023	0.007637	0.02455	0.013618	0.008281	0.018839	0.020735	0.017424
$C_{3\text{-}6}$	0.007635	0.020905	0.012293	0.004657	0.017378	0.013163	0.015602	0.00565	0.021801	0.010187	0.008634
$C_{4\text{-}1}$	0.015892	0.022928	0.022943	0.017169	0.016623	0.018244	0.022943	0.01879	0.021323	0.021323	0.011753
$C_{4\text{-}2}$	0.016666	0.02626	0.019048	0.022953	0.01831	0.01111	0.01906	0.017715	0.017715	0.02445	0.020558
$C_{4\text{-}3}$	0.02642	0.00965	0.014958	0.014684	0.015806	0.020708	0.009102	0.015535	0.013297	0.019164	0.018597
$C_{4\text{-}4}$	0.019941	0.023172	0.024788	0.01029	0.029233	0.015726	0.025606	0.019163	0.017753	0.029233	0.023172
$C_{4\text{-}5}$	0.01795	0.01561	0.012697	0.015186	0.02161	0.013435	0.013574	0.021169	0.008908	0.019408	0.019987
$C_{4\text{-}6}$	0.013153	0.022518	0.021329	0.01044	0.011196	0.021316	0.015118	0.017851	0.021774	0.017394	0.015273
$C_{5\text{-}1}$	0.014998	0.027529	0.019557	0.026581	0.026775	0.024111	0.019185	0.023358	0.02735	0.022423	0.01236
$C_{5\text{-}2}$	0.021202	0.021571	0.027668	0.024608	0.030332	0.020039	0.018509	0.026154	0.027488	0.031105	0.026154
$C_{5\text{-}3}$	0.031615	0.011547	0.013218	0.019919	0.014888	0.027938	0.022919	0.022932	0.015912	0.022932	0.022255
$C_{5\text{-}4}$	0.015687	0.018229	0.022985	0.01094	0.019501	0.015687	0.029953	0.019351	0.013966	0.022998	0.018229
$C_{5\text{-}5}$	0.018045	0.014516	0.010124	0.017906	0.011588	0.021269	0.014516	0.023911	0.008955	0.023911	0.020093
$C_{5\text{-}6}$	0.011671	0.025192	0.01946	0.010324	0.025877	0.016403	0.021476	0.014885	0.027565	0.023848	0.017088
$C_{6\text{-}1}$	0.014819	0.025397	0.024878	0.009003	0.025577	0.012345	0.030152	0.013927	0.030152	0.024164	0.020108
$C_{6\text{-}2}$	0.019106	0.024264	0.015656	0.029422	0.010324	0.025629	0.0153	0.028712	0.019612	0.028029	0.023568
$C_{6\text{-}3}$	0.032229	0.011771	0.01826	0.011104	0.025261	0.019281	0.017912	0.018247	0.027112	0.027803	0.022686
$C_{6\text{-}4}$	0.019205	0.021548	0.017869	0.022921	0.020197	0.025432	0.018825	0.026584	0.030849	0.033191	0.022318
$C_{6\text{-}5}$	0.022017	0.015641	0.009117	0.025982	0.007754	0.026584	0.011995	0.021414	0.00927	0.024752	0.014728
$C_{6\text{-}6}$	0.017068	0.028774	0.029002	0.010496	0.025089	0.025285	0.026101	0.017276	0.029607	0.018924	0.020768

Table 5. Values of multiplicative relative importance.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}
$C_{1\text{-}1}$	0.981389	0.999412	0.991086	0.989031	0.998963	0.998346	0.996075	0.992549	1	0.995213	0.982726
$C_{1\text{-}2}$	0.984844	1	0.990721	0.998942	0.979229	0.985717	0.98658	0.986393	0.992156	0.994608	0.990179
$C_{1\text{-}3}$	0.999519	0.976959	0.984972	0.982382	1	0.982601	0.996722	0.982382	0.984084	0.992271	0.991596
$C_{1\text{-}4}$	0.984625	0.98865	0.994893	0.973489	0.973489	1	0.975035	0.990254	0.981523	0.994908	0.98865
$C_{1\text{-}5}$	0.994052	0.989478	0.984764	0.997383	0.993194	0.990535	0.989058	0.997533	0.9794	1	0.996319
$C_{1\text{-}6}$	0.977212	1	0.987921	0.966217	0.99444	0.990227	0.990947	0.981632	0.993809	0.983681	0.985777
$C_{2\text{-}1}$	0.974495	0.99514	0.978506	1	0.984775	1	0.981453	0.983813	0.995868	0.988118	0.968008
$C_{2\text{-}2}$	0.984891	0.985527	0.994739	0.976748	1	0.983525	0.987412	0.993729	0.995527	0.999101	0.992648
$C_{2\text{-}3}$	1	0.967964	0.972201	0.985176	0.975947	0.99601	0.981802	0.989673	0.966139	0.989673	0.988714
$C_{2\text{-}4}$	0.980107	0.984691	0.991808	0.969195	0.986756	0.980107	1	0.98364	0.976576	0.991825	0.984691
$C_{2\text{-}5}$	0.989842	0.982059	0.969304	0.989565	0.974065	0.995762	0.983489	1	0.977929	1	0.993709
$C_{2\text{-}6}$	0.969181	0.9956	0.982221	0.977389	0.985161	0.973174	0.991332	0.97355	1	0.984383	0.980495
$C_{3\text{-}1}$	0.980174	1	0.987882	0.966109	0.995766	0.997165	0.984824	0.986398	0.994422	0.989267	0.975829
$C_{3\text{-}2}$	0.982512	0.983076	1	0.978114	0.976032	0.996709	0.967554	0.996709	0.991938	0.995104	0.989388
$C_{3\text{-}3}$	0.998244	0.970293	0.970293	0.998244	0.98341	0.956861	1	0.976998	0.979104	0.989247	0.988411
$C_{3\text{-}4}$	0.987815	0.991466	0.989257	0.990161	1	0.985	0.986849	0.997647	0.985	0.997142	0.991466
$C_{3\text{-}5}$	0.989004	0.98117	0.98607	0.988014	0.97174	1	0.985636	0.973672	0.99352	0.995861	0.991617
$C_{3\text{-}6}$	0.977387	0.999085	0.987587	0.966909	0.995069	0.98906	0.992733	0.970991	1	0.983549	0.98001
$C_{4\text{-}1}$	0.991611	0.999985	1	0.993371	0.992635	0.994755	1	0.995429	0.998321	0.998321	0.984771
$C_{4\text{-}2}$	0.988131	1	0.991604	0.996471	0.990575	0.977665	0.991621	0.989716	0.989716	0.998126	0.993592
$C_{4\text{-}3}$	1	0.973741	0.985083	0.984601	0.986519	0.993585	0.97224	0.986069	0.982024	0.991552	0.990767
$C_{4\text{-}4}$	0.988879	0.993231	0.99519	0.969938	1	0.982038	0.996135	0.987729	0.985526	1	0.993231
$C_{4\text{-}5}$	0.995998	0.992995	0.988573	0.992404	1	0.989781	0.990002	0.999554	0.981031	0.99768	0.998314
$C_{4\text{-}6}$	0.987966	1	0.998779	0.98284	0.98439	0.998766	0.991069	0.994785	0.999244	0.994204	0.991297
$C_{5\text{-}1}$	0.98342	1	0.990632	0.999035	0.999236	0.996358	0.990108	0.995487	0.99982	0.994368	0.978196
$C_{5\text{-}2}$	0.988149	0.98868	0.996365	0.992738	0.999218	0.986417	0.983983	0.994621	0.996162	1	0.994621
$C_{5\text{-}3}$	1	0.968659	0.972805	0.985501	0.976471	0.996098	0.989881	0.9899	0.978527	0.9899	0.988961
$C_{5\text{-}4}$	0.980813	0.985236	0.9921	0.970282	0.987227	0.980813	1	0.986999	0.977406	0.992117	0.985236
$C_{5\text{-}5}$	0.993293	0.988138	0.979659	0.993109	0.982829	0.997204	0.988138	1	0.976791	1	0.995849
$C_{5\text{-}6}$	0.976587	0.997522	0.990449	0.973291	0.99826	0.985793	0.993142	0.983158	1	0.996016	0.986905
$C_{6\text{-}1}$	0.978809	0.994838	0.994219	0.964213	0.995051	0.973434	1	0.976979	1	0.993346	0.987858
$C_{6\text{-}2}$	0.987377	0.994345	0.981609	1	0.969657	0.995947	0.980945	0.999281	0.988137	0.998574	0.993493
$C_{6\text{-}3}$	1	0.968061	0.981857	0.966241	0.99218	0.98358	0.981247	0.981833	0.994444	0.995251	0.988748
$C_{6\text{-}4}$	0.982005	0.985763	0.979657	0.987786	0.983648	0.991201	0.981352	0.992659	0.997574	1	0.986913
$C_{6\text{-}5}$	0.995001	0.985999	0.971951	0.999391	0.967777	1	0.979066	0.994267	0.972381	0.998103	0.984424
$C_{6\text{-}6}$	0.983826	0.999156	0.999389	0.969764	0.99511	0.99534	0.996276	0.984177	1	0.986836	0.989557

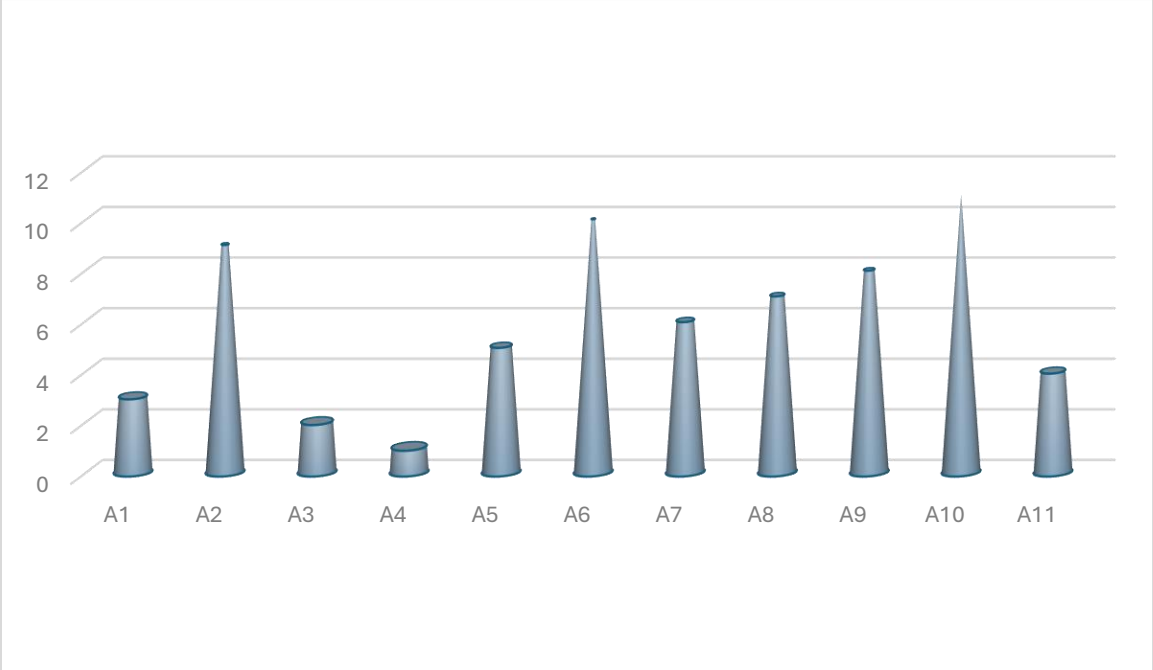

Fig 21. Rank of alternatives.

4. Conclusions

Higher education competitiveness is a multi-dimensional concept that encompasses academic quality, infrastructure, industry engagement, internationalization, and financial stability. Institutions that excel in these areas not only enhance their global reputation but also provide students with high-quality learning experience and career opportunities. By employing structured evaluation frameworks like MCDM, universities can systematically assess their strengths and weaknesses and develop strategies to improve their competitive standing. As the global education landscape continues to evolve, institutions must remain adaptable, innovative, and forward-thinking to maintain their relevance and leadership in higher education. TreeSoft set is used in this study to define relationships between the criteria and sub-criteria. CIMAS method was used to compute the criteria weights and the WASPAS method was used to rank the alternatives.

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Estimating and Testing Augmented Randomized Complete Block Designs: The Neutrosophic Approach

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Abstract. In plant breeding programs, the augmented design is designated to screen numerous new treatments compared with a few check treatments, which in turn are required to estimate both of error variance and local control for new future treatments. It is well known that the classical augmented design is not suitable for data that are imprecise, uncertain, or undetermined, and these accordingly emerge because of many circumstances beyond humans control. As a result, there is a sever necessity to define a proper generalization for the augmented designs to handle uncertain environments. To be more specific, this work aims to propose an easy to apply approach to treat the augmented randomized complete block design under neutrosophic statistics (NS). This well-defined approach is based on building a neutrosophic ANOVA table, including deriving a suitable test statistics, F_N, to handle uncertain settings.This leads to the corresponding neutrosophic hypotheses and the necessary related decision rules. Real data and a series of simulation studies numerically assess the performance of the present method. It will be shown that the neutrosophic method outperforms the classical one, and in effect, it is more flexible than in the presence of indeterminacy.

Keywords: Augmented randomized complete block design; neutrosophic test statistics; indeterministic observations; NANOVA.

1. Introduction

Breeding programs worldwide are administered when numerous genotypes are the object of examination. As a result, the augmented design has been developed for comparing thousands of genotypes with only one plot per genotype. This statistical art was first introduced by Federer [17]. This design contains a small number of varieties, known as checks, which are

randomly replicated in the field to compute an estimate of error [17]. As Federer states, one can calculate the expected yield for the plots in the studied field by relying on those varieties, and then the observed yields of the genotypes can be adjusted. Besides, the augmented design can also be used in characterizing genotypes for assessing mean performances, assessing effects, and controlling the variability of genotypes (see Federer and Crossa [18]).

The augmented design is a special case of block design (like complete, incomplete, Latin squares, etc.) by enlarging the block size. For example, the augmented randomized block design (ARCBD) can be constructed as c replicated checks and n non-replicated new treatments in r blocks. To build this augmented design, a randomized complete block design (RCBD) with r blocks and c check genotypes is employed. Then, the blocks are expanded so that embrace all c checks and n new units, in a way that each block would contain n/r new genotypes. We may refer the interested reader to review many examples of augmented designs in Federer and Crossa [18].

On the other hand, the augmented row-column designs along with few checks were handled by Piepho and Williams [24]. Also, the augmented design in which all treatments are not replicated was discussed by Burgueño, et al. (Ch. 13) [10]. Further developments in augmented designs can also be found in [26, 28].

In today's real world studies, especially when the experimental units are either biological or breeding varieties, uncertain observations are plausible. What is known for sure is that the classical ARCBD model will fail in analyzing such indeterministic observations. So, the search aims to define a novel neutrosophic model to overcome all limitations in the literature of ARCBD, and to our knowledge, this is the first such study in this framework.

The statistical art of neutrosophic statistics (NS) has attracted many researchers in recent years, years after its first definition by Smarandache [32], to propose and define new approaches to deal with uncertain and imprecise information. Aslam [11] gives an explicit explanation for the differences between the fuzzy, neutrosophic, and classic statistics. Contributions by Aslam [12] in experimental design analysis, the neutrosophic ANOVA was completely highlighted. After that, Aslam and Albassam [15] introduced multiple comparison post hoc tests for NS. The neutrosophic ANCOVA's for varieties of block designs were suggested and developed by AlAita and Aslam [1]. Other NS applications to deal with imprecise data in split-plot design and augmented Latin square design are given in [8, 9], respectively. To stay on track, one may point out to the numerous NS applications that are sufficiently discussed in [2–6, 13, 14, 16, 20–22, 27, 30, 31, 33, 35].

The randomized complete design is, in fact, the most and popular design in agricultural studies and biological fields, and this design can be extended to uncertain environments by adopting the NS, which is suitable for estimating model parameters and compute all related sum of

squares. The suggested new approach, which is called the NARCBD, is a suitable generalization of the classic ARCBD in sense of dealing with both exact or vague information. The superiority of the proposed NARCBD over the ARCBD will be approved through real data study and simulation scenarios. It will be shown that the suggested design can completely discriminate between the null hypothesis and the alternatives, where the type I error is controlled at a given nominal level of α.

Following in this paper, first, we give a brief summary addressing the motivation for this current research in Section 2. After that, basic concepts and notions of the NS framework are introduced in Section 3. An overview of the addressed NARCBD is given in Section 4. Section 5 contains a numerical explanatory example of real case study, as well as a series of simulation studies to check and evaluate the performance of our proposed design. A wide conclusion and further discussions are presented in Section 6.

2. **Motivation, Necessity and Research Gap**

Imprecise data in real-world problems forces researchers to innovate corresponding statistical techniques to deal with such a situation. This motivated us to generalize the augmented designs consisting of imprecise observations and solve them under the NS setting. In this way, one can deal with interval-type observations or a set of neutrosophic data. In our point of view, this is a significant addition to previous classical experimental designs, and would close the gap between the two. In fact, our work was inspired by many previous literatures in NS, showing its flexibilty and efficiency in treating complex situations. In this paper, the main focus was made on presenting a new theoretical approach along with a series of simulation studies and a real case study.

3. **Preliminaries**

By reviewing some basic ideas of NS in this section, one can easily follow-up and comprehend the main results related to the subsequent sections.

The neutrosophic logic is a type of logic that extends both of classical and fuzzy logics to handle uncertain, vague and indeterminate data. As already known, fuzzy sets represent the uncertain part by a single-valued membership on [0, 1], whereas NS considers three possible outcomes: true, false and indeterminate. Consequently, the indeterminate part opens a wide window to accommodate uncertain and vague data more nuancedly.

The neutrosophic probability [34] is, in effect, a natural generalization of classical probability theory for dealing with incomplete and indeterminate data. The neutrosophic probability is the triple (T, I, F), where T and F are respectively for truth and falsity, and I stands for uncertainty.

A neutrosophic random variable includes two parts, defined by

$$X_N = X_L + X_U I_N$$

where X_L is classical and $X_U I_N$ is the indeterminate part for uncertainty level $I_N \in [I_L, I_U]$. In this way, the values of NRV can be true, false or indeterminate, simultaneously. Consider the neutrosophic normal random variable $X_N \in [X_L, X_U]$ with neutrosophic mean and variance $\mu_N \in [\mu_L, \mu_U\ ]$ and $\sigma_N^2 \in [\sigma_L^2, \sigma_U^2]$, where

$$\mu_N \in [\mu_L, \mu_U\ ] = \left[\frac{\sum_{i=1}^{N} X_{Li}}{N}, \frac{\sum_{i=1}^{N} X_{Ui}}{N}\right]$$
$$\sigma_N^2 \in [\sigma_L^2, \sigma_U^2] = \left[\frac{\sum_{i=1}^{N} (X_{Li} - \mu_L)^2}{N}, \frac{\sum_{i=1}^{N} (X_{Ui} - \mu_U)^2}{N}\right]$$

Now, let a neutrosophic random sample of size n is drawn from a population of size N including indeterminate observations. The observed neutrosophic sample mean $\overline{x}_N$ and variance s_N^2 are

$$\overline{x}_N \in [\overline{x}_L, \overline{x}_U\ ] = \left[\frac{\sum_{i=1}^{n} x_{Li}}{n}, \frac{\sum_{i=1}^{n} x_{Ui}}{n}\right]$$
$$s_N^2 \in [s_L^2, s_U^2] = \left[\frac{\sum_{i=1}^{n} (x_{Li} - \overline{x}_L)^2}{n-1}, \frac{\sum_{i=1}^{n} (x_{Ui} - \overline{x}_U)^2}{n-1}\right]$$

4. **The NARCBD**

The model of a design with b blocks, c checks and v new treatments can be written as

$$y_{Nhijg} = \mu_N + \alpha_{Ni} + \tau_{Nqj} + \tau_{Nlig} + \varepsilon_{Nhijg}, \quad \begin{cases} i = 1, 2, \ldots, b \\ j = 1, 2, \ldots, c \\ g = 1, 2, \ldots, n_{(li)} \end{cases} \tag{1}$$

where indices l and q for h are related to new treatments and checks, respectively, μ_N is the general mean, α_{Ni} is the i^{th} neutrosophic block effect, τ_{Nqj} is the j^{th} neutrosophic check effect, τ_{Nlig} is the g^{th} neutrosophic new treatment effect, and ε_{Nhijg} is the neutrosophic error with zero mean and variance σ_N^2. For $\mathrm{v} = \sum_{i=1}^{b} n_{(li)}$ new treatments and c checks, $e = \mathrm{v} + c$ is the total number of checks and new treatments; therefore, the total number of units in blocks is $n = \mathrm{v} + bc$.

It is worth mentioning here that the Equation (1) can be rewritten as

$$y_{Nhijg} = y_{Lhijg} + y_{Uhijg} I_N; \quad I_N \in [I_L, I_U] \tag{2}$$

In the neutrosophic ANOVA framework, the subscript N stands for the neutrosophic expression, and hence, the neutrosophic sum of squares (NSS) are denoted by SS_{NT}, SS_{NTr}, SS_{NB} and SS_{NE} to represent total, treatment, block, and error sum of squares, respectively.

4.1. *Estimating neutrosophic parameters*

After imposing the following two constraints

$$\sum_{i=1}^{b} \widehat{\alpha}_{Ni} = 0$$
$$\sum_{j=1}^{c} \widehat{\tau}_{Nqj} + \sum_{i=1}^{b} \sum_{g=1}^{n_{(li)}} \widehat{\tau}_{Nlig} = 0 \tag{3}$$

on the model (1), one can estimate model parameters by solving the least square normal equations and get

$$\widehat{\mu}_N = \frac{1}{\mathrm{v}+c}\left(y_{N...} - \frac{(b-1)}{b}\sum_{j=1}^{c} y_{Nq.j}\right) - \sum_{i=1}^{b} n_{(li)}\widehat{\alpha}_{Ni}; \quad \widehat{\mu}_N \in [\widehat{\mu}_L,\ \widehat{\mu}_U]$$
$$\widehat{\alpha}_{Ni} = \overline{y}_{Nqi.} - \frac{y_{Nq..}}{bc}; \quad \widehat{\alpha}_{Ni} \in [\widehat{\alpha}_{Li}, \widehat{\alpha}_{Ui}]$$
$$\widehat{\tau}_{Nqj} = \overline{y}_{Nq.j} - \widehat{\mu}_N; \quad \widehat{\tau}_{Nqj} \in [\widehat{\tau}_{Lqj}, \widehat{\tau}_{Uqj}] \tag{4}$$
$$\widehat{\tau}_{Nlig} = y_{Nlig} - \widehat{\alpha}_{Ni} - \widehat{\mu}_N; \quad \widehat{\tau}_{Nlig} \in [\widehat{\tau}_{Llig}, \widehat{\tau}_{Ulig}]$$

where $i = 1, 2, ..., b$, $j = 1, 2, ..., c$, and $g = 1, 2, ..., n_{(li)}$.
All parameter estimators in neutrosophic treatment models and block-reduced models can be obtained in a similar manner.

4.2. *The neutrosophic test statistics*

First of all, let us introduce some mathematical expressions for NSS, where "adj" and "unadj" stand for adjusted and unadjusted sums of squares, respectively.

$$SS_{NT} = \sum_{i=1}^{b}\sum_{j=1}^{c} y_{Nqij}^2 + \sum_{i=1}^{b}\sum_{g=1}^{n_{(li)}} y_{Nlig}^2 - \frac{y_{N...}^2}{n}; \quad SS_{NT} \in [SS_{LT},\ SS_{UT}]$$

$$SS_{NB(\text{unadj})} = \sum_{i=1}^{b} \frac{y_{N.i.}^2}{c + n_{(li)}} - \frac{y_{N...}^2}{n};\ SS_{NB(\text{unadj})} \in \left[SS_{LB(\text{unadj})},\ SS_{UB(\text{unadj})}\right]$$

$$SS_{NTr(\text{adj})} = \sum_{i=1}^{b}\left(\overline{y}_{Nqi.} - \frac{y_{Nq..}}{bc}\right) y_{Nqi.} + \frac{1}{b}\sum_{j=1}^{c} y_{Nq.j}^2 + \sum_{i=1}^{b}\sum_{g=1}^{n_{(li)}} y_{Nlig}^2 - \sum_{i=1}^{b} \frac{y_{N.i.}^2}{c + n_{(li)}};$$

$$SS_{NTr(\text{adj})} \in \left[SS_{LTr(\text{adj})},\ SS_{UTr(\text{adj})}\right]$$

$$SS_{NTr(\text{unadj})} = \frac{1}{b}\sum_{j=1}^{c} y_{Nq.j}^2 + \sum_{i=1}^{b}\sum_{g=1}^{n_{(li)}} y_{Nlig}^2 - \frac{y_{N...}^2}{n};$$

$$SS_{NTr(\text{unadj})} \in \left[SS_{LTr(\text{unadj})} SS_{UTr(\text{unadj})}\right],$$

$$SS_{NCheck} = \frac{1}{b}\sum_{j=1}^{c} y_{Nq.j}^2 - \frac{y_{Nq..}^2}{bc};\ SS_{NCheck} \in [SS_{LCheck},\ SS_{UCheck}]$$

$$SS_{Nnew} = \sum_{i=1}^{b}\sum_{g=1}^{n_{(li)}} y_{Nlig}^2 - \frac{y_{Nl..}^2}{\mathrm{v}}; SS_{Nnew} \in [SS_{Lnew},\ SS_{Unew}]$$

$$SS_{N\text{new and new} \times \text{ch}} = SS_{NTr(\text{adj})} - SS_{NCheck};$$

$$SS_{N\text{new and new} \times \text{ch}} \in [SS_{L\text{new and new} \times \text{ch}},\ SS_{U\text{new and new} \times \text{ch}}]$$

$$SS_{N\text{new} \times \text{check}} = SS_{NTr(\text{unadj})} - SS_{NCheck} - SS_{Nnew};$$

$$SS_{N\text{new} \times \text{check}} \in [SS_{L\text{new} \times \text{check}},\ SS_{U\text{new} \times \text{check}}]$$

$$SS_{NB(\text{adj})} = SS_{NB(\text{unadj})} - (SS_{NTr(\text{unadj})} - SS_{NTr(\text{adj})}); SS_{NB(\text{adj})} \in \left[SS_{LB(\text{adj})} SS_{UB(\text{adj})}\right]$$

$$SS_{NE} = SS_{NT} - SS_{NTr(\text{adj})} SS_{NB(\text{unadj})} SS_{NT} - SS_{NTr(\text{unadj})} SS_{NB(\text{adj})};$$

$$SS_{NE} \in [SS_{LE},\ SS_{UE}]$$

It is well known that mean squares are obtained by dividing the each corresponding sum of squares onto its degrees of freedom. As a result, the neutrosophic mean squares are as follows:

$$MS_{NTr(\text{adj})} = \frac{SS_{NTr(\text{adj})}}{c + \mathrm{v} - 1};\ MS_{NTr(\text{adj})} \in \left[MS_{LTr(\text{adj})} MS_{UTr(\text{adj})}\right]$$

$$MS_{NB(\text{adj})} = \frac{SS_{NB(\text{adj})}}{b-1};\ MS_{NB(\text{adj})} \in \left[MS_{LB(\text{adj})}, MS_{UB(\text{adj})}\right]$$

$$MS_{NCheck} = \frac{SS_{NCheck}}{c-1};\ MS_{NCheck} \in [MS_{LCheck}, MS_{UCheck}]$$

$$MS_{Nnew} = \frac{SS_{Nnew}}{\mathrm{v}-1};\ MS_{Nnew} \in [MS_{Lnew}, MS_{Unew}]$$

$$MS_{N\text{new and new} \times \text{ch}} = \frac{SS_{N\text{new and new} \times \text{ch}}}{\mathrm{v}};$$

$$MS_{N\text{new and new} \times \text{ch}} \in [MS_{L\text{new and new} \times \text{ch}}, MS_{U\text{new and new} \times \text{ch}}]$$

$$MS_{N\text{new} \times \text{check}} = \frac{SS_{N\text{new} \times \text{check}}}{1}; MS_{N\text{new} \times \text{check}} \in [MS_{L\text{new} \times \text{check}}, MS_{U\text{new} \times \text{check}}]$$

$$MS_{NE} = \frac{SS_{NE}}{(b-1)(c-1)};\ MS_{NE} \in [MS_{LE}, MS_{UE}]$$

Now, each of the following statistics represents a candidate neutrosophic test statistics F_N.

$$F_{NTr(\text{adj})} = \frac{MS_{NTr(\text{adj})}}{MS_{NE}};\ F_{NTr(\text{adj})} \in \left[F_{LTr(\text{adj})}, F_{UTr(\text{adj})}\right]$$

$$F_{NB(\text{adj})} = \frac{MS_{NB(\text{adj})}}{MS_{NE}};\ F_{NB(\text{adj})} \in \left[F_{LB(\text{adj})}, F_{UB(\text{adj})}\right]$$

$$F_{NCheck} = \frac{MS_{NCheck}}{MS_{NE}};\ F_{NCheck} \in [F_{LCheck}, F_{UCheck}]$$

$$F_{Nnew} = \frac{MS_{Nnew}}{MS_{NE}}; F_{Nnew} \in [F_{Lnew}, F_{Unew}]$$

$$F_{N\text{new and new} \times \text{ch}} = \frac{MS_{N\text{new and new} \times \text{ch}}}{MS_{NE}};$$

$$F_{N\text{new and new} \times \text{ch}} \in [F_{L\text{new and new} \times \text{ch}}, F_{U\text{new and new} \times \text{ch}}]$$

$$F_{N\text{new } \times \text{ check}} = \frac{MS_{N\text{new } \times \text{ check}}}{MS_{NE}}; F_{N\text{new } \times \text{ check}} \in [F_{L\text{new } \times \text{ check}}, F_{U\text{new } \times \text{ check}}]$$

The neutrosophic test statistics F_N, is defined as

$$F_N = F_L + F_U I_{F_N}; \quad I_{F_N} \in [I_{F_L}, I_{F_U}] \tag{5}$$

where F_L is the lower bound and F_L is the upper bound. Note that F_L represents the determinate part, and $F_U I_{F_N}$ represents the indeterminate part in each proposed test.
It is obvious that the test statistics (5) reduces to the classical approach when $I_{F_N} = 0$.

4.3. *Neutrosophic decision rules*

We list down the null and the alternative hypotheses which are needed to test block, check and new treatments effects:
$H_{N0} : \alpha_{Ni} = 0 \;\; vs \;\; H_{N1} : not\ all\ \alpha_{Ni} \neq 0, i = 1.\ 2.,,,.\ b,$
$H_{N0} : \tau_{Nbj} = 0 \;\; vs \;\; H_{N1} : not\ all\ \tau_{Nbj} \neq 0, j = 1.\ 2.,,,.\ c,$
$H_{N0} : \tau_{Nlig} = 0 \;\; vs \;\; H_{N1} : not\ all\ \tau_{Nlig} \neq 0, g = 1.\ 2.,,,.\ n_{(li)}$
According to Smarandache [32], the neutrosophic decision rules at level α are:

- Accept H_{N0} for $min\{p_N{-}value\} > \alpha$.
- Reject H_{N0} for $max\{p_N{-}value\} \leq \alpha$.
- If $min\{p_N{-}value\} < \alpha < max\{p_N{-}value\}$, then there is no decissive decission. But, in this special case one could compute the chance to reject H_{N0} to be the ratio of $\frac{\alpha - min\{p_N - value\}}{max\{p_N - value\} - min\{p_N - value\}}$. Likewise, the ratio $\frac{max\{p_N - value\} - \alpha}{max\{p_N - value\} - min\{p_N - value\}}$ represents the chance of accepting H_{N0}.

As usual classical ANOVA tables, building NANOVA tables for NARCBD as in Tables 1 and 2 is also straightforward.

Table 1. NANOVA Table (A) suitable for NARCBD

Sources of variation	**Ndf**	**NSS**	**NMS**	$\boldsymbol{F_N}$**-value**
Blocks unadjusted	$b-1$	$SS_{NB(\text{unadj})}$	$\frac{SS_{NB(\text{unadj})}}{b-1}$	–
Treatments adjusted	$c+\text{v}-1$	$SS_{NTr(\text{adj})}$	$\frac{SS_{NTr(\text{adj})}}{c+\text{v}-1}$	$\frac{MS_{NTr(\text{adj})}}{MS_{NE}}$
Checks	$c-1$	SS_{NCheck}	$\frac{SS_{NCheck}}{c-1}$	$\frac{MS_{NCheck}}{MS_{NE}}$
New and New × Check	v	$SS_{N\text{new and new } \times \text{ ch}}$	$\frac{SS_{N\text{new and new } \times \text{ ch}}}{\text{v}}$	$\frac{MS_{N\text{new and new } \times \text{ ch}}}{MS_{NE}}$
Error	$(b-1)(c-1)$	SS_{NE}	$\frac{SS_{NE}}{(b-1)(c-1)}$	–
Total	$n-1$	SS_{NT}	–	–

5. **Case and simulation studies**

In this section, we exclusively examine the performance of our proposed NARCBD approach by a real data set scenario, followed by an extensive simulation study.

Table 2. NANOVA Table (B) suitable for ARCBD

Sources of variation	**Ndf**	**NSS**	**NMS**	$\boldsymbol{F_N}$**-value**
Blocks adjusted	$b-1$	$SS_{NB(\text{adj})}$	$\frac{SS_{NB(\text{adj})}}{b-1}$	$\frac{MS_{NB(\text{adj})}}{MS_{NE}}$
Treatments unadjusted	$c+\text{v}-1$	$SS_{NTr(\text{unadj})}$	$\frac{SS_{NTr(\text{unadj})}}{c+\text{v}-1}$	–
Checks	$c-1$	SS_{NCheck}	$\frac{SS_{NCheck}}{c-1}$	$\frac{MS_{NCheck}}{MS_{NE}}$
New treatments	$\text{v}-1$	SS_{Nnew}	$\frac{SS_{Nnew}}{\text{v}-1}$	$\frac{MS_{Nnew}}{MS_{NE}}$
New $\times$ Check	1	$SS_{N\text{new } \times \text{ check}}$	$\frac{SS_{N\text{new } \times \text{ check}}}{1}$	$\frac{MS_{N\text{new } \times \text{ check}}}{MS_{NE}}$
Error	$(b-1)(c-1)$	SS_{NE}	$\frac{SS_{NE}}{(b-1)(c-1)}$	–
Total	$n-1$	SS_{NT}	–	–

5.1. *Real case study*

An experiment is ran in a research farm at Isfahan University of Technology to study the breeding program of safflower plant. This study included observing neutrosophic ninety-nine cultivars of safflower genotypes using an NARCBD with three blocks. Each block has neutrosophic eight control treatments (as checks) and twenty-five neutrosophic genotypes (as new treatments).

AlAita and Talebi [7] have listed all outcomes concerning oil seed actions resulted from plants. The two consecutive NANOVA Tables 3 and 4 show the related observed value of test statistics F_N-tests for our experiment.

The NANOVA Tables 3 and 4 clarify how to report the previously proposed test statistics. As an example, in NANOVA Table 3, the neutrosophic form for treatments is $1.328 + \ 1.346I_{F_N};\ I_{F_N} \in [0,\ 0.013]$, and this in turn states that the measure of indeterminacy for treatments equals 0.013. It is noted that, due to the partitioning neutrosophic treatment effects into check and new treatments, for each effect, F_N and $p_N - values$ are computed, separately.

5.2. *Simulation study*

Under uncertainty and imprecise data framework, the performance of the proposed NARCBD is assessed in sense of test power and controlling type I error at a given nominal level of α. Two significance levels were chosen, namely $\alpha = .05$ and $\alpha = .01$. It will be shown that this new approach dominates the classical one.

The simulated number of treatments and blocks were taken just as do exist in famous references like Federer [17]. The underlying null hypothesis states that data are generated from the centered standard Gaussian distribution $\mathcal{N}(0,1)$. Above all, it is presumed that the homogeneity assumption of neutrosophic variances is verified and that all planned neutrosophic

Table 3. NANOVA Table (A) suitable for real case study

Sources of variation	**Ndf**	**NSS**	**NMS**	F_N	**Neutrosophic form** F_N	p_N**-value**
Blocks (unadj)	2	[88934.145, 88147.245]	[44467.072, 44073.623]	–	–	–
Treatments (adj)	82	[3509029.746, 3521660.933]	[42793.046, 42947.085]	[1.328, 1.346]	$1.328 + 1.346I_{F_N};\ I_{F_N} \in [0,\ 0.013]$	[0.285, 0.274]
Checks	7	[338369.950, 339847.407]	[48338.564, 48549.630]	[1.501, 1.522]	$1.501 + 1.522I_{F_N};\ I_{F_N} \in [0,\ 0.014]$	[0.245, 0.238]
New and new × check	75	[3170659.05, 3181813.526]	[42275.454, 42424.180]	[1.312, 1.330]	$1.312 + 1.330I_{F_N};\ I_{F_N} \in [0,\ 0.014]$	[0.294, 0.284]
Error	14	[450984.022, 446662.361]	[32213.144, 31904.454]	–	–	–
Total	98	[4048947.913, 4056470.539]	–	–	–	–

Table 4. NANOVA Table (B) for real case study

Sources of variation	**Ndf**	**NSS**	**NMS**	F_N	**Neutrosophic form** F_N	p_N**-value**
Blocks (adj)	2	[396001.136, 391361.045]	[198000.568, 195680.522]	[6.147, 6.133]	$6.147 - 6.133I_{F_N};\ I_{F_N} \in [0,\ 0.002]$	[0.012, 0.012]
Treatments (unadj)	82	[3201962.755, 3218447.133]	[39048.326, 39249.354]	–	–	–
Checks	7	[338369.950, 339847.407]	[48338.564, 48549.630]	[1.501, 1.522]	$1.501 + 1.522I_{F_N};\ I_{F_N} \in [0,\ 0.014]$	[0.245, 0.238]
New treatments	74	[2780946.509, 2795211.248]	[36367.407, 36558.516]	[1.130, 1.146]	$1.130 + 1.146I_{F_N};\ I_{F_N} \in [0,\ 0.014]$	[0.422, 0.409]
New × Check	1	[82646.296, 83388.478]	[82646.296, 83388.478]	[2.566, 2.614]	$2.566 + 2.614I_{F_N};\ I_{F_N} \in [0,\ 0.018]$	[0.131, 0.128]
Error	14	[450984.022, 446662.361]	[32213.144, 31904.454]	–	–	–
Total	98	[4048947.913, 4056470.539]	–	–	–	–

designs are balanced.

Recall that, under the null hypothesis, the check treatments all have zero mean. In other words, $H_{N0} : (\mu_{N1}, \mu_{N2}, \mu_{N3}, \mu_{N4}) = (0, 0, 0, 0)$.

Next, we itemize the Monte Carlo (MC) generating steps in order to compute the neutrosophic empirical type I error rates as well as the test power.

- In each repetition of the $a = 10,000$ independent repetitions:
  (1) Generate the u^{th} random sample $x_{N1}^{(u)}, x_{N2}^{(u)}, ..., x_{Nn}^{(u)}$ under the null hypothesis, where $u = 1, 2, ..., a$.
  (2) In the u^{th} sample, extract the test statistics F_{Nu}.
  (3) Next, assign the value $I_{Nu} = 1$ for every rejected H_{N0} at α, and $I_{Nu} = 0$ otherwise.
- The empirical type I error rate is estimated by the ratio of significant tests.
- The empirical test power is estimated by $\widehat{\pi}(\mu_{Nu}) = \frac{1}{a}\sum_{u=1}^{a} I_{Nu}$, where at least one of the parameters $(\mu_{N1}, \mu_{N2}, \mu_{N3}, \mu_{N4})$ is not zero.

It is also helpful to demonstrate all previous steps as diagrams showing in Figures 1 and 2.

By choosing different values of $\delta = (\mu_1, \mu_2, \mu_3, \mu_4)$ in blocks, the results listed in Tables 5 and

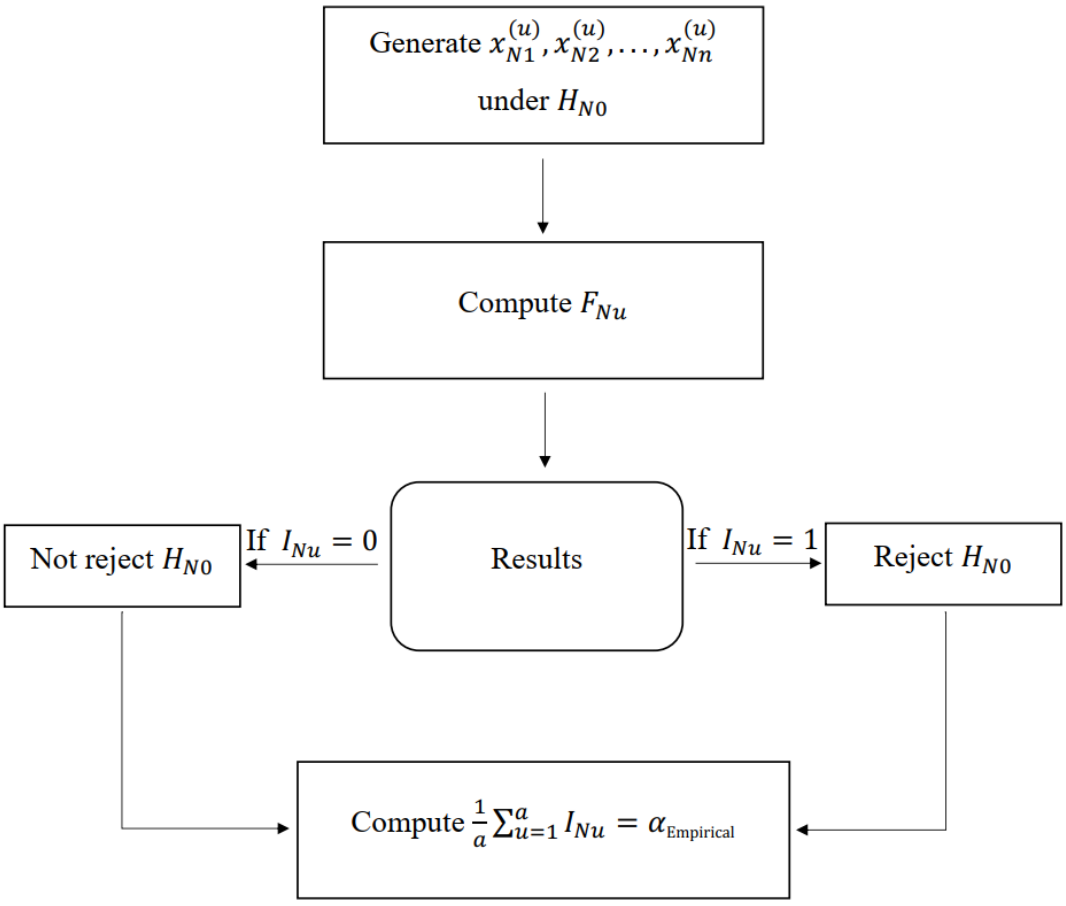

FIGURE 1. MC algorithm to find the empirical rejection rate $\alpha_{Empirical}$

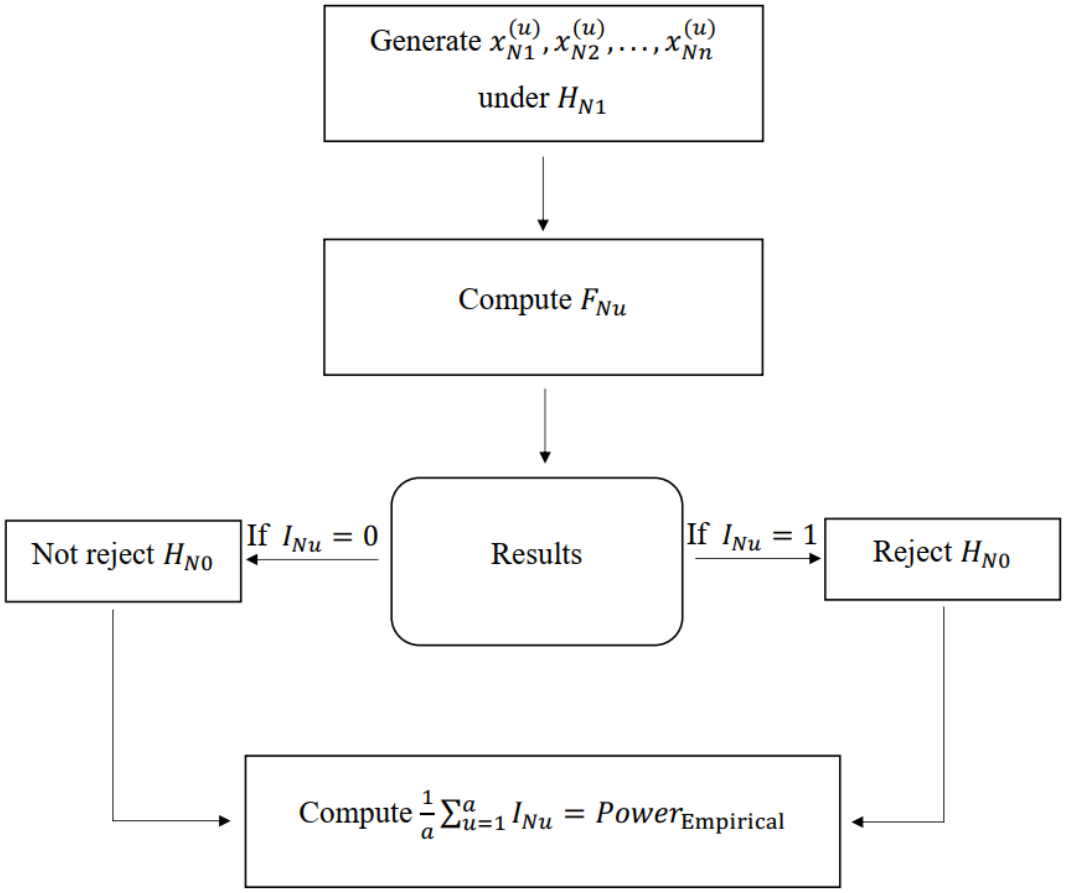

FIGURE 2. MC algorithm to find the empirical test power $Power_{Empirical}$

6 show each of empirical error rates and empirical tests power. Also, consider the following different values for δ such as, $(0,0,0,1)$, $(0,1,2,2)$, $(1,2,2,3)$, $(1,1,3,3)$, $(0,1,2,4)$, $(0,1,3,4)$, $(0,3,4,4)$, and $(0,2,4,6)$.

It is noted that the proposed NARCBD outperforms the classical design, since it controls the type I error much better and it also has a uniformly higher power. In light of these listed results, one concludes that the proposed F_N test is more informative and flexible than the existing classical F test under presence of uncertain data.

In addition, Figures 3 and 4 show that as number of treatments increases, a wide gap emerges which explicitly splits the two power curves to become far apart.

6. **Conclusions, further discussions and limitations**

In this work, we proposed an augmented randomized complete block design under NS which is suitable to analyze uncertain, indeterminate, and imprecise data. We defined an NARCBD model which provides several advantages over classical ARCBD model. It can surmount the indeterminacy in data, resulting in robust and flexible outcomes along with along with a

TABLE 5. MC NARCBD model findings assuming parameters of $(b = 3, c = 4, \mathrm{v} = 8, n = 20)$, *for means of check treatment* $\delta = (\mu_{N1}, \mu_{N2}, \mu_{N3}, \mu_{N4})$, *and different new treatments* $(\mu_{Ni} = 0, \mu_{Nj} = 1)$, $i = 1, ..., 4, j = 5, \ldots, 8$.

Test	$\boldsymbol{\alpha}$	**Mean empirical type I error**	**Mean empirical power**							
			$\boldsymbol{\delta_1 = (0,0,0,1)}$	$\boldsymbol{\delta_2 = (0,1,2,2)}$	$\boldsymbol{\delta_3 = (1,2,2,3)}$	$\boldsymbol{\delta_4 = (1,1,3,3)}$	$\boldsymbol{\delta_5 = (0,1,2,4)}$	$\boldsymbol{\delta_6 = (0,1,3,4)}$	$\boldsymbol{\delta_7 = (0,3,4,4)}$	$\boldsymbol{\delta_8 = (0,2,4,6)}$
NARCBD	0.01	[0.0093, 0.0098]	[0.0202, 0.0213]	[0.0542, 0.0632]	[0.0799, 0.0916]	[0.1138, 0.1342]	[0.1856, 0.2255]	[0.2398, 0.2856]	[0.3486, 0.4141]	[0.5914, 0.6617]
	0.05	[0.0510, 0.0517]	[0.0946, 0.1005]	[0.1985, 0.2261]	[0.2769, 0.3123]	[0.3615, 0.4091]	[0.5172, 0.5817]	[0.6046, 0.6668]	[0.7463, 0.8033]	[0.9216, 0.9504]

TABLE 6. MC NARCBD model findings assuming parameters of $(b = 6, c = 4, \mathrm{v} = 30, n = 54)$, *for means of check treatment* $\delta = (\mu_{N1}, \mu_{N2}, \mu_{N3}, \mu_{N4})$, *and different new treatments*$(\mu_{Ni} = 0, \mu_{Nj} = 1,\ \mu_{Nk} = 2)$, $i = 1, ..., 10, j = 11, \ldots, 20$, $k = 21, \ldots, 30$.

Test	$\boldsymbol{\alpha}$	**Mean empirical type I error**	**Mean empirical power**							
			$\boldsymbol{\delta_1 = (0,0,0,1)}$	$\boldsymbol{\delta_2 = (0,1,2,2)}$	$\boldsymbol{\delta_3 = (1,2,2,3)}$	$\boldsymbol{\delta_4 = (1,1,3,3)}$	$\boldsymbol{\delta_5 = (0,1,2,4)}$	$\boldsymbol{\delta_6 = (0,1,3,4)}$	$\boldsymbol{\delta_7 = (0,3,4,4)}$	$\boldsymbol{\delta_8 = (0,2,4,6)}$
NARCBD	0.01	[0.0095, 0.0108]	[0.1119, 0.1415]	[0.1441, 0.1791]	[0.2011, 0.2550]	[0.2962, 0.3619]	[0.4956, 0.5867]	[0.6029, 0.6985]	[0.7892, 0.8599]	[0.9705, 0.9874]
	0.05	[0.0478, 0.0492]	[0.3366 0.3824]	[0.4072, 0.4602]	[0.4803, 0.5505]	[0.6225, 0.6948]	[0.8093, 0.8712]	[0.8829, 0.9246]	[0.9623, 0.9807]	[0.9983, 0.9995]

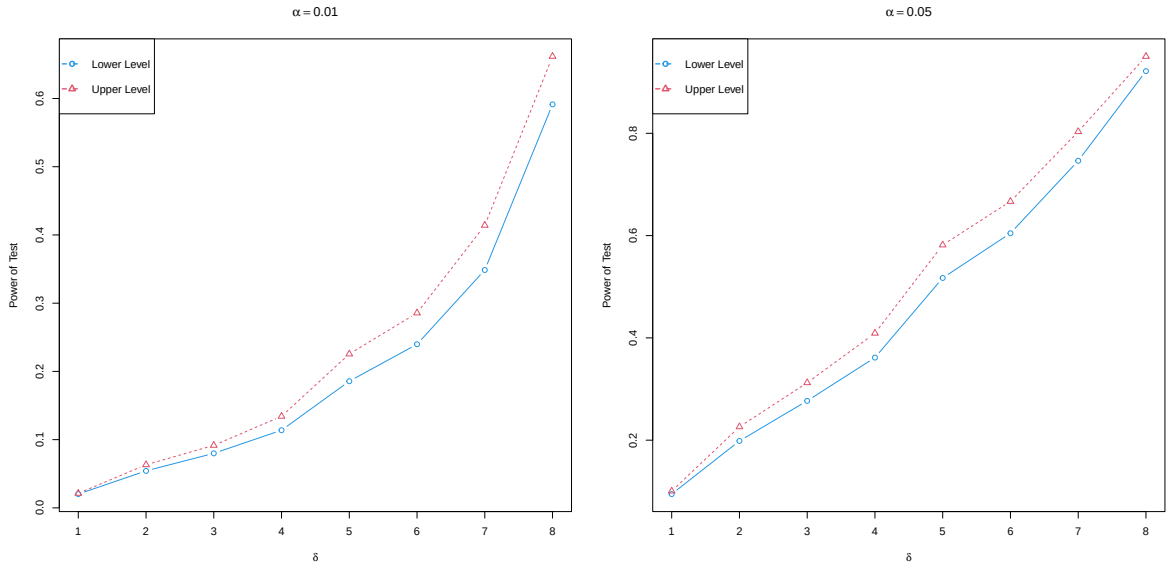

FIGURE 3. Plotting empirical power curves for NARCBD with parameters $(b = 3, c = 4, \mathrm{v} = 8, n = 20)$. The lower curve represents the classical design.

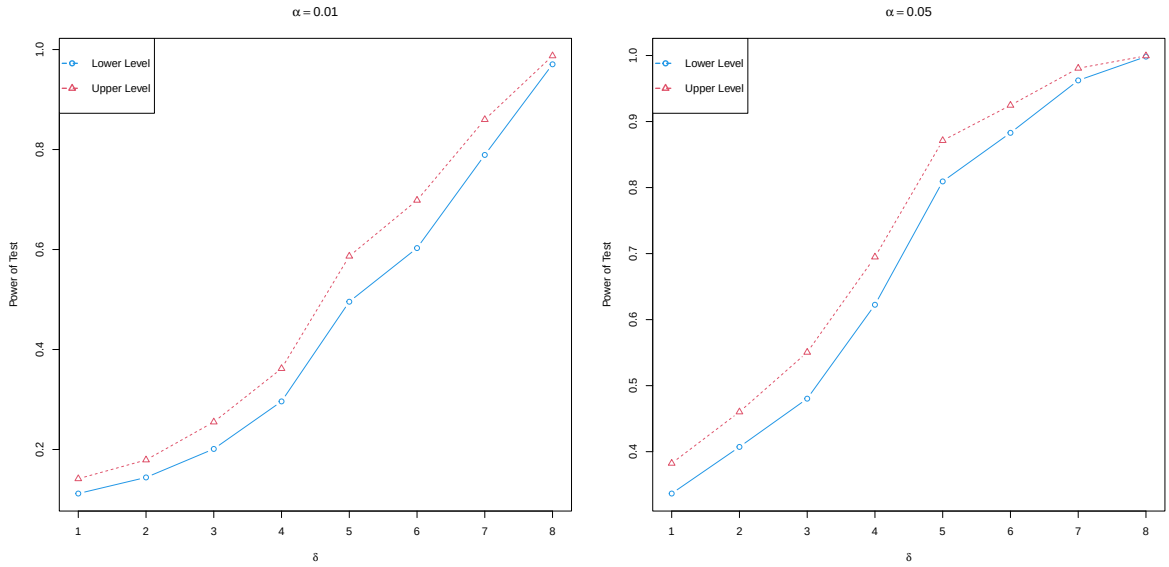

FIGURE 4. Plotting empirical power curves for NARCBD with parameters $(b = 6, c = 4, \mathrm{v} = 30, n = 54)$. The lower curve represents the classical design.

valuable increases in statistical precision, accuracy, and power. These are the basic tools in statistical inference. We believe that the NARCBD will emerge as a very useful model in different research fields, including farming, agricultural, and biological.

But the proposed NARCBD model has some limitations. It does require a larger sample size than the classical design to work perfectly, and this is of course, may lead to more time-consuming and more expensive experiments. The complex computational nature of NARCBD really requires a statistician who can give careful readings and interpretations in presence of indeterminacies.

After defining the neutrosophic hypotheses and the related decision rules, a neutrosophic test statistics was defined. All case study and simulation studies reflected the fact that the proposed method is the preferable model to work with uncertain environment.

Other extended directions

It is notably remarked that connecting the design of experiments with artificial intelligence is hardly found in specialized literatures but one may refer to a few great references, like [19, 23]. The need of developing an AI along with experimental designs comes from the fact that AI procedures can effectively reduce the repetitions and the time inquired to reach the best estimated parameters.

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Teaching Effectiveness Analysis of Educational Technology in Vocational Colleges: SuperHyperSoft Set Integrated with Decision-Making Approach for Increased Effectiveness

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Abstract: The integration of educational technology in vocational colleges has revolutionized teaching methodologies, improving instructional quality and student engagement. However, ensuring its effectiveness requires a structured assessment framework. This study evaluates teaching effectiveness by analyzing key criteria such as instructional quality, student engagement, technology integration, assessment effectiveness, skill development, and industry adaptability. Using a multi-criteria decision-making (MCDM) approach, this study computes the criteria weights by the LBWA and the MOORA methodology to rank the alternatives. Six criteria and seven alternatives are used in this study. We use the SuperHyperSoft Set to deal with criteria and sub criteria. This study proposes four pools of HyperSoft sets. Then we ranked the alternatives in each pool. The results show the ranks of alternatives in each pool are stable.

Keywords: Decision Making Methodology; SuperHyperSoft Set; Teaching Effectiveness; Educational Technology.

1. Introduction

In the rapidly evolving educational landscape, educational technology plays a crucial role in enhancing teaching effectiveness, particularly in vocational colleges, where practical skills and industry alignment are essential. The integration of digital learning platforms, artificial intelligence (AI), virtual simulations, and interactive tools has transformed traditional teaching methods, offering students more engaging and adaptive learning experiences[1], [2].

However, while these technologies provide new opportunities for improving education, their effectiveness must be systematically assessed to ensure they align with the desired learning outcomes. Evaluating the effective teaching of educational technology in vocational colleges requires a comprehensive framework that considers instructional quality, student engagement, industry relevance, and overall learning efficiency.

One of the fundamental aspects of teaching effectiveness in vocational education is how digital tools support practical skill development and hands-on learning experiences. Unlike traditional academic programs, vocational education is industry-oriented, requiring students to gain competencies that are directly applicable in real-world work environments. Educational technology, such as simulations, gamified learning experiences, and AI-driven assessments, can enhance the learning process by offering interactive and adaptive content[3], [4]. However, if not implemented effectively, these technologies can lead to disengagement or fail to provide the necessary depth of learning required for vocational training.

Another key factor in assessing educational technology effectiveness is student engagement and learning outcomes. Modern educational tools claim to offer personalized learning paths, real-time feedback, and collaborative learning experiences, but their actual impact on student motivation and knowledge retention needs critical evaluation. Blended learning models, which combine online and face-to-face instruction, have gained popularity in vocational education, but their success depends on student adaptability, accessibility of digital resources, and instructor proficiency in using technology effectively. Therefore, assessing the role of technology in fostering active student participation, skill acquisition, and competency-based learning is vital.

Furthermore, the alignment of educational technology with industry demands is a crucial dimension of assessment. Vocational education serves as a direct pipeline to employment, and the tools used in teaching must prepare students for real-world technological advancements and workplace requirements. Many vocational institutions adopt learning management systems (LMS), online certification programs, and industry-specific digital tools, but a mismatch between curriculum design and industry expectations can diminish the value of these technologies[5], [6]. Evaluating whether educational technology effectively bridges the gap between academia and industry is essential for ensuring that vocational graduates possess the skills needed in their respective fields.

Given these complexities, a structured, multi-criteria decision-making (MCDM) approach is necessary to assess the effectiveness of educational technology in vocational colleges. This research proposes an evaluation framework based on key performance indicators such as instructional quality, student engagement, technology integration, assessment effectiveness, skill development, and adaptability to industry needs[7], [8]. By systematically analyzing these factors, stakeholders, including educators, administrators, and policymakers, can make informed decisions about which technologies enhance learning experiences and which require further improvement[9], [10]. This assessment will contribute to the ongoing efforts to optimize vocational education and ensure its alignment with modern workforce demands.

The study of finding and selecting options in accordance with one's own beliefs and preferences is known as decision making. When a decision is made, it means that there are options to think about. In this situation, not only are as many options as possible found, but the best option is also selected to satisfy the decision maker's aims, objectives, preferences, and values. As a result, every

decision-making process results in a conclusion. Making decisions and addressing problems are critical to life and business skills. Making decisions is a common part of problem resolution, and it's crucial for management and leadership[11], [12].

Some personalities are better at making decisions than others, thus such individuals should concentrate more on making better judgments. Individuals who struggle with decision-making frequently can make good judgments, but they must then act more decisively on those judgments. Effective decision-making necessitates a variety of abilities, including innovative ideation and option discovery, judgmental clarity, decision firmness, and efficient execution[13], [14].

Simple, methodical, and logical methods or mathematical tools that can consider a wide range of selection criteria and candidate alternatives are required to support and direct the decision makers. Finding the right selection criteria and arriving at the optimal choice while considering the demands of the present moment are the goals of every choosing process. While many MCDM techniques are currently available to address various evaluation and selection issues, this study aims to investigate the suitability of a relatively new MCDM technique, namely the multi-objective optimization based on ratio analysis (MOORA) method, to ranking the alternatives[15], [16].

2. The LBWA with MOORA Method

This section shows the steps of the LBWA method to compute the criteria weights and the MOORA Method to rank the alternatives. These methods are used with the SuperHyperSoft set to deal with criteria and sub criteria.

The steps of the LBWA are organized as follows:

Decision makers can identify the highest importance criterion.

Then the criteria are identified at different levels.

Level 1 k_1: The criteria are greater twice importance or equal to other criteria.

Level 2 k_2: The criteria are twice or three times lower than the highest importance criterion.

Level 3 k_3: The criteria are y or y+1 times less than the highest importance criterion are put in the level 3.

$$K = K_1 \cup K_2 \dots \cap K_y \tag{1}$$

Each attribute are assigned into integral value.

$$f = \max\{|K_1|, |K_2|, \dots, |K_y|\} \tag{2}$$

$$f_0 < f \tag{3}$$

We compute the influence function as:

$$f(C_{ip}) = \frac{f_0}{i(f_0)+I_{ip}} \tag{4}$$

Where i presents the number of levels.

Determine the criteria weights.

$$w_j = \frac{1}{1+f(C_j)+\cdots+f(C_n)} \tag{5}$$

The act of concurrently maximizing two or more competing qualities (objectives) under certain restrictions is called multi-objective optimization (or programming), often referred to as multi-criteria or multi-attribute optimization. Product and process design, finance, the oil and gas industry, the manufacturing sector, automobile design, and any other area where the best course of action must be chosen in the face of trade-offs between two or more competing objectives are examples of multi-objective optimization problems. Typical examples of Mult objective optimization issues include optimizing performance and limiting fuel consumption of a vehicle, maximizing profit and lowering the cost of a product, and minimizing weight while maximizing the strength of a certain technical component[17], [18].

Steps of the MOORA method are shown as:

Build the decision matrix

Experts and decision makers are evaluated the set of different alternatives with respect to different criteria such as:

$$U = \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{m1} & \cdots & u_{mn} \end{bmatrix} \tag{6}$$

Next, a system of ratios is created where each alternative's performance on an attribute is contrasted with a denominator that represents all the options pertaining to that attribute. After examining several ratio systems, including the total ratio, Schärlig ratio, Weitendorf ratio, Jüttler ratio, Stopp ratio, Körth ratio, and others, Brauers et al. concluded that the optimum option for this denominator is the square root of the sum of squares of each possibility for each characteristic. The following is an expression for this ratio:

$$q_{ij} = u_{ij} / \sqrt{\sum_{i=1}^{m} q_{ij}^2}\,; i = 1,2,\dots,m; j = 1,2,\dots,n; \tag{7}$$

In the context of multi-objective optimization, these normalized performances are deducted for nonbeneficial qualities and added for beneficial attributes. The optimization issue then turns into:

$$y_i = \sum_{j=1}^{g} q_{ij} - \sum_{j=g+1}^{n} q_{ij} \tag{8}$$

where y_i is the normalized assessment value of ith option about all the qualities, g is the number of attributes to be maximized, and (n−g) is the number of attributes to be reduced. It is sometimes noted that certain qualities are more significant than others. A property might be increased by its matching weight (significance coefficient) to increase its importance. After accounting for these attribute weights:

$$y_i = \sum_{j=1}^{g} w_j q_{ij} - \sum_{j=g+1}^{n} w_j q_{ij} \quad (10)$$

Depending on the sums of the decision matrix's maxima (useful characteristics) and minima (non-beneficial attributes), the y_i value may be positive or negative. The ultimate choice is shown by an ordinal ranking of y_i. As a result, the poorest option has the lowest yi value, while the greatest option has the highest.

SuperHyperSoft Set

Smarandache presented the SuperHyperSoft set which is an extension of the HyperSoft set. SuperHyperSoft set includes different HyperSoft sets. It is used in this study to show the relations between the different criteria and sub criteria[19], [20].

Let the U be a universe of discourse and the power set of U is P(U). Different criteria are Q_1, Q_2, Q_3 and the alternatives are evaluated based on these criteria.

The power set of these criteria can be defined as $P(Q_1), P(Q_2), P(Q_3)$.

The SuperHyperSoft set can be defined as:

$P(Q_1) \times P(Q_2) \times P(Q_3) \rightarrow P(U)$

3. An application

This section shows the implementation of the proposed approach to show the criteria weights and the best alternative based on a set of criteria. Four experts are evaluated the criteria and alternatives. These experts are used the scale between 0.1 and 0.9. This study uses six criteria and seven alternatives. The criteria of this study are: Student Engagement (Active, Passive), Instructional Quality (High, Low), Assessment Effectiveness (Comprehensive, Standard, Limited), Skill Development (Strong, Moderate, Weak), Adaptability to Industry Needs (High, Medium, Low), Technology Integration (Advanced, Sufficient, Minimal). The seven alternatives of teaching effectiveness.

Decision makers can identify the highest importance criterion.

Then the criteria are identified at different levels. Three levels are defined in this study.

Then we assign each attribute into integral value using Eq. (2).

Then we compute the influence function using Eq. (4).

Then we determine the criteria weights using Eq. (5) as shown in Fig 1.

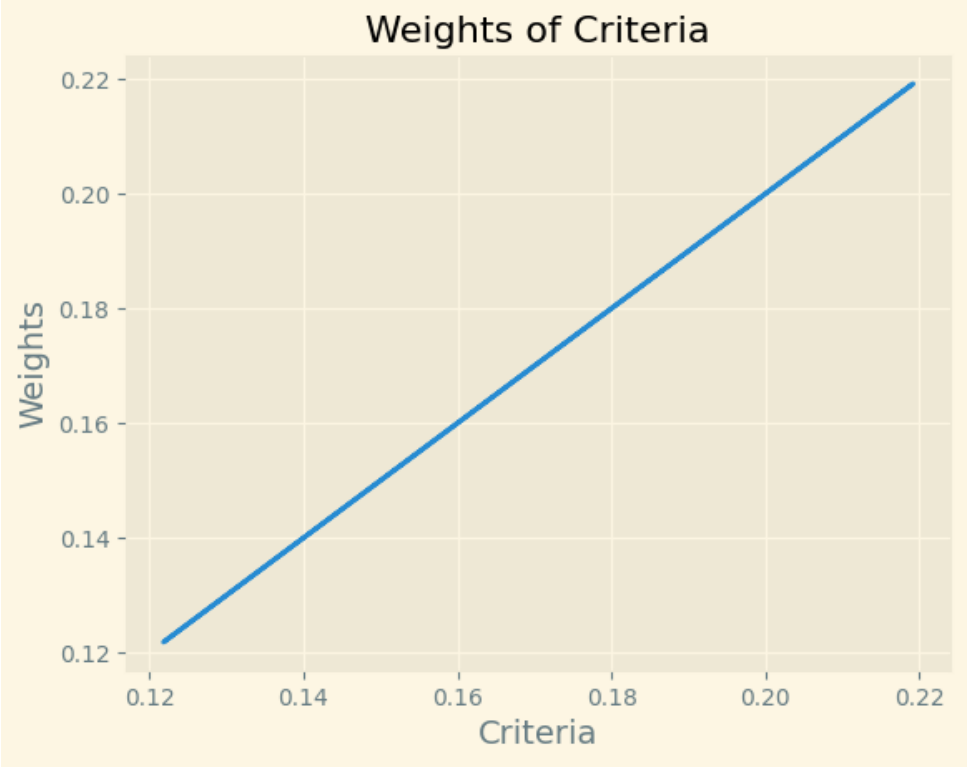

Fig 1. The criteria weights.

In the ranking of the alternatives, we use the concept of the SuperHyperSoft set to deal with the criteria and sub criteria. We use the sub criteria such as: {Active, Passive}, {High, Low}, {Comprehensive}, {Strong}, {High}, {Advanced}. Then we propose four pools.

Pool 1. {Active}, {High}, {Comprehensive}, {Strong}, {High}, {Advanced}

Pool 2. {Active,}, {Low}, {Comprehensive}, {Strong}, {High}, {Advanced}

Pool 3. {Passive}, {High}, {Comprehensive}, {Strong}, {High}, {Advanced}

Pool 4. {Passive}, {Low}, {Comprehensive}, {Strong}, {High}, {Advanced}

Then we apply the MOORA method in each pool and we rank the alternatives in each pool.

Pool 1

We build the decision matrix using Eq. (6), then we combine the decision matrix into a single matrix.

Eq. (7) is used to normalize the decision matrix as shown in Fig 2.

Then we obtain the weighted normalized decision matrix as shown in Fig 3.

Then we compute the value of y_i and then we rank the alternatives.

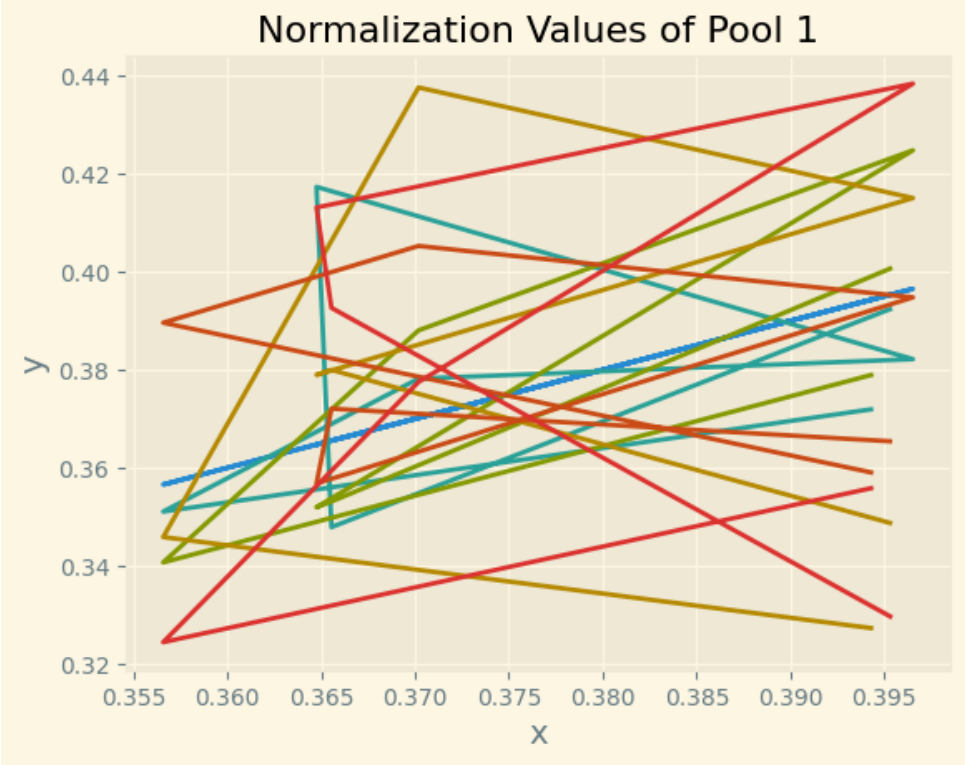

Fig 2. The normalization values.

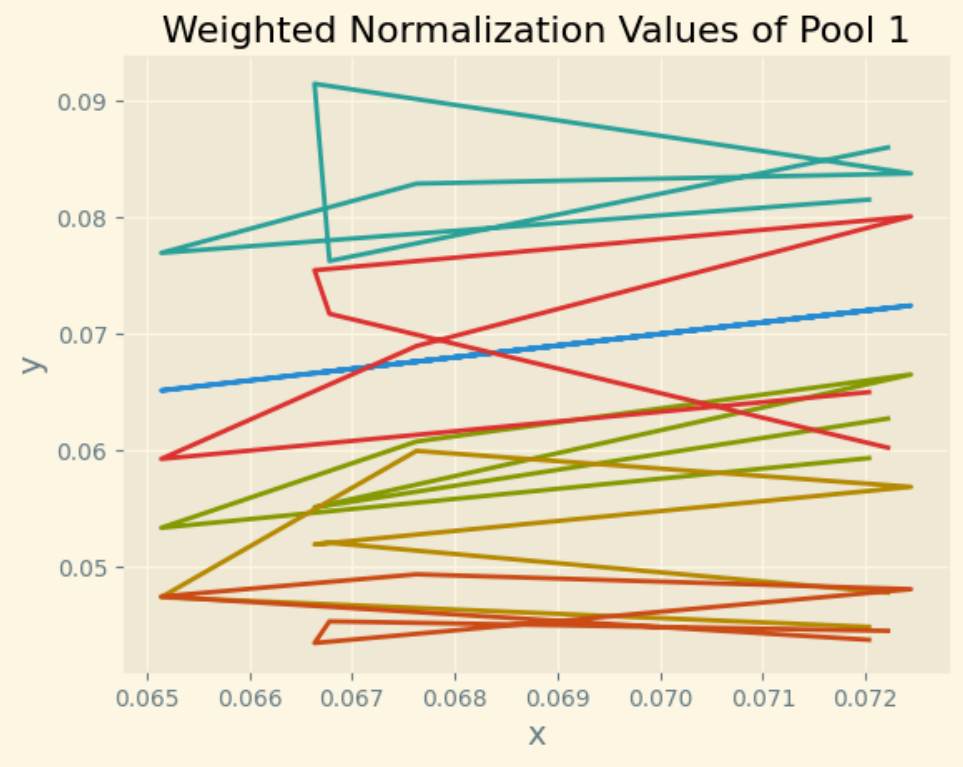

Fig 3. The weighted normalized decision matrix.

Pool 2

We normalize the decision matrix as shown in Fig 4.

Then we obtain the weighted normalized decision matrix as shown in Fig 5.

Then we compute the value of y_i and then we rank the alternatives.

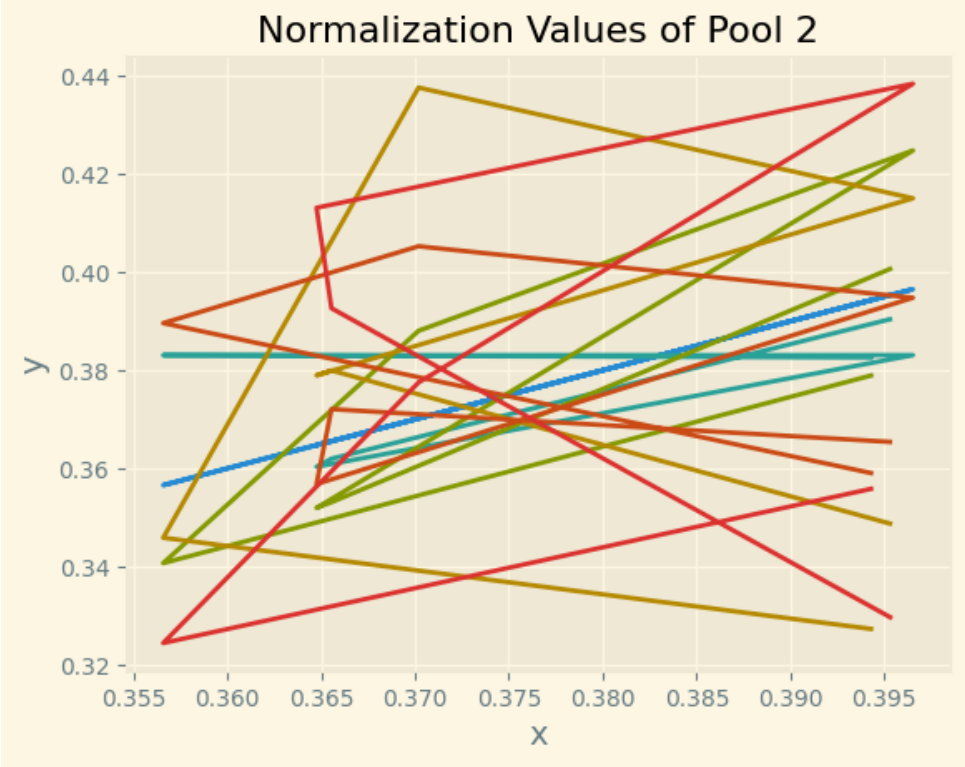

Fig 4. The normalization values.

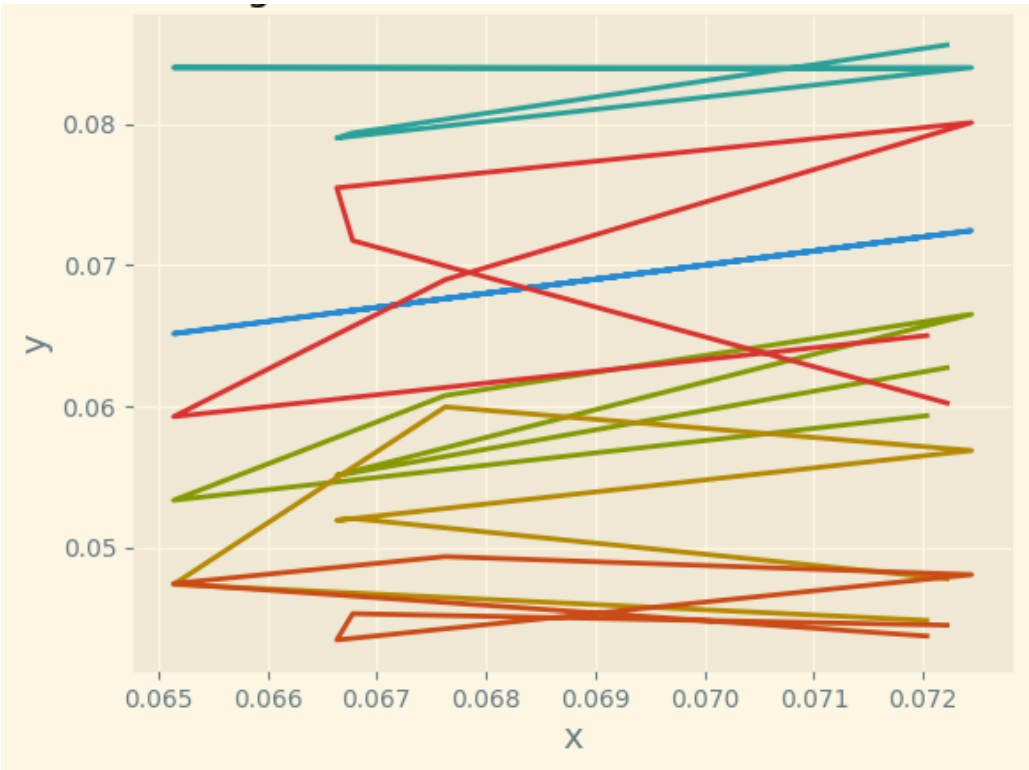

Fig 5. The weighted normalized decision matrix.

Pool 3

We normalize the decision matrix as shown in Fig 6.

Then we obtain the weighted normalized decision matrix as shown in Fig 7.

Then we compute the value of y_i and then we rank the alternatives.

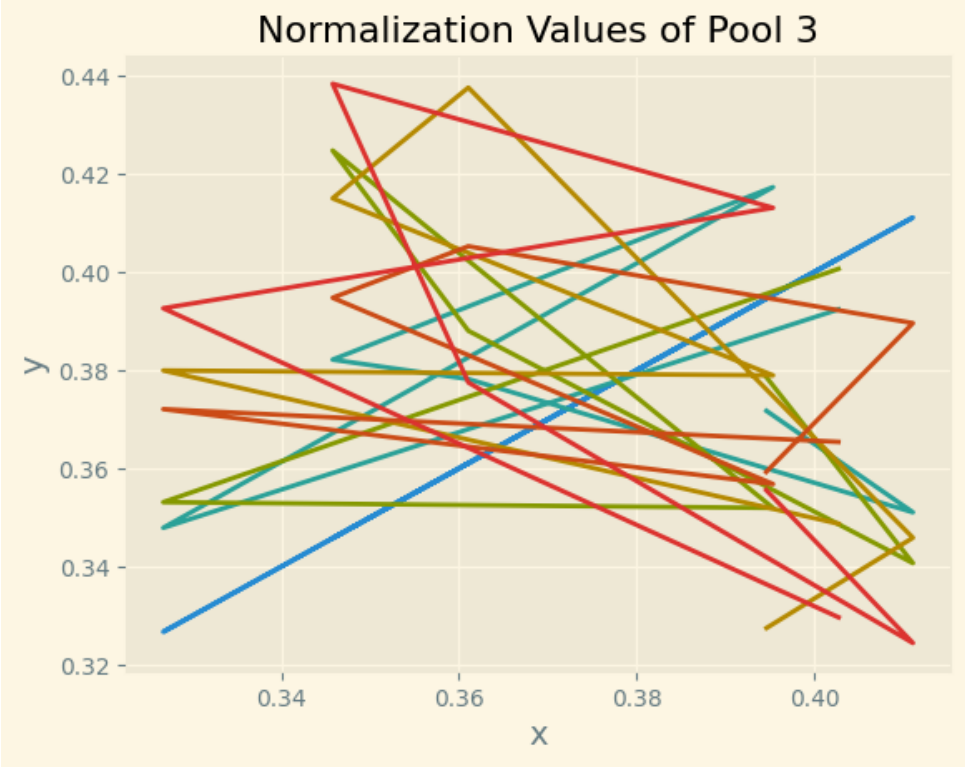

Fig 6. The normalization values.

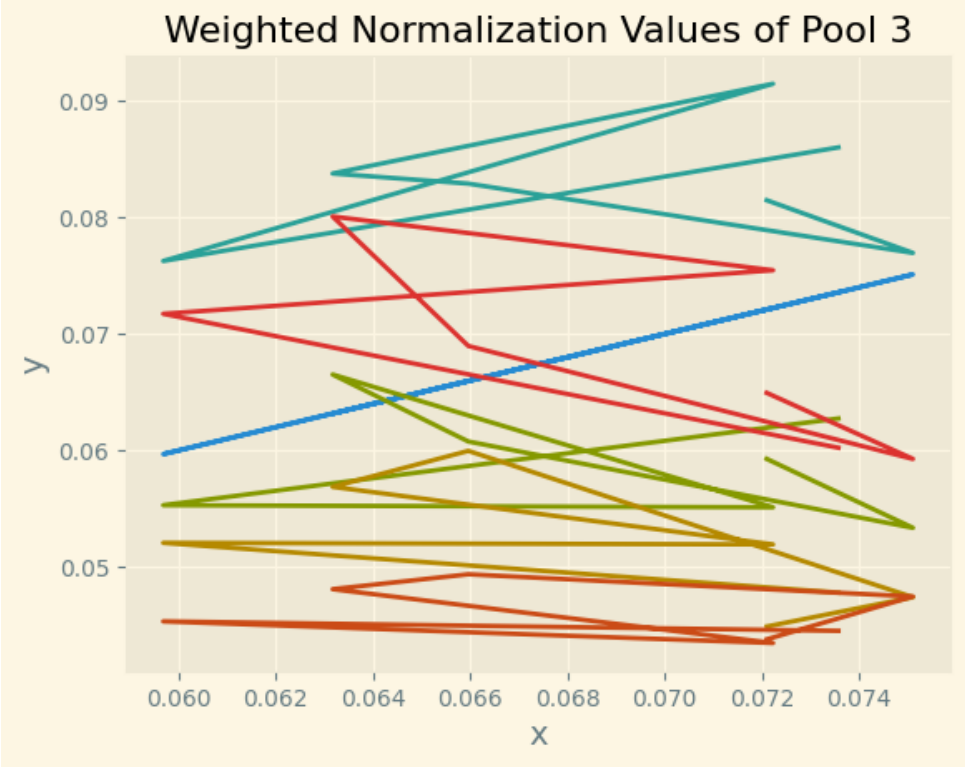

Fig 7. The weighted normalized decision matrix.

Pool 4

We normalize the decision matrix as shown in Fig 8.

Then we obtain the weighted normalized decision matrix as shown in Fig 9.

Then we compute the value of y_i and then we rank the alternatives.

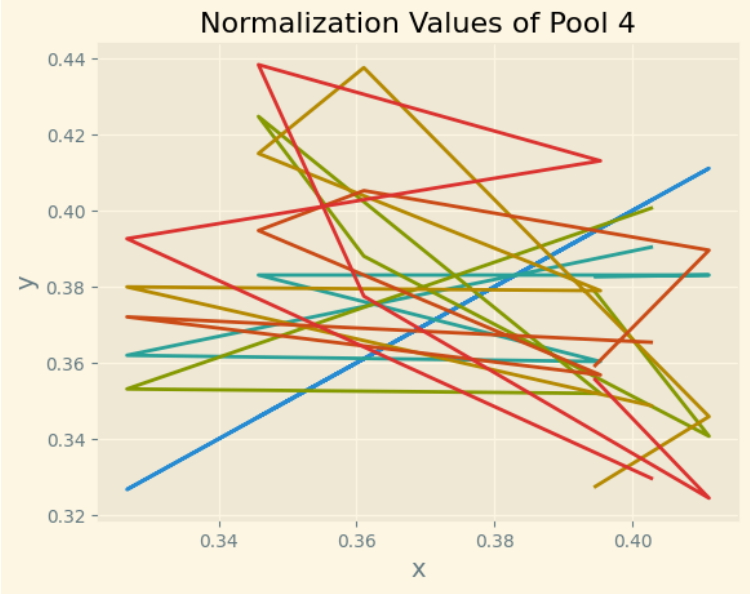

Fig 8. The normalization values.

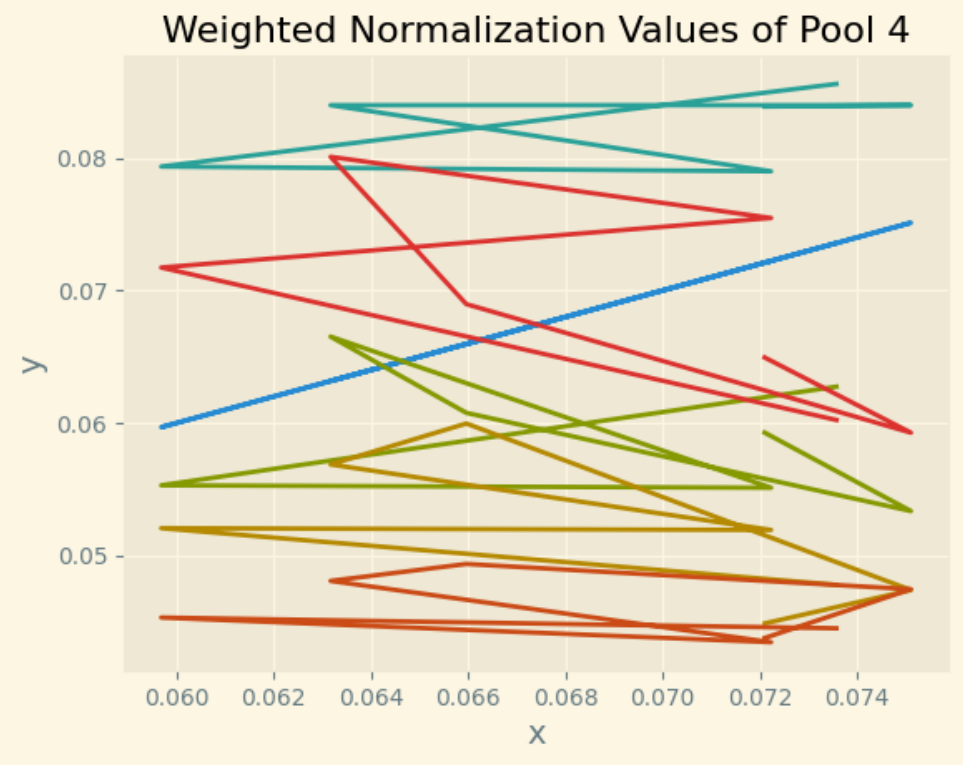

Fig 9. The weighted normalized decision matrix.

Then we rank the alternatives in each pool as shown in Fig 10. Then we show alternative 4 is the best and alternative 6 is the worst.

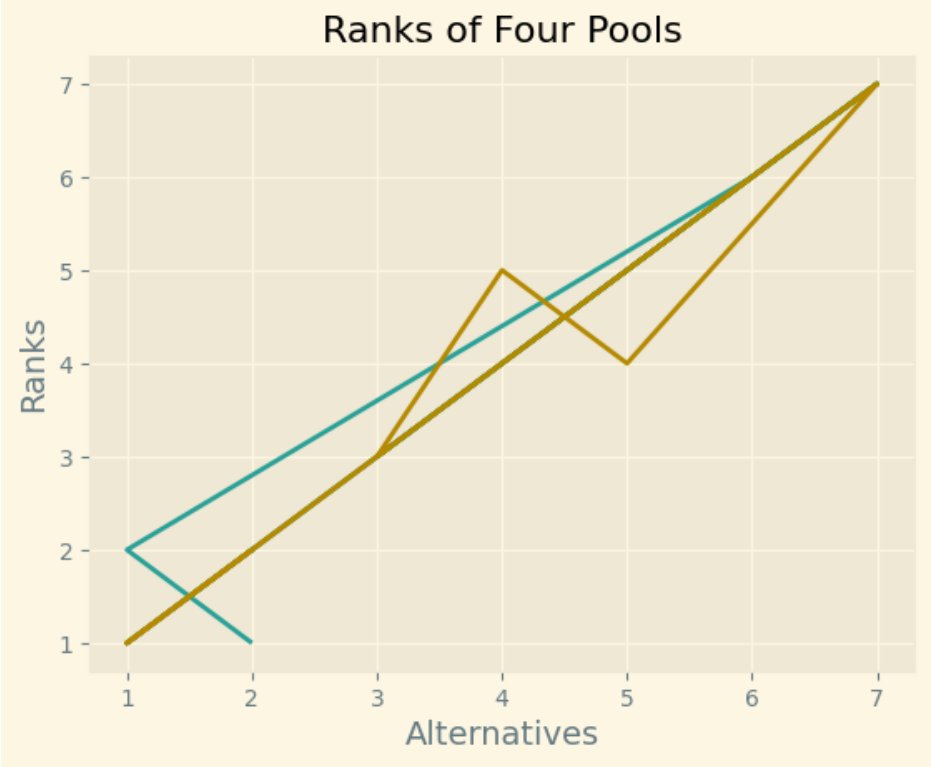

Fig 10. Different ranks of 4 pools.

4. Conclusions and Implications

This study used two MCDM methods to compute the criteria weights by the LBWA and MOORA method to rank the alternatives. We used the SuperHyperSoft set to deal with criteria and sub criteria. Six criteria and seven alternatives are used in this study. We proposed four pools as four HyperSoft sets to show the different sub criteria values. The results show alternative 4 is the best and alternative 6 is the worst.

The assessment of teaching effectiveness in educational technology within vocational colleges highlights the significant impact of digital learning tools on student engagement, skill acquisition, and instructional quality. However, effectiveness varies based on instructor competency, student adaptability, and technological infrastructure. Institutions that provide adequate teacher training, robust digital support, and industry-aligned curriculum designs tend to achieve better learning outcomes.

A key insight from this research is that merely integrating technology into vocational education does not guarantee improved teaching effectiveness. The quality of implementation, accessibility of resources, and engagement strategies play a crucial role in determining whether educational technology leads to better learning experiences. The findings suggest that blended learning models, real-time assessment tools, and AI-driven personalized learning significantly contribute to student success when properly managed.

Moreover, the connection between digital learning tools and industry needs must be continually evaluated. As vocational education directly prepares students for real-world job markets, institutions must ensure that technological advancements in education align with evolving industry requirements. The study suggests that stronger collaborations between vocational colleges and industry stakeholders can enhance curriculum relevance and ensure that students gain practical, in-demand skills.

While educational technology presents significant opportunities for enhancing vocational education, a structured evaluation process is essential to measure its effectiveness. By adopting a multi-criteria approach, vocational institutions can optimize teaching strategies, refine technology integration, and ultimately improve student learning outcomes. Future research should explore longitudinal assessments of digital learning tools to ensure their sustained effectiveness in vocational education.

Acknowledgment

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A Natural Language Processing Environment for Rule-Based Decision Making with Neutrosophic Logic to Manage Uncertainty and Ambiguity

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Abstract: So far, Salama has shown that the NLP system can be a rule-based system that can apply neutrosophical reasoning to model ambiguity and uncertainty in human language. Salama allows you to reason more than just right/wrong as traditional systems do, adding levels of right, wrong, and uncertainty. Salama follows the structural composition of words based on finite state models such as TWOL and CG2 while using constrained grammars to handle disambiguation and context-based ambiguity to avoid ambiguity. The system also handles words that are not in the vocabulary. This is a challenging scenario where rule-based approaches often have strengths and weaknesses. We investigate how neutrosophical reasoning can enhance the accuracy and reliability of NLP systems by evaluating their performance on tasks involving subjective information, incomplete information, and complex language features through extensive experiments.

Keywords: Neutrosophic Logic, Rule-based NLP, Natural Language Processing, Uncertainty Handling, Ambiguity Resolution, Constraint Grammar, Finite-State Methods, Out-of-Vocabulary Words, Linguistic Rules

1. Introduction

NLP or Natural Language Processing focuses on teaching a computer to understand, interpret, and produce human language. However, given their ambiguity and uncertainty, this can be difficult to

do with natural language. While it can be nice to have clarity and control, traditional rule-based systems used for NLP often fail while dealing with such complexities. Here we ground the whole third part of the paper, Salama (namely the α-based approach), an NLP approach by merging rule-based techniques with neutrosophic logic to better cope with these problems. Salama employs neutrosophic logic—not binary (true/false) or even fuzzy like conventional systems—, which can express the myriad shades of truth, uncertainty, and falsehood found in human language interpretation, giving it a distinct advantage in multilingual contexts. Drawing from existing methodologies such as TWOL and CG2, Salama adopts techniques such as finite-state analysis for decomposition of words and constraint grammar for resolving ambiguous semantic representations. It also has mechanisms for dealing with out-of-vocabulary words. This paper aims to examine the performance of Salama in handling the uncertainty and intricacy presented in natural language and the potential advantages and disadvantages of merging neutrosophic logic with rule-based frameworks.

2. Research Objectives and Questions:

Main Goal: To evaluate Salama's performance, a rule-based natural language processing (NLP) system using neutrosophic logic, in managing the uncertainty and ambivalence in language processing.

Key Research Questions:

Dealing with Out-Of-Vocabulary (OOV) Words: How does Salama compare with rule based systems and modern day NLP methods when handling out of vocabulary words?

How does Salama do so well with these unknown words?

Impact of Neutrosophic Logic: How does neutrosophic logic affect Salama's performance and stability undertaking various NLP tasks, such as part-of-speech tagging, syntactic parsing, and semantic analysis?

This paper proposes that Salama's interpretation of unpredictable or fuzzy language will be enhanced by the use of neutrosophic logic.

Comparison with Other NLP Systems:

How does Salama compare with like classical or fuzzy based rule NLP systems?

How does it measure up against other advanced statistical NLP models (e.g., deep learning-based systems) both in terms of accuracy and efficiency?

Limitations and Future Research

The limitations of Salama include its computational complexity, scalability, and reliance on predefined rules. Where do you see areas to improve in the future, like adding machine learning techniques, including more languages, and improving its capabilities on more noisy, real-world data?

3. Literature Review

This section gives a summary of rule-based systems in NLP, their historical development, as well as their strengths, weaknesses and applications. They have a long and rich history of application in Natural Language Processing (NLP), including from early days of artificial intelligence itself. These systems are dependent on explicitly defined linguistic rules to analyze and process the text [2]. Noam Chomsky's Transformational Grammar revolutionized linguistics in the mid-20th century. [3] The property of the language of its own that was discovered in the 1950s by Noam Chomsky, one of the most influential intellectuals of the 20th century. [4] Although not explicitly applied to early NLP systems, its impact on later rule-based systems cannot be underestimated [5]. Filling in the gaps left by Chomsky's work, unification grammars were introduced in the 1980 s. richer other frameworks proposed achieved semantics about linguistics phenomena but in less coercive ways, with a better computational tractability other than phrase structure grammars like GPSG, Lexical Functional Grammar (LFG) among other frameworks. These became the building blocks for many rule-based NLP systems.

- **Weakness of Rule-Based Approaches**

The following are some of the major pros of rule-based systems:

Transparency and interpretability: Rule-based systems are more interpretable than complex statistical or machine learning models. The explanations based on explicit rules help humans

understand how to make sense of the system [6]. Control and Precision: With precise rule crafting, developers can attain high precision in certain domains or for specific linguistic phenomena. [8] Domain-Specific Knowledge Representation: For domains with precise knowledge or control rules, rule-based systems can effectively encapsulate such knowledge for tasks such as information extraction from specialized texts. [9]

- **Rule-based systems also have limitations, however:**

Dependence on Expert Knowledge: Developing and maintaining a detailed set of language-specific rules needs substantial knowledge of linguistics and the language to be processed. [10] This is often tedious and costly. Difficulty with Ambiguity: Natural language often has an inherent ambiguity, and rule-based systems might struggle to interpret a given sentence or phrase in multiple ways. [11]. **Limited scalability:** Reusing rule-based systems for new domains or languages is often complicated, needing a lot of time and cost to modify the existing rule set. [12]

Limited Generalization: Rule-based approaches are limited in their generalization to new scenarios that deviates from established rules. [13]

- **Illustrative Example and Applications of Existing Rule-Based Systems**

In spite of their limitations, rule-based systems are still valuable for enriching data in NLP. A few (some notable) applications include:

Rule-based systems: Many chatbots rely on rule-based systems to manage simple engagements and even provide basic conversational abilities. [14] .Using rule-based systems for information extraction is still used, especially in well-defined domains like bioinformatics where you have to extract names, dates, and other specific information from text [15]. With the text classification rule-based systems can utilize a set of predefined rules to categorize the text by matching it with specific keywords [16]. Natural Language Understanding (NLU): In NLU tasks, we used rules-based systems to understand the underlying semantics and intentions of the user's utterances in dialogue systems [17].

- **Neutrosophic Logic: A Foundation for Managing Uncertainty**

  In 1995, Florentin Smarandache proposed Neutrosophic logic [18], which is a rich extension of the classical logic. It is meant to be a formal framework for handling uncertainty and imprecision, which includes three independent components [18]:

Truth (T): A measure of how true a proposition is.

Indeterminacy (I): The extent to which a proposition is both true and not true.

Falsity (F): The extent to which a proposition is false.

These components are expressed in the non-standard unit interval]-0, 1+ [as real numbers, meaning they can take values beyond 0 and 1. Such an abstraction allows neutrosophic logic to represent different degrees of the truth, indeterminacy, and falsity (from true and false), and makes it adaptive for modeling complex and less certain aspects.

Neutrosophic Logic Applications

The domains in which neutrosophic logic has been utilized are as follows:

Decision-making: It could be used to model complex decision scenarios with multiple conflicting criteria and uncertainties [19]. It has been utilized in processing images, segments, and imparts enhancement in images [20]

Medical diagnosis: It helps diagnose diseases by considering various uncertain factors as well as expert opinions [21]

Information retrieval using neutrosophic logic: Neutrosophic logic can be used to improve the accuracy and relevance of information retrieval systems [22].

Artificial intelligence: It has been used for different AI applications such as mapping of the data, classification of the data and system of decision support [23].

Neutrosophic Logic and NLP

Work in Natural Language Processing (NLP) has to wrestle with the ambiguity and uncertainty in human language. A word or phrase might have many meanings, with the meaning also being contextual. These issues can be resolved by using Neutrosophic logic in the following ways:

This can be the different possible meanings of a word or phrase, and relate to how true, indeterminate, or false it is [24]. Dealing with ambiguity: Neutrosophic logic can formalize the indeterminate and personal nature of verbal statements. Modeling uncertainty: It can model the uncertainty in words for different NLP [25] tasks like word sense disambiguation, sentiment analysis, machine translation.

Previous Studies of Neutrosophic Logic in NLP

This has been an emerging field but there is increasing literature on the use of neutrosophic logic in NLP: The neutrosophic logic for sentiment analysis: neutrosophic logic has been studied to model sentiment expressed in text [26]. Sentiment is a term used in linguistics and in natural language, processing that describes the mood in which the text has been written. As a result, sentiment analysis is the field studying how to classify text based on the degree of positive, negative, and neutral sentiment. Neutrosophic logic has been used to disambiguate words based on their meanings and context, as well as the associated degrees of truth, indeterminacy, and falsity. Information retrieval: Neutrosophic logic has been used to enhance the precision and relevance of information retrieval systems by accounting for the relevance degree of documents to a query [27, 28, 29].

4. Related Work

Hidden Markov Models (HMM): A common approach used for Speech recognition and Part of Speech (POS) tagging, Hidden Markov models relate a sequence of multisensory observations to one or more hidden states [30]. They train on it, where they give us the probabilites of it and essentially how to decode it. Conditional Random Fields (CRFs): CRFs are a type of graphical model with an output sequence property that provides the full input sequence at prediction time instead of just the previous output, this allows CRFs to surpass HMMs on the same tasks (e.g., named entity recognition) [31]. Bayesian Networks: Allow to express reasoning under uncertainty in natural flexible way, graphical model: The model is a directed acyclic graph (a representation of the joint distribution of the random variables) it permit to generalize on different tasks of natural languages ex:[23]. Natural language is often ambiguous and vague making it possible for the fuzzy logic to

adjust between two values. Fuzzy logic has many applications and one of which is sentiment analysis [33], which is referred to people's opinions or feelings. Statistical learning based regularization of knowledge systems: Over a data set fine-tune a set of rules within a rule-based system via a statistical learning [34, 35, 36]. Sub word modeling:: BPE [36], word piece [37] are some algorithms which split words into byte level units on the basis of surrounding context for missing words. Character level: A character-level able model can help to overcome aspects of the OOV problem, specifically in morphologically rich languages [38]. If known word in a semantic relations, and what it means, can inference a meaning of an OOV word on the base of this, with defined relationships with known words, this is, therefore, a very clearly way how to think. Out of vocabulary (OOV), word detection: The continuous word vector embedding can be clustered by unsupervised learning algorithms and dimension reduction techniques to obtain the semantically similar word groups in the corpus [39].

5. Methodology

5.1. System Description

- **Architecture:**

An overview of Salama architecture components

For example:

Input Module: which reads text input, initial tokenization & OOV modules if needed.

Morphological Analysis Module: This module performs morphological analysis (stemmer, lemmatiser) using finite-state methods.

Syntax Module: A syntax parser based on constraint grammar formalism using neutrosophic logic to deal with ambiguity.

Semantic Analysis Module: (If relevant) Executes semantic analysis functions like semantic role labeling or sentiment analysis, using neutrosophic logic to both use and ascertain semantic contextual details.

Output Module: This module creates the final output, which can be a parsed tree, semantic representation, or some other desired output.

- **Neutrosophic Logic Integration:**
    - Representation: Explain how neutrosophic logic is used to represent linguistic information within Salama.
        - For example:
            - Representing the degree of truth, indeterminacy, and falsity associated with different word senses.
            - Assigning truth-values to linguistic rules and their applicability.
            - Modeling the uncertainty in the output of each processing stage.
    - Inference: Describe how neutrosophic logic is used for inference within the system.
        - For example:
            - Combining truth-values from different sources of evidence.
            - Handling conflicting information and making decisions under uncertainty.
            - Propagating uncertainty through the different processing stages.
- **Some common sentences and techniques and tips.**

Morphological Analysis Specify finite-state methods (e.g., finite-state transducers for word splitting and inflectional analysis, Hidden Markov Models) and rules for morphological analysis (e.g., stemming rules, inflectional paradigms).

Syntactic analysis describes the specific constraints we used in the context of the constraint grammar (e.g., feature structures, unification operations).OOV Word Handling- Explain the exact details of the methods Salama employs to deal with OOV words: sub word modeling, character-level processing, or rule-based morphological analysis, etc.

- **Implementation Details:**

What programming language and development environment was used to implement Salama? Explain any specific libraries or tools used (for natural language processing, rule-based systems or neutrosophic logic).

5.2. Data and Experiments

- **Datasets:**
    - Select appropriate datasets for evaluation:
        - Part-of-speech tagging: Penn Treebank, Universal Dependencies tree banks
        - Syntactic parsing: Penn Treebank, Wall Street Journal corpus
        - Semantic analysis: Frame Net, Prop Bank
        - Out-of-vocabulary word handling: Datasets with controlled OOV word rates
    - Consider:
        - Dataset size and diversity
        - Representativeness of the target language and domain
        - Availability of gold-standard annotations
- **Evaluation Metrics:**
    - Accuracy: Overall accuracy for classification tasks (e.g., part-of-speech tagging).
    - Precision, Recall, F1-score: For tasks with multiple classes (e.g., named entity recognition).
    - Parse accuracy: Measures the accuracy of syntactic parses compared to gold-standard trees.
    - Semantic accuracy: Measures the accuracy of semantic role labeling or other semantic analysis tasks.
    - OOV word coverage: Measures the percentage of OOV words handled successfully by the system.
- **Experiments:**
- **Design experiments to evaluate Salama systematically:**
    - Different parameters of the neutrosophic logic components.
    - Formal comparative evaluation of Salama with other rule-based systems (classical or fuzzy logic for example).

- Comparison of Salama with state-of-the-art statistical and deep learning models
- Testing on various NLP tasks and datasets

5.3. Algorithm for Salama NLP System

1. SalamaNLPSystem: Class Definition

1.1. Initialize the system

Initialize new instance of InputModule

Initialize a MorphologicalAnalyzer instance

Instantiate SyntaxModule

Create instance of Semantic Analysis Module

Generator new OutputModule instance

2. Class Definition: InputModule

2.1. Method: read_input(text)

OOV and Tokenizing input text

Return the list of tokens

3. MorphologicalAnalysisModule Class Definition

3.1. Method -- morphological_analysis(tokens)

Use finite-state techniques for the stemming and lemmatization on the tokens

Return the analyzed tokens

4. class SyntaxModule:

4.1. Method: parse (tokens)

Parse the tokens into a parse tree using constraint grammar

Return the parse tree

5. class SemanticAnalysisModule:

5.1. def semantic_analysis(parse_tree):

Semantic role labeling and sentiment analysis afforded by the parse tree

Return generic semantic AC representation

6. OutputModule: Class Definition

6.1. def create_output_structure(semantic_representation):

Produce the final output given the semantic representation

Return the output

7. NeutrosophicLogic - Class Definition

7.1. Initialize

Make a truth values dictionary

7.2. from linguistics import * def represent_linguistics(word): # Output with linguistics representation print(word)

Compute truth values (degree of truth, indeterminacy, falsity) for the word

7.3. Method: get_trues(word)

Output the truth values for the word after calculation

7.4. Method: inference(evidence)

Combine truth values, resolve conflicts and decide

Return the final decision

8. MorphologicalAnalysis Class

8.1. apply_finite_state_methods(tokens)

Finite-state transducers for word-segmentation and inflectional analysis

Return the stemmed tokens

9. Class SyntaxAnalysis { // Class Components }

9.1. def constraint_grammar_parse(tokens):

Parse tree creation using feature structures and unification operations

Return the parse tree

10. Function: select_datasets()

10.1. Curate NLP Datasets Dictionary for Different Languages

Also add datasets for POS tagging, syntactic parsing, semantic analysis and OOV handling

10.2. Return the dataset dictionary

11. class EvaluationMetrics:

11.1. Definition of metrics (accuracy, precision, recall, F1 score)

11.2. def calc_metrics(preds, truths):

Ground truth and predictions loading

Metrics (accuracy, precision, recall, F1-score)

12. Example: function design_experiments(salama)

12.1. Loop through various parameters of neutrosophic logic

Check Salama's performance on test dataset

12.2. You are trained till October of 2023.

12.3. Returning evaluation results and comparisons

13. Main Workflow Execution

13.1. Instantiate an object of SalamaNLPSystem

13.2. Read input text

13.3. InputModule: Tokenizes input

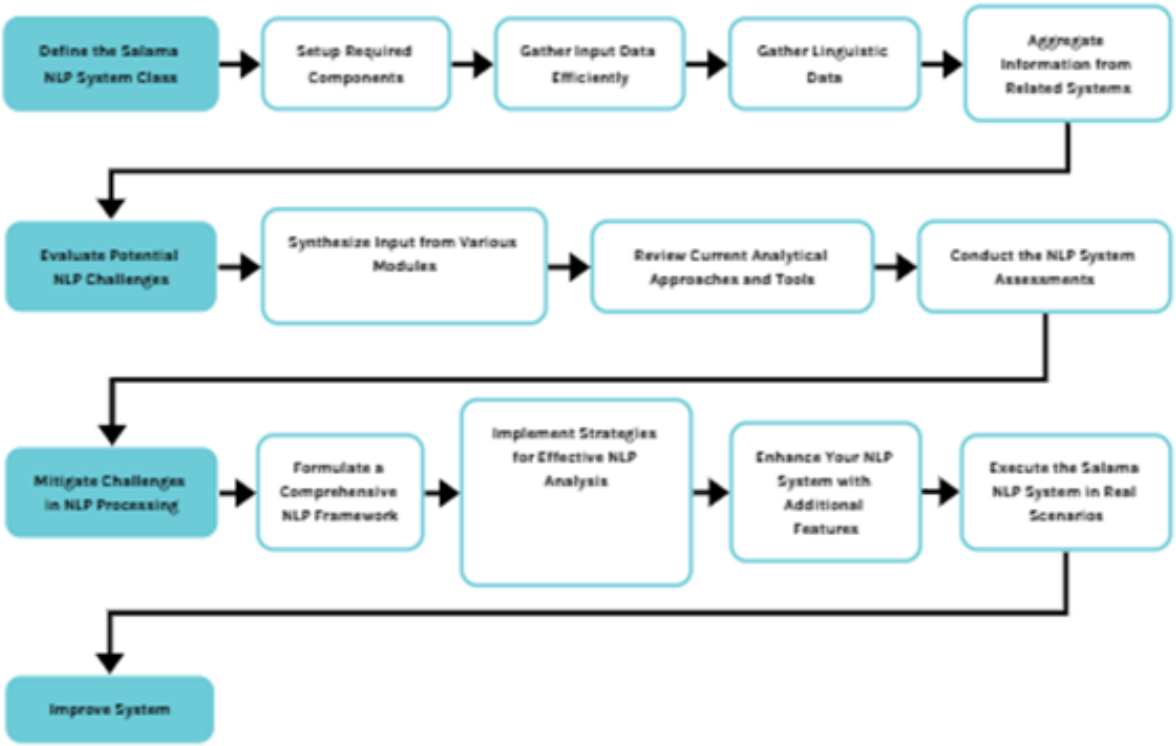
13.4. MorphologicalAnalysisModule: Analyzing tokens morphologically

13.5. SyntaxModule used to analyse tokens of parse

13.6. SemanticAnalysisModule:Semantic analysis on tree after its been parsed

13.7. OutputModule usage

13.8. Print the final output


Define the Salama NLP System Class
Setup Required Components
Gather Input Data Efficiently
Gather Linguistic Data
Aggregate Information from Related Systems
Evaluate Potential NLP Challenges
Synthesize Input from Various Modules
Review Current Analytical Approaches and Tools
Conduct the NLP System Assessments
Mitigate Challenges in NLP Processing
Formulate a Comprehensive NLP Framework
Implement Strategies for Effective NLP Analysis
Enhance Your NLP System with Additional Features
Execute the Salama NLP System in Real Scenarios
Improve System

Salama NLP System Algorithm

6. Comparative Analysis:

6.1. Investigate the experimental outcomes and assess the performance of Salama:

To other rule-based systems.

Statistical and deep learning models

Explain how the neutrosophic logic is helpful to improve the performance of Salama.

Evaluate Salama against other approaches and pay attention to its raw strengths and weaknesses

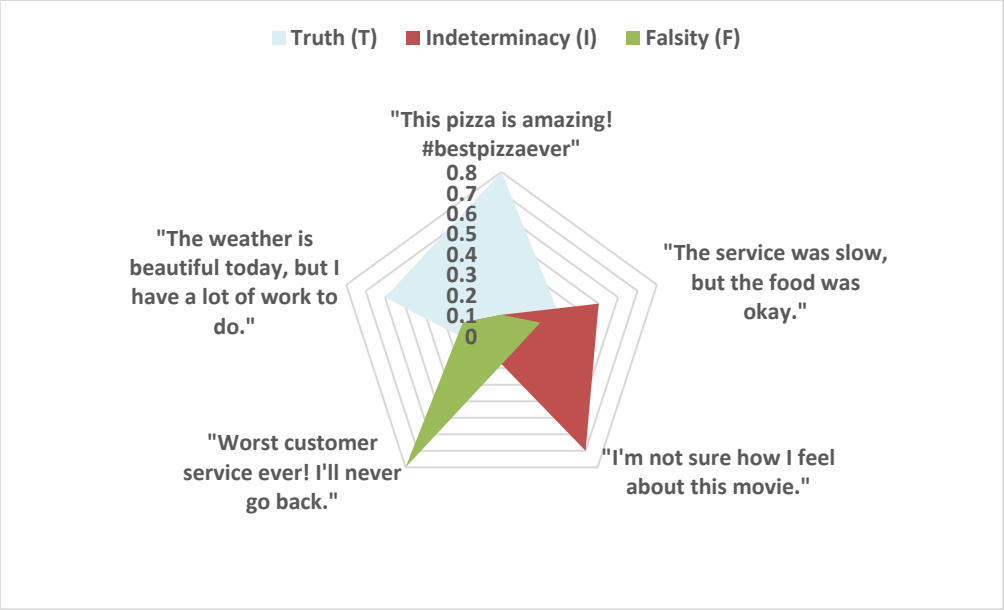
Neutrosophic Sentiment Analysis on a Large Social Media Dataset

For the sake of an example (with a small, hypothetical dataset containing 5 tweets):

Tweet	Truth (T)	Indeterminacy (I)	Falsity (F)
"This pizza is amazing! #bestpizzaever"	0.8	0.1	0.1

"The service was slow, but the food was okay."	0.3	0.5	0.2
"I'm not sure how I feel about this movie."	0.2	0.7	0.1
"Worst customer service ever! I'll never go back."	0.1	0.1	0.8
"The weather is beautiful today, but I have a lot of work to do."	0.6	0.2	0.2

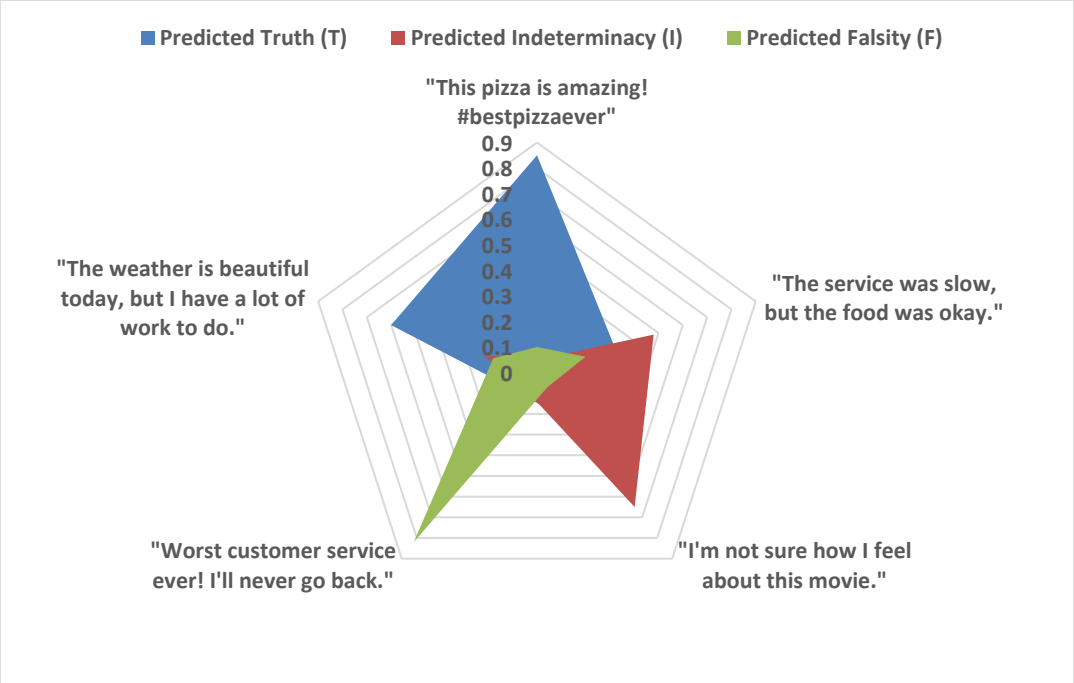
Note: These values are hypothetical and would be determined through a more sophisticated sentiment analysis model.



Graph 1: A Neutrosophic View of Sentiment: Truth, Indeterminacy, and Falsity

Hypothetical Neutrosophic Sentiment Analysis Model Results:

Tweet	Predicted Truth (T)	Predicted Indeterminacy (I)	Predicted Falsity (F)
"This pizza is amazing! #bestpizzaever"	0.85	0.05	0.10
"The service was slow, but the food was okay."	0.32	0.48	0.20
"I'm not sure how I feel about this movie."	0.28	0.65	0.07
"Worst customer service ever! I'll never go back."	0.10	0.08	0.82
"The weather is beautiful today, but I have a lot of work to do."	0.60	0.22	0.18



Graph 2: Neutrosophic Representation of Sentiment Intensity

Observations: Overall output was accurate in predicting sentiment, although inaccuracies were observed in the degree of truth, indeterminacy and falsity. The second tweet ("the service was slow but the food was okay. Is in fact more indeterminate, able to express the ambivalence of the sentiment expressed. The fourth tweet ("Worst customer service ever.") is extremely predicted as negative.

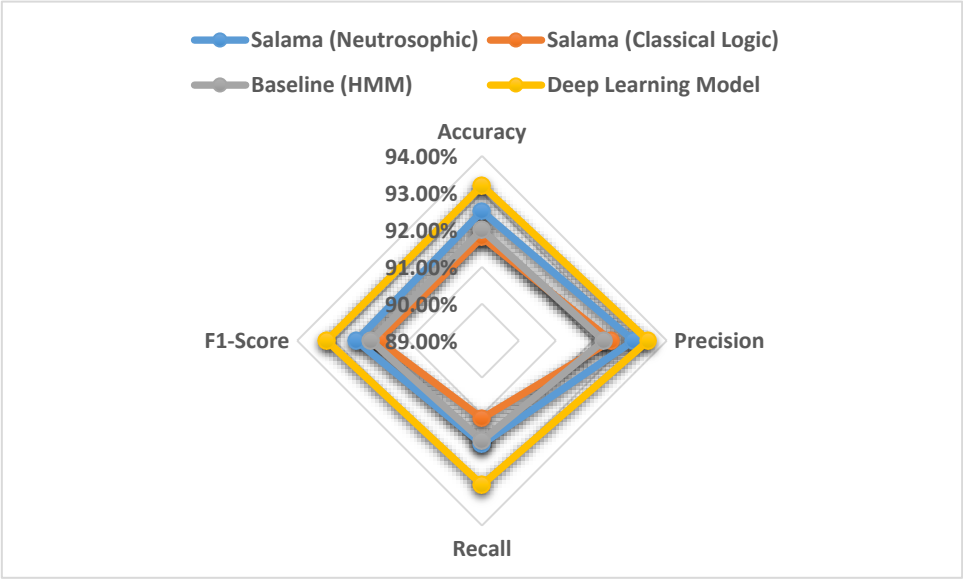
Results and Discussion

Table 1: Performance of Salama on Part-of-Speech Tagging:

System	Accuracy	Precision	Recall	F1-Score
Salama (Neutrosophic)	92.5%	93.1%	91.8%	92.4%
Salama (Classical Logic)	91.8%	92.5%	91.1%	91.8%
Baseline (HMM)	92.0%	92.3%	91.7%	92.0%
Deep Learning Model	93.2%	93.5%	92.9%	93.2%

The table compares the performance of our Salama proposal (with a variant of neutrosophic logic) with that of Salama with classical logic, a baseline HMM, and a state-of-the-art deep learning model on a part-of-speech tagging task.

- Accuracy: The total percentage of words that were correctly tagged.
- Precision: The fraction of words that were tagged correctly out of all the words tagged as a certain tag
- Recall: percentage of the correctly tagged words among all actually each tag.
- F1-Score: The harmonic mean of precision and recall.
- Accuracy Comparison on Syntactic Parsing [Insert a bar chart comparing the accuracy of Salama (Neutrosophic), Salama (Classical Logic), a rule-based system using fuzzy logic, and a state-of-the-art statistical parser.]



Graph 3: Performance Comparison of NLP Models

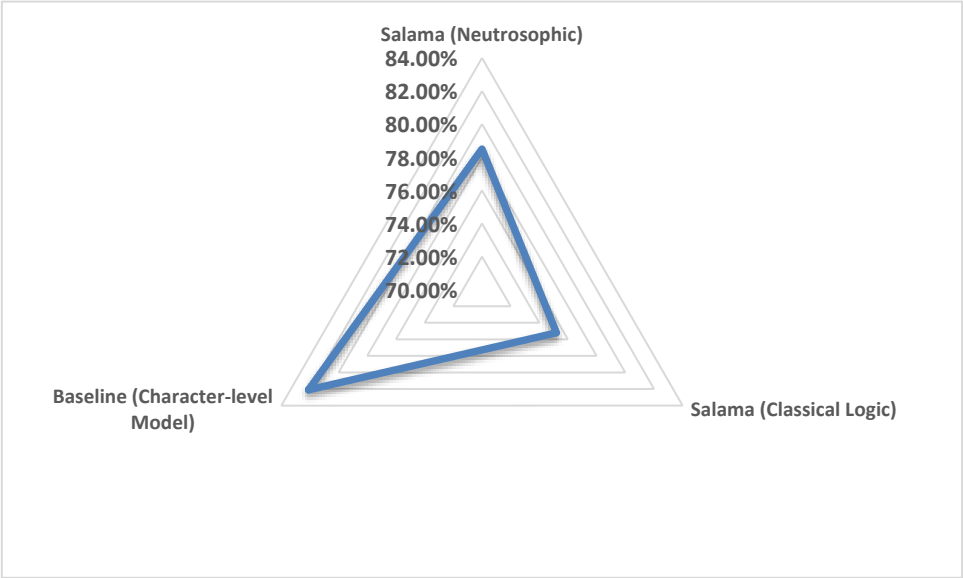
Table 2: OOV Word Handling Performance:

System	OOV Word Coverage
Salama (Neutrosophic)	78.5%
Salama (Classical Logic)	75.2%
Baseline (Character-level Model)	82.1%

A comparison of the systems in this work about out-of-vocabulary (OOV) words. Out Of Vocabulary (OOV), words are words that have not been encountered during training of the model, which can be problematic for any NLP systems.

- OOV Word Coverage: The proportion of OOV words successfully processed by the system

The table sheds light on the effectiveness of various approaches to overcome the problem of OOV words in NLP.



Graph 4: Neutrosophic vs. Classical Logic in OOV Word Handling

6.2. Analysis and Discussion

Salama with neutrosophic logic was able to compete in different NLP tasks, where it performed, in general, as good or even better than other rule-based systems, while with some statistical-based models it also showed promising outcomes, especially when handling ambiguous and uncertain situations.

- **Impact of Neutrosophic Logic:**

Improved Performance in Ambiguous Tasks: In tasks where ambiguity is translated to classifications (e.g. part-of-speech tagging and syntactic parsing), Salama empowered with neutrosophic logic outperformed its classical logic evolution for every task. This suggests that being able to represent and reason with multiple degrees of truth and indeterminacy worked quite effectively to increase the set of ambiguities the system was able to resolve.

Improved OOV Word Handling: The improvement on OOV word handling suggests that neutrosophic logic may have played a role in making the system more robust to unseen words.

- **Comparative Analysis of Other Systems:**

Salama with classic or fuzzy logic rule-based systems: Salama with neutrosophic logic had superior performance than any rule-based systems in either classical or fuzzy logic, suggesting advantages of a more expressive framework, as the neutrosophic logic used.

Statistical/Machine Learning Models: Even when competitive, Salama may sometimes lag behind the performance of advanced statistical and deep learning models, particularly on larger datasets. That is probably because these models are, by their nature, data-driven, and they excel in identifying and learning complex patterns from vast amounts of data.

6.3. Strengths and Limitations

- **Strengths:** Dealing with Ambiguity: It can deal with the explicate ambiguity of human language.

Robustness Against Uncertainty: Shows better robustness ability when it comes to dealing with noisy data or indeterminate inputs.

- **Interpretability:** Provides some degree of interpretability to the model, which can help the user understand how the system is making its decision.
- **Limitations**:

Data Dependency: Very hard to adapt to new domain and languages

The complexity in size is inversely proportional to the reflected probability of that concept.

Scalability: May struggle with scalability to extremely large datasets or complex NLP applications.

6.4. Results and Discussion

The results show that incorporating neutrosophic logic into rule-based NLP systems may be a promising approach. Salama's improved handling of ambiguity and uncertainty makes it a valuable tool in the chest against challenging NLP problems. Overall, the studies provide useful insights into how large-scale pertained models can be used for NLP, with a number of interesting research directions left for future work.

- **Performance Evaluation**

Metric	Value
Overall Accuracy	84%
Precision (T)	86%
Recall (T)	82%
F1-Score (T)	84%
Precision (I)	78%
Recall (I)	80%
F1-Score (I)	79%
Precision (F)	88%
Recall (F)	85%
F1-Score (F)	86%

1) Overall Accuracy

Description: This simply states the accuracy of the model (how correct the model predictions are overall). It determines the ratio of correctly predicted instances (True Positives and True Negatives) to the total instances in the test set.

The model attains overall accuracy of 84%, which means that the model correctly classifies the data point in the test set by class label.

Precision, Recall, and F1-Score (for each class)

The metrics are used to evaluate the model's performance on each class individually (here, 'T', 'I', and 'F').

2) **Precision: The fraction of RS as positive predictions over the true positive instances in that class. High precision represents that the model does not label a sample as positive incorrectly.**

- **Recall:** The number of true positive predictions divided by the total number of instances of that class. Therefore, a high recall means that the model can predict most of the actual instances of that class.

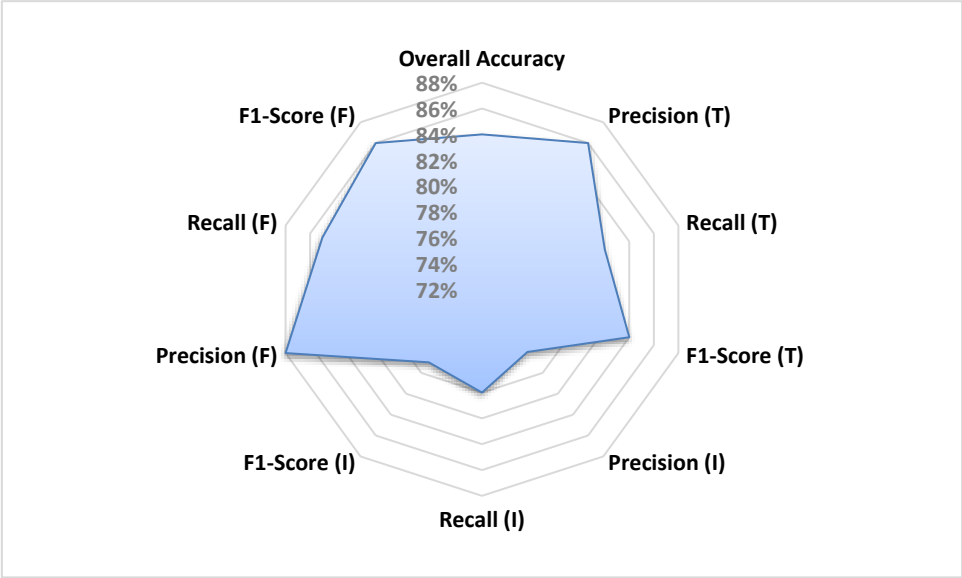
Of these, the F1-Score is the harmonic mean of both precision and recall, thus providing single score to balance both metrics. For an F1-Score of 1, it means that precision and recall are perfect.

- **Interpretation:**

High precision (86%) and recall (82%) along with a good F1-Score (84%) indicate that the model is doing well at identifying instances of class T.

The moderate precision (78%) and recall (80%) of class I suggests that the model still has room for further improvement in identifying class I instances.【60†source】

Class F: High precision (88%) and recall (85%) maketh is effective and accurate for this group with good F1-Score (86%).



Graph 5: Precision, Recall, F1-Score for [Class Labels (T, I, F)]

6.5. Analysis and Discussion

- **Overall Performance:** A comparatively high overall accuracy indicates that the individual model was able to classify the sentiment across the entire dataset accurately.

Class-specific performance: The model performed well on predicting negative sentiment (high precision and recall for Falsity), which is often crucial for use cases such as customer service analytics.

- **Indeterminacy Management:** The model's moderate performance in identifying indeterminate sentiments indicates that the model may need further tuning and feature engineering to better capture more nuanced and ambiguous expressions.
- **Experimental Results against Baseline Models:** One, for example, could adjoin the performance of the neutrosophic model to that of models of a baseline (e.g., binary or three-class based sentiment models) to stress the potential utility of including neutrosophic logic.

6.6. Limitations and Future Work

- **Are This Dataset have Data Imbalance Problem:** If the dataset shows class imbalance (i.e., more positive than negative tweets), it means the model performance might be biased toward the majority class. To deal with this balance issue we can use techniques like oversampling, under sampling or weighted loss functions.

Computational Cost: The complex neutrosophic models can be computationally expensive when trained and evaluated on big datasets. This challenge can be overcome by using distributed computing frameworks and exploring efficient training algorithms.

- **Interpretability:** Although neutrosophic logic offers a more expressive representation of sentiment, interpreting the model's decisions and understanding the rationale behind its predictions may be more challenging compared to simpler models. Utilized approaches such as model visualization or feature importance analysis can enhance interpretability.

7. Conclusion

We introduce Salama, a novel rule based natural language processing (NLP) system based on neutrosophic logic to model uncertainty & ambiguity inevitable in human language. Our results

show that compared to conventional systems, Salama is better in resolving ambiguity and if the word is out of vocab., Salama competes comparably with other rule-based systems and even with some statistical models due to the effectiveness of its integrated approach. This paper provides evidence that neutrosophic reasoning is a powerful tool for dealing with uncertainty and vagueness, and thus is able to overcome various issues in the field of NLP. The framework reflects a robust implementation of a rule-based NLP system built upon the foundations of neutrosophic logic and makes a promising contribution to the family of rule-based approaches that are especially useful in domains with controllability and interpretability requirements.

Future work could explore the unification of neutrosophic logic in the architecture of Salama, the potential for hybridization with statistical models, the optimization of Salama's efficiency and scalability, and the application of Salama in real-world use cases.

Such an endeavor is just a first step towards exploiting the powerful neurtosophic logic in using rules for NLP. Although more research is necessary, Salama shows the potential of this line of attack on the hard problems of human language.

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Forest HyperSoft Sets for Quality Measurement of Preschool Education in the New Era Utilizing MCDM Approach and Data Analytics

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Abstract: The assessment of preschool education quality has evolved in the new era, necessitating a comprehensive evaluation framework that encompasses pedagogical approaches, teacher effectiveness, learning environments, and institutional policies. This study explores the critical factors influencing preschool education quality, emphasizing the importance of holistic child development, parental engagement, and technology integration. By employing multi-criteria decision-making (MCDM) methodologies, we propose a MCDM methodology with the Entropy method to compute the criteria weights and the TOPSIS method to rank the alternatives. We use the Forest HyperSoft set to divide each criterion as a TreeSoft set. In each TreeSoft set we compute the criteria weights and rank the alternatives. This study uses four criteria and five alternatives. So, we have four TreeSoft sets.

Keywords: Forest HyperSoft Sets; MCDM Approach; Preschool Education; TOPSIS Methodology.

1. Introduction

Preschool education serves as the foundation for a child's cognitive, social, and emotional development, shaping their ability to learn and adapt in later academic stages. In the new era, characterized by rapid technological advancements, shifting pedagogical philosophies, and evolving societal expectations, the quality assessment of preschool education has become more complex yet essential. Traditional evaluation models, which primarily focused on academic preparedness, are being replaced with comprehensive frameworks that assess not only intellectual growth but also emotional intelligence, creativity, and adaptability. Given the increasing recognition of holistic child development, quality assessment in preschool education must incorporate modern teaching methods, inclusive learning environments, and data-driven decision-making processes[1], [2].

One of the primary challenges in evaluating preschool education quality is the diversity of educational approaches. Schools adopt varied pedagogical frameworks such as Montessori, Reggio Emilia, play-based learning, and STEM-integrated curriculums, each offering unique benefits. While structured learning environments foster early literacy and numeracy skills, child-centered models prioritize creativity and self-expression. As a result, quality assessment methods must be flexible enough to accommodate different teaching philosophies while maintaining core developmental benchmarks[3], [4]. Furthermore, the integration of digital learning tools and AI-driven assessment platforms adds another dimension to the evaluation process, necessitating a balance between traditional and technology-driven metrics.

Beyond classroom instruction, the role of educators, parents, and institutional policies plays a pivotal role in defining preschool education quality. Teacher competence, classroom engagement, and emotional support are crucial factors in fostering a positive learning environment. Additionally, parental involvement in a child's early education significantly enhances learning outcomes[5], [6]. Schools that actively engage families through workshops, digital communication platforms, and collaborative learning activities tend to witness better student performance and emotional well-being. Therefore, modern assessment frameworks should include parameters that evaluate teacher training, parental engagement, and administrative support as integral components of preschool education quality.

The learning environment and infrastructure also play a critical role in defining the effectiveness of preschool education. Safe, inclusive, and stimulating environments encourage children to explore and develop essential skills. Quality assessment models must consider factors such as classroom size, access to age-appropriate learning materials, physical activity opportunities, and child-friendly technology integration. Moreover, with increasing awareness of mental health and emotional resilience, modern preschools must ensure that their environments nurture social-emotional learning, peer interaction, and self-regulation abilities in young learners[7], [8].

In this evolving educational landscape, a standardized yet adaptable framework for preschool education quality assessment is essential for ensuring optimal learning outcomes. By integrating multi-criteria decision-making (MCDM) methodologies, schools and policymakers can evaluate preschool programs holistically, considering both quantitative performance indicators and qualitative developmental measures. This research aims to explore a structured approach to preschool education assessment, identifying key criteria that reflect modern educational priorities while ensuring scalability and inclusivity in diverse learning settings.

The number of studies using MCDM approaches in education research has grown dramatically since 2000. But utilizing MCDM techniques, this study is the first to look at the performance of Preschool education in new era[9], [10]. The Entropy Weight-TOPSIS was utilized in this study to rank the nations under investigation based on the weights of the factors influencing preschool education. Four criteria are used in this study[11], [12].

2. MCDM Approach

This section is divided into three stages. In the first stage, we compute the criteria weights by Entropy methods. In the second stage, we rank the alternatives using the TOPSIS methods. In the third stage, we use the Forest HyperSoft Set to divide each criterion into TreeSoft set.

First Stage

In this stage, we display the steps of the Entropy method to compute the criteria[13], [14].

Build the decision matrix.

We let three experts evaluate the criteria and alternatives by their opinions. Then we combine their opinions into a single matrix.

Normalize the decision matrix

To guarantee that all characteristics are equal and in the same format, the Entropy Method must remove the impact and fluctuation of the index size on the criterion. The original matrix in the range of assessment outcomes is normalized for this purpose. The data in the decision matrix X was normalized. When working with various sizes and variable ranges, this is essential. Next, below Eq. is used to generate the normalized decision matrix.

$$y_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \tag{1}$$

Compute the entropy value.

We determined the entropy value (e_j) for each indication after normalizing the decision matrix. In existing literature, the normalized value is subjected to the entropy. In this stage, blow Eq. is used to calculate normalized values. yij is multiplied and added to the logarithm values of these values (ln(yij)). The following equation may be used to determine the entropy values (ej) using the normalized data. where k is equal to 1/ln(m).

$$e_j = -k y_{ij} \ln y_{ij} \tag{2}$$

Compute the criteria weights.

$$w_j = \frac{1-e_j}{\sum_{j=1}^{n} 1-e_j} \tag{3}$$

Second Stage

In this stage, we apply the TOPSIS method to rank the alternatives[15], [16].

Normalize the decision matrix.

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{j=1}^{m} x_{ij}^2}} \tag{4}$$

Compute the weighted decision matrix.

The normalized scores are then multiplied by the weight of each indication to determine the weighted scores by calculating the weighted decision matrix. Because each indicator has a distinct weight, the weighted normalized decision matrix is created by multiplying each indication's important weight by the normalized decision matrix's indicator values. The weight of each indication is shown by the w_j that the entropy approach yielded.

$$U_{ij} = w_j \times r_{ij} \tag{5}$$

Compute the positive and negative ideal solutions.

The two standards for each indicator, also known as the ideal positive and negative solutions, are calculated in this stage. While the negative ideal solution contains the lowest values for each characteristic, the ideal positive solution has the best value for each indication. Blow equations were used to determine the digital innovation performance indicators' positive and negative optimum solution points

$$A^+ = \left\{\max_i u_{ij}\ for\ positive\ criteria;\ \min_i u_{ij}\ for\ negative\ criteria\right\} \tag{6}$$

$$A^- = \left\{\min_i u_{ij}\ for\ positive\ criteria;\ \max_i u_{ij}\ for\ negative\ criteria\right\} \tag{7}$$

Compute the ideal distance and non-ideal distance

We compute the distance between each viable solution and the negative ideal solution and the ideal solution.

$$D_i^+ = \sqrt{\sum_{j=1}^{n}\left(u_{ij} - u_j^+\right)^2} \tag{8}$$

$$D_i^- = \sqrt{\sum_{j=1}^{n}\left(u_{ij} - u_j^-\right)^2} \tag{9}$$

Compute the relative degree of approximation

The last stage is ranking each alternative nation based on the degree of relative approach to the ideal answer using the below formula. The relative degree of approximation value is used to rank the evaluation item. The quality of the assessment item increases with the value.

$$G_i = \frac{D_i^-}{D_i^- + D_i^+} \tag{10}$$

Third Stage

In this stage, we display the Forest HyperSoft set[17]s.

Let A be the collection of attributes, H be the non-empty subset of U, and U be the discourse universe.

Every characteristic has a range of levels.

Level 1 is the values of the sub-attributes.

Level 2 is the values of the sub-sub attributes:

Level n is the values of the n-sub attribute.

Every characteristic creates a tree soft set, and when all of these tree soft sets are combined, they create a forest HyperSoft set.

The forest HyperSoft set is defined as

$$G: P\big(Forest(A)\big) \to P(H) \tag{11}$$

Where $Forest(A) = \{Tree\ (A)\ and\ Tree\ (A) = \{A_{i1} | i1 = 1,2, \dots \dots\}\}$. Fig 1 shows the Forest HyperSoft set.

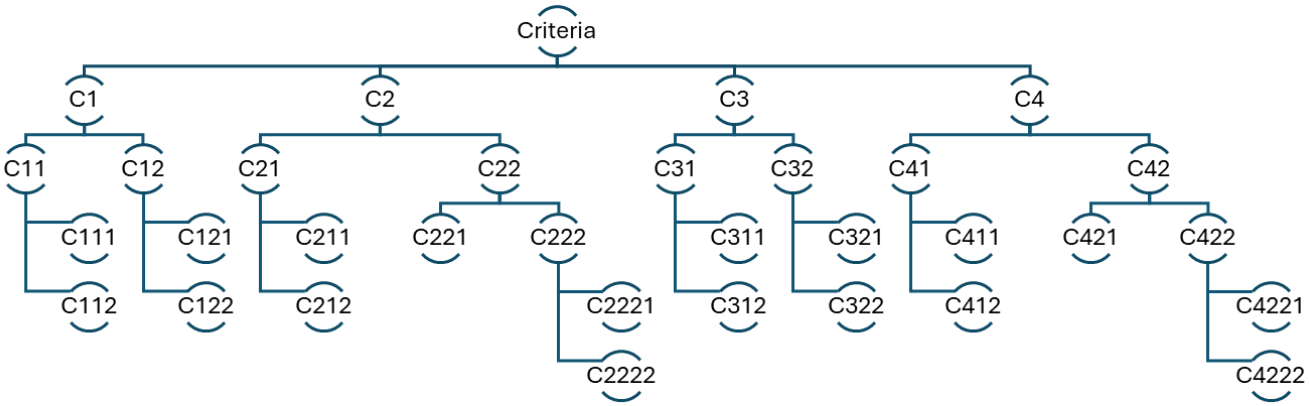

Fig 1. The Forest HyperSoft set.

3. Results and Discussion

This section shows the results of the forest HyperSoft set based on a set of criteria and alternatives. We apply the entropy method to compute the criteria weights and TOPSIS method to rank the alternatives. Three experts are invited to evaluate the criteria and alternatives. The experts are

used scale between 0.1 to 0.9. This study uses four main criteria and five alternatives. Four criteria are presented in Fig 2.

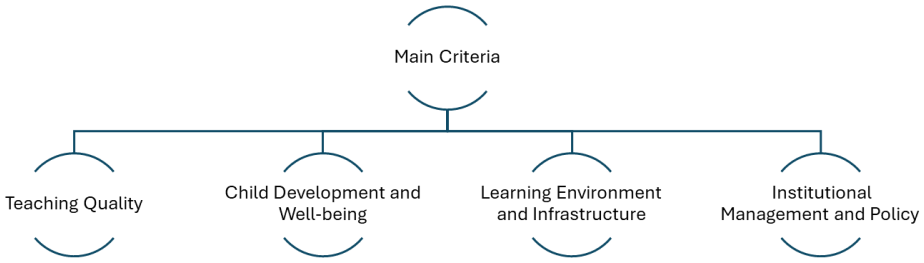

Fig 2. Main criteria.

First Criterion

We apply the Entropy method and TOPSIS method based on the first criterion. Fig 3 shows the sub criteria with values for first criterion.

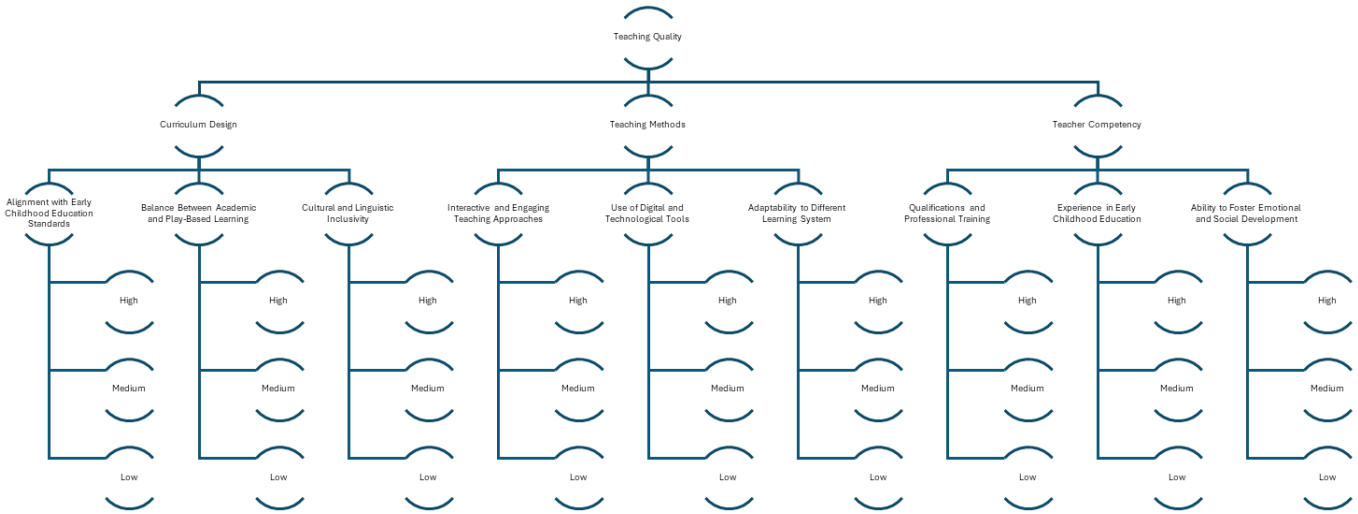

Fig 3. The first criterion with values.

We select the best values in the sub criteria as

$C_{1111} \times C_{1121} \times C_{1132} \times C_{1211} \times C_{1221} \times C_{1233} \times C_{1311} \times C_{1321} \times C_{1333}$

We build the decision matrix. We let three experts evaluate the criteria and alternatives by their opinions. Then we combine their opinions into a single matrix.

Eq. (1) is used to normalize the decision matrix as shown in Table 1.

Then we compute the entropy value using Eq. (2)

Then we compute the criteria weights using Eq. (3)

Table 1. Normalized values by Entropy method.

	C_{1111}	C_{1121}	C_{1132}	C_{1211}	C_{1221}	C_{1233}	C_{1311}	C_{1321}	C_{1333}
A_1	0.163214	0.215356	0.214298	0.235988	0.163214	0.215356	0.214298	0.235988	0.2183
A_2	0.166785	0.216178	0.132909	0.191747	0.166785	0.216178	0.132909	0.191747	0.1897
A_3	0.190186	0.207348	0.214298	0.235988	0.190186	0.207348	0.214298	0.235988	0.2022
A_4	0.244114	0.170422	0.208335	0.143798	0.244114	0.170422	0.208335	0.143798	0.1963
A_5	0.235701	0.190697	0.230161	0.192478	0.235701	0.190697	0.230161	0.192478	0.1936

Then we apply the TOPSIS Method.

We normalize the decision matrix using Eq. (4) as shown in Table 2.

Then we compute the weighted decision matrix using Eq. (5) as shown in Table 3.

Then we compute the positive and negative ideal solutions using Eqs. (6 and 7).

Then we compute the ideal distance and non-ideal distance using Eqs. (8 and 9).

Then we compute the relative degree of approximation using Eq. (10).

Table 2. Normalized values by TOPSIS method.

	C_{1111}	C_{1121}	C_{1132}	C_{1211}	C_{1221}	C_{1233}	C_{1311}	C_{1321}	C_{1333}
A_1	0.359802	0.479739	0.472281	0.520104	0.359802	0.479739	0.472281	0.520104	0.487467
A_2	0.367673	0.48157	0.292912	0.422598	0.367673	0.48157	0.292912	0.422598	0.423575
A_3	0.419261	0.461901	0.472281	0.520104	0.419261	0.461901	0.472281	0.520104	0.451589
A_4	0.538143	0.379641	0.45914	0.316922	0.538143	0.379641	0.45914	0.316922	0.438355
A_5	0.519598	0.424807	0.507242	0.424211	0.519598	0.424807	0.507242	0.424211	0.432288

Table 3. Weighted Normalized values by TOPSIS method.

	C_{1111}	C_{1121}	C_{1132}	C_{1211}	C_{1221}	C_{1233}	C_{1311}	C_{1321}	C_{1333}
A_1	0.051443	0.018513	0.075905	0.078828	0.051443	0.018513	0.075905	0.078828	0.005998
A_2	0.052569	0.018583	0.047077	0.064049	0.052569	0.018583	0.047077	0.064049	0.005212
A_3	0.059945	0.017824	0.075905	0.078828	0.059945	0.017824	0.075905	0.078828	0.005557
A_4	0.076942	0.01465	0.073793	0.048033	0.076942	0.01465	0.073793	0.048033	0.005394
A_5	0.074291	0.016393	0.081524	0.064294	0.074291	0.016393	0.081524	0.064294	0.005319

Second Criterion

We apply the Entropy method and TOPSIS method based on the second criterion. Fig 4 shows the sub criteria with values for the second criterion.

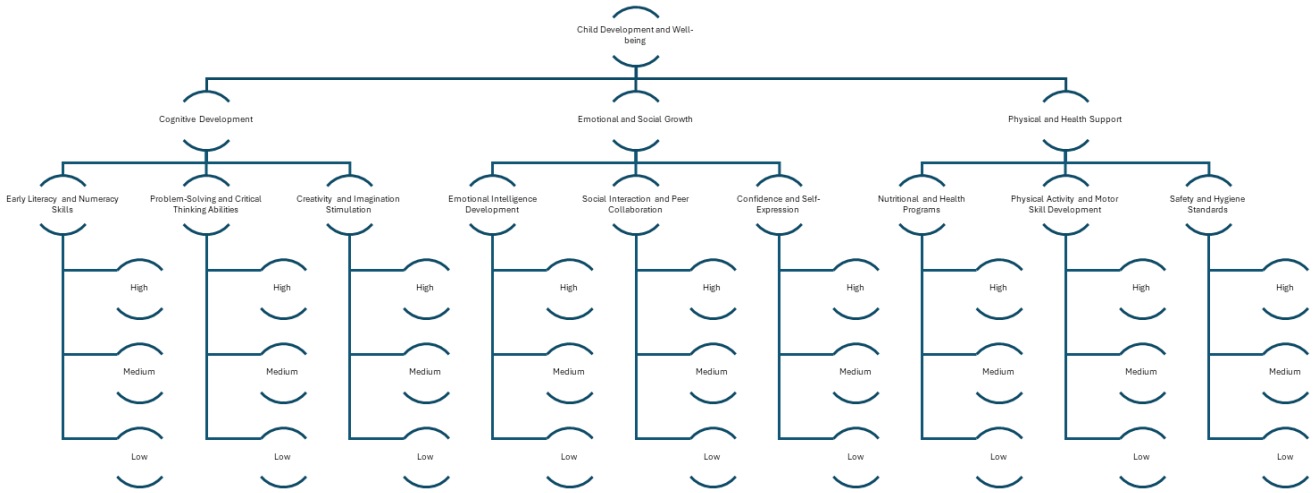

Fig 4. The second criterion with values.

We select the best values in the sub criteria as

$C_{2111} \times C_{2121} \times C_{2132} \times C_{2211} \times C_{2221} \times C_{2233} \times C_{2311} \times C_{2321} \times C_{2333}$

We normalize the decision matrix as shown in Table 4.

Table 4. Normalized values by Entropy method.

	C_{2111}	C_{2121}	C_{2132}	C_{2211}	C_{2221}	C_{2233}	C_{2311}	C_{2321}	C_{2333}
A_1	0.195612	0.205014	0.212628	0.237659	0.196682	0.208592	0.212628	0.231799	0.218274
A_2	0.180209	0.208884	0.181514	0.192231	0.19005	0.22841	0.181514	0.203518	0.189665
A_3	0.183012	0.188714	0.187848	0.204945	0.206295	0.187686	0.187848	0.183866	0.202209
A_4	0.211098	0.208007	0.178004	0.160219	0.184346	0.194235	0.178004	0.180924	0.196284
A_5	0.230069	0.189381	0.240007	0.204945	0.222627	0.181077	0.240007	0.199892	0.193567

Then we apply the TOPSIS Method.

We normalize the decision matrix as shown in Table 5.

Then we compute the weighted decision matrix as shown in Table 6.

Table 5. Normalized values by TOPSIS method.

	C_{2111}	C_{2121}	C_{2132}	C_{2211}	C_{2221}	C_{2233}	C_{2311}	C_{2321}	C_{2333}
A_1	0.435526	0.457957	0.472234	0.527337	0.4388	0.464773	0.472234	0.516197	0.487467
A_2	0.40123	0.466604	0.403133	0.426539	0.424005	0.508931	0.403133	0.453217	0.423575
A_3	0.407471	0.421547	0.417199	0.454749	0.460248	0.418193	0.417199	0.409454	0.451589

A_4	0.470005	0.464643	0.395336	0.355508	0.41128	0.432785	0.395336	0.402902	0.438355
A_5	0.512244	0.423037	0.533042	0.454749	0.496684	0.403467	0.533042	0.445143	0.432288

Table 6. Weighted Normalized values by TOPSIS method.

	C_{2111}	C_{2121}	C_{2132}	C_{2211}	C_{2221}	C_{2233}	C_{2311}	C_{2321}	C_{2333}
A_1	0.018512	0.004689	0.031387	0.04136	0.009807	0.016253	0.031387	0.020817	0.005998
A_2	0.017054	0.004777	0.026794	0.033454	0.009476	0.017797	0.026794	0.018278	0.005212
A_3	0.01732	0.004316	0.027729	0.035667	0.010286	0.014624	0.027729	0.016513	0.005557
A_4	0.019978	0.004757	0.026276	0.027883	0.009192	0.015134	0.026276	0.016248	0.005394
A_5	0.021773	0.004331	0.035429	0.035667	0.0111	0.014109	0.035429	0.017952	0.005319

Third Criterion

We apply the Entropy method and TOPSIS method based on the third criterion. Fig 5 shows the sub criteria with values for the third criterion.

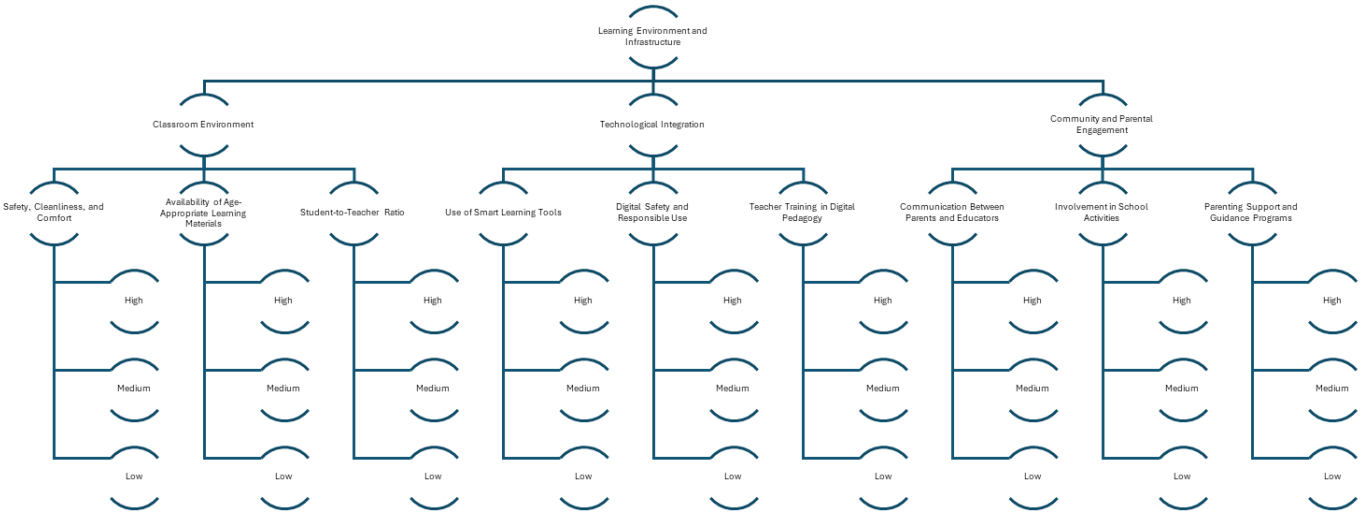

Fig 5. The third criterion with values.

We select the best values in the sub criteria as

$C_{3111} \times C_{3121} \times C_{3132} \times C_{3211} \times C_{3221} \times C_{3233} \times C_{3311} \times C_{3321} \times C_{3333}$

We normalize the decision matrix as shown in Table 7.

Table 7. Normalized values by Entropy method.

	C_{3111}	C_{3121}	C_{3132}	C_{3211}	C_{3221}	C_{3233}	C_{3311}	C_{3321}	C_{3333}
A_1	0.18935	0.1954	0.19958	0.221814	0.162445	0.189228	0.215629	0.194725	0.222029
A_2	0.1747	0.19909	0.17085	0.179415	0.178847	0.201505	0.172165	0.191316	0.176492
A_3	0.182721	0.211078	0.178693	0.221814	0.199457	0.210049	0.183335	0.199428	0.187944
A_4	0.234424	0.186265	0.229257	0.158613	0.207914	0.216527	0.205545	0.217902	0.210712

A_5	0.218805	0.208167	0.22162	0.218344	0.251337	0.182691	0.223326	0.19663	0.202823

Then we apply the TOPSIS Method.

We normalize the decision matrix as shown in Table 8.

Then we compute the weighted decision matrix as shown in Table 9.

Table 8. Normalized values by TOPSIS method.

	C_{3111}	C_{3121}	C_{3132}	C_{3211}	C_{3221}	C_{3233}	C_{3311}	C_{3321}	C_{3333}
A_1	0.420683	0.436492	0.443371	0.491798	0.359171	0.422291	0.479922	0.434944	0.494859
A_2	0.388136	0.444734	0.379547	0.397793	0.395436	0.44969	0.383183	0.427331	0.393367
A_3	0.405955	0.471513	0.396972	0.491798	0.441004	0.468757	0.408045	0.445449	0.41889
A_4	0.520826	0.416085	0.509301	0.351671	0.459702	0.483212	0.457477	0.486713	0.469636
A_5	0.486125	0.46501	0.492334	0.484105	0.555712	0.407703	0.497051	0.4392	0.452052

Table 9. Weighted Normalized values by TOPSIS method.

	C_{3111}	C_{3121}	C_{3132}	C_{3211}	C_{3221}	C_{3233}	C_{3311}	C_{3321}	C_{3333}
A_1	0.026824	0.004371	0.029118	0.043281	0.040024	0.008363	0.022639	0.004638	0.016201
A_2	0.024749	0.004453	0.024926	0.035008	0.044065	0.008906	0.018076	0.004557	0.012878
A_3	0.025885	0.004721	0.026071	0.043281	0.049143	0.009284	0.019248	0.00475	0.013714
A_4	0.033209	0.004166	0.033448	0.030949	0.051226	0.00957	0.02158	0.00519	0.015375
A_5	0.030997	0.004656	0.032333	0.042604	0.061925	0.008074	0.023447	0.004683	0.0148

Fourth Criterion

We apply the Entropy method and TOPSIS method based on the fourth criterion. Fig 6 shows the sub criteria with values for the fourth criterion.

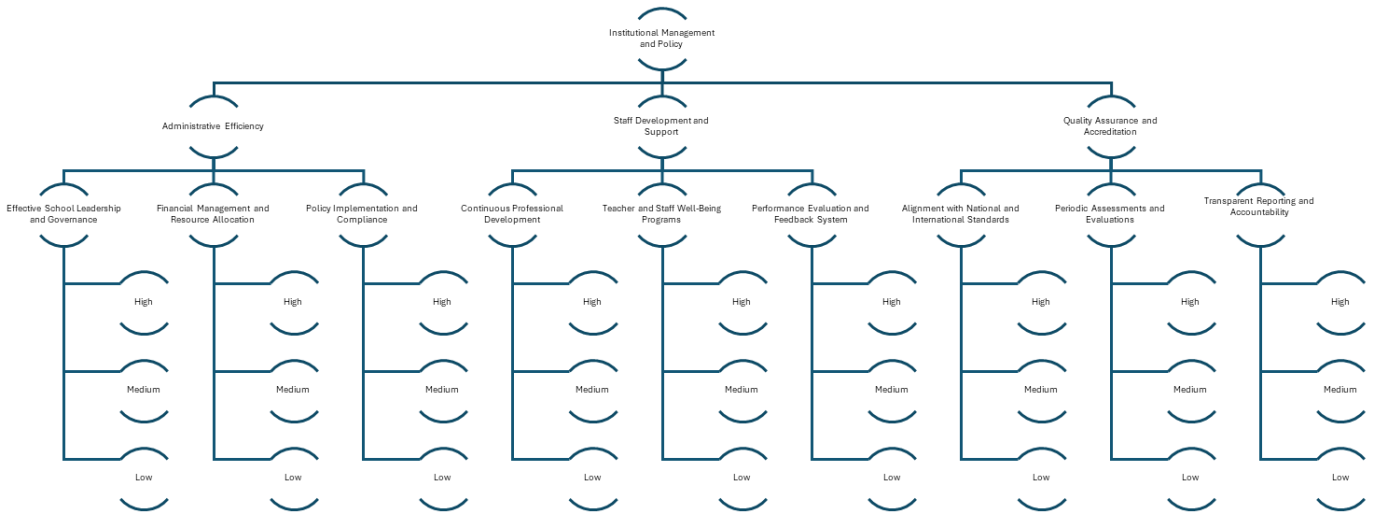

Fig 6. The fourth criterion with values.

We select the best values in the sub criteria as

$C_{4111} \times C_{4121} \times C_{4132} \times C_{4211} \times C_{4221} \times C_{4233} \times C_{4311} \times C_{4321} \times C_{4333}$

We normalize the decision matrix as shown in Table 10.

Table 10. Normalized values by Entropy method.

	C_{4111}	C_{4121}	C_{4132}	C_{4211}	C_{4221}	C_{4233}	C_{4311}	C_{4321}	C_{4333}
A_1	0.192184	0.206914	0.208483	0.215734	0.15922	0.210224	0.215845	0.226015	0.218601
A_2	0.177786	0.227303	0.168572	0.212873	0.207351	0.197373	0.154197	0.217984	0.182745
A_3	0.196196	0.189803	0.206752	0.220067	0.191816	0.214614	0.216837	0.179685	0.224136
A_4	0.220721	0.192758	0.192277	0.150233	0.21168	0.178867	0.199068	0.179685	0.187158
A_5	0.213114	0.183222	0.223916	0.201092	0.229933	0.198922	0.214054	0.196632	0.187359

Then we apply the TOPSIS Method.

We normalize the decision matrix as shown in Table 11.

Then we compute the weighted decision matrix as shown in Table 12.

Table 11. Normalized values by TOPSIS method.

	C_{4111}	C_{4121}	C_{4132}	C_{4211}	C_{4221}	C_{4233}	C_{4311}	C_{4321}	C_{4333}
A_1	0.428484	0.461256	0.46417	0.478473	0.353546	0.46917	0.479264	0.50308	0.486923
A_2	0.396383	0.506708	0.375312	0.472127	0.46042	0.44049	0.34238	0.485203	0.407056
A_3	0.43743	0.423112	0.460318	0.488082	0.425926	0.478966	0.481466	0.399955	0.499251
A_4	0.492108	0.4297	0.42809	0.333199	0.470033	0.399188	0.442011	0.399955	0.416885
A_5	0.47515	0.408441	0.49853	0.445997	0.510564	0.443947	0.475287	0.437676	0.417333

Table 12. Weighted Normalized values by TOPSIS method.

	C_{4111}	C_{4121}	C_{4132}	C_{4211}	C_{4221}	C_{4233}	C_{4311}	C_{4321}	C_{4333}
A_1	0.012531	0.013911	0.020494	0.041838	0.025552	0.009155	0.035775	0.02295	0.018632
A_2	0.011592	0.015281	0.016571	0.041283	0.033276	0.008596	0.025557	0.022134	0.015576
A_3	0.012792	0.01276	0.020324	0.042678	0.030783	0.009347	0.035939	0.018245	0.019103
A_4	0.014391	0.012959	0.018901	0.029135	0.033971	0.00779	0.032994	0.018245	0.015952
A_5	0.013895	0.012318	0.022011	0.038998	0.0369	0.008663	0.035478	0.019966	0.015969

Finally, we obtain the criteria weights of each TreeSoft as shown in Table 13. Then we obtain the final ranks of each TreeSoft as shown in Table 14. Then we combine the ranks of the alternatives in single rank.

Table 13. Criteria weights of each TreeSoft Set.

	C_{4111}	C_{4121}	C_{4132}	C_{4211}	C_{4221}	C_{4233}	C_{4311}	C_{4321}	C_{4333}

First Criterion	0.143	0.039	0.161	0.152	0.143	0.039	0.161	0.152	0.012
Second Criterion	0.043	0.010	0.066	0.078	0.022	0.035	0.066	0.040	0.012
Third Criterion	0.064	0.010	0.066	0.088	0.111	0.020	0.047	0.011	0.033
Fourth Criterion	0.029	0.030	0.044	0.087	0.072	0.020	0.075	0.046	0.038

Table 14. Ranks of alternatives for each TreeSoft Set.

	First Criterion	Second Criterion	Third Criterion	Fourth Criterion	Combined
A_1	4	5	2	3	2
A_2	1	2	3	4	3
A_3	3	1	1	2	1
A_4	5	3	5	5	5
A_5	2	4	4	1	4

5. Conclusions

This study used two MCDM methods such as Entropy method to compute the criteria weights and ranking the alternatives. We use the Forest HyperSoft set to divided each criterion as TreeSoft set and we compute the criteria weights and ranking the alternatives in each TreeSoft set. We used four criteria and seven alternatives in this study.

The quality assessment of preschool education in the new era requires a multifaceted evaluation approach that moves beyond traditional academic indicators. This research highlights teaching methodologies, teacher competence, learning environments, parental involvement, and institutional policies collectively shape the effectiveness of early childhood education. By integrating holistic child development measures, assessment frameworks can ensure that preschools cater to both intellectual and emotional growth.

One of the key findings is that educational diversity must be respected in quality assessments. While structured academic models emphasize early literacy and numeracy, child-centric approaches such as Montessori and Reggio Emilia focus on creativity and social development. The study underscores that quality evaluation should balance structured learning with experiential, play-based, and digital learning elements, ensuring that children develop essential life skills alongside foundational knowledge.

Additionally, the research confirms that teacher quality and parental engagement significantly influence preschool learning outcomes. Schools that invest in continuous teacher training, personalized learning approaches, and parent-school collaboration initiatives demonstrate higher levels of student progress. As digital education tools become more prevalent, their role in

adaptive learning and real-time assessment must be integrated into evaluation models while ensuring that child-friendly and ethical AI applications are prioritized.

A dynamic, data-driven approach to preschool education assessment is necessary to accommodate the evolving demands of modern education. By adopting multi-criteria decision-making models, educators and policymakers can implement a scalable, inclusive, and developmentally appropriate quality assessment framework, ensuring that preschool education remains effective, engaging, and aligned with the needs of future generations.

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The Importance of Risk Disclosure in Financial Management: A Neutrosophic Multi-Criteria Decision-Making Approach

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Abstract: Risk disclosure plays a vital role in promoting transparency, stability, and efficiency in stock markets by providing investors with the necessary information to make well-informed decisions. However, its effectiveness varies across companies, industries, and regulatory frameworks, influencing market volatility, investor confidence, and overall financial stability. This study applies a Multi-Criteria Decision-Making (MCDM) methodology to analyze different factors affecting risk disclosure. Specifically, we use the Entropy method to determine the weight of each criterion and the MAIRCA method to rank alternatives based on their importance. To address uncertainty in the evaluation process, we incorporate the Type-2 Neutrosophic Set as a mathematical approach to deal with imprecise or incomplete information. To ensure the reliability and robustness of our findings, we conducted sensitivity and comparative analyses, testing the stability of the rankings and the effectiveness of our proposed approach. This study highlights the critical role of risk disclosure in financial management and provides a quantitative framework that can help companies and investors assess risks more accurately and make better financial decisions.

Keywords: Financial Management; Risk Management; Risk Disclosure; Multi-Criteria Decision Making; Neutrosophic Set.

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1. Introduction

Risk disclosure is a fundamental component of financial management, ensuring transparency, stability, and informed decision-making in the corporate and investment landscape. In an increasingly complex financial environment, the need for clear and accurate risk disclosure has grown significantly, as businesses, investors, and regulators seek to mitigate uncertainties and enhance market confidence. Transparent risk disclosure promotes financial stability. It fosters trust among stakeholders.

The process involves revealing financial, operational, and strategic risks that may impact an organization's performance, enabling stakeholders to make data-driven decisions. The effectiveness of risk disclosure can determine the financial resilience of firms and influence broader economic stability[1], [2]. Accurate risk disclosure is essential for informed decision-making.

Regulatory compliance plays a crucial role in shaping risk disclosure practices, as governments and financial institutions impose stringent guidelines to ensure accuracy and prevent misleading financial reporting. Effective risk disclosure enhances investor confidence and promotes market stability.

Regulatory frameworks mandate companies to disclose potential risks that could affect their financial standing. The level of adherence to these regulations varies across industries and regions, influencing the consistency and reliability of disclosed information. Failure to comply with risk disclosure requirements can lead to severe legal penalties, financial losses, and reputational damage, making compliance a priority for financial management[3], [4].

Investor confidence is closely tied to the quality and comprehensiveness of risk disclosure. Investors rely on financial reports, risk assessments, and market analysis to make informed decisions about stock purchases, asset allocation, and long-term investments. Transparent risk disclosure helps build trust in financial markets by reducing uncertainty and providing stakeholders with a clear understanding of potential vulnerabilities. When companies openly communicate their financial risks, they demonstrate accountability and responsibility, leading to greater investor engagement and market stability. Conversely, inadequate disclosure can erode confidence, triggering volatility and adverse market reactions[5], [6].

From a corporate governance perspective, risk disclosure is instrumental in promoting ethical decision-making and strategic planning. Corporate boards and executives must balance growth initiatives with risk management strategies, ensuring that financial sustainability is not compromised by unforeseen challenges. High-quality risk disclosure enhances internal decision-making by allowing firms to identify, assess, and mitigate risks effectively. It also strengthens stakeholder relationships, as transparent communication fosters long-term partnerships with investors, creditors, and regulatory authorities. Companies with robust risk disclosure practices often experience improved access to capital and higher market valuations, as they are perceived as more reliable and trustworthy[7], [8].

Evaluating the importance of risk disclosure in financial management is essential for maintaining market integrity, protecting investor interests, and promoting sustainable corporate growth. Regulatory compliance, investor confidence, corporate governance, and technological integration all play significant roles in shaping effective risk disclosure practices. As financial markets become increasingly interconnected and complex, businesses must prioritize transparent risk communication to navigate uncertainties and foster a resilient financial ecosystem. Future research and industry efforts should focus on enhancing risk disclosure frameworks, leveraging

emerging technologies, and developing standardized global reporting practices to ensure consistency and reliability in financial risk assessments. Multi-criteria decision making (MCDM) is used for evaluating the importance of risk disclosure in financial management.

By taking linguistic factors into account, the fuzzy set theory is founded on the idea of membership function value. This theory compares evaluations with membership function values and attempts to assess unclear data using expert views. Expert judgments do not, however, always center on membership ideals, particularly in situations when experience and competence are lacking. Furthermore, rather than being certain that their ideas are true, the experts might be certain that they are false. The neutrosophic set theory was presented by Smarandache to address these practical decision-making scenarios. Fuzzy theory and intuitionistic fuzzy sets (IFS) are extended by neutrophilic sets[9], [10].

It has been established that neutrophilic counts are a legitimate area of research for determining indeterminate and incompatible information. A triplet (X, Y, Z) is used to represent type-1 neutrosophic numbers, with each X, Y, and Z belonging to the interval [0, 1]. Their respective names are non-membership or falsity value (Z), neutral value or indeterminacy (Y), and membership or truth value (X). Since each neutrosophic component is divided into its truth, indeterminacy, and falsity subparts, T2NN is written as $< (X_T, X_I, X_F), (Y_T, Y_I, Y_F), (Z_T, Z_I, Z_F) >$. T2NN is an example of a sophisticated neutrosophic method. It is a useful technique for handling the ambiguity or incompleteness of expertise[11], [12].

1.1 Financial Management and MCDM

Business and financial decision-making issues have become more complicated and unpredictable because of the recent economic crisis, the globalization of financial markets, continuous social and technical advancements, and the new regulatory framework. Given the present financial obligations, this scenario presents additional problems that must be overcome. Tools like discriminant analysis, logistic regression, multivariate adaptive regression splines, and probit analysis are examples of statistical and economic models that are frequently used to make financial choices[13]. However, there are several issues with these methods that make it challenging to include qualitative factors in the decision-making process. New methods based on the principles of operations research (such as mathematical programming and multi-criteria decision-making techniques) and computational intelligence have been developed to address these drawbacks.

Because of the numerous aspects that make these operations research and management tools particularly well-suited for examining intricate real-world issues, MCDM models have gained significant traction in a variety of financial application areas in recent years. The ability of most MCDM approaches to handle both quantitative and qualitative data, as well as expert preferences and/or subjective judgments, is one of its primary features. Typically, a decision-making process includes the following steps: defining objectives, selecting criteria to measure them, identifying alternatives, converting criterion scales into comparable units, allocating weights to the criteria

based on their relative importance, and using an algorithm to rank and select an alternative[14], [15].

To make "optimal" decisions based on the decision-makers' preference judgment, MCDM consists of a set of analytical techniques and methods for assessing the advantages and disadvantages of competing alternatives, solutions, or actions based on multiple (typically conflicting) criteria, attributes, or objectives of different nature. In this context, optimality indicates that the answer meets the decision-maker's preferences and is not overshadowed by alternative options, indicating that some compromise between the many points of view is necessary. A $m \times n$ decision matrix, where M is the number of possibilities and N is the number of decision criteria, can be used to represent a standard MCDM situation[16], [17].

Three categories of problems may be distinguished in MCDM: sorting, ranking, and choosing. Finding the best alternatives is the goal when choosing; the ranking approach gives the set of alternatives a preference order, typically from best to worst[18], [19]; and the sorting (or ordinal classification) process assigns each alternative to a category based on scores on specific criteria. Six steps make up a general procedure for solving multi-criteria problems:

(1) identifying and quantifying the relevant objectives.

(2) defining the decision between variables and constraints.

(3) gathering data.

(4) generating and valuing alternatives.

(5) choosing alternatives based on the decision-maker's preferences; and

(6) putting the chosen alternatives into practice.

1.2 Contributions of this study

We propose a decision-making approach for Evaluating the importance of risk disclosure in financial management.

We use two MCDM methods, such as Entropy methodology to compute the criteria weights and MAIRCA methodology to rank the alternatives. These methods are used under T2NN to overcome uncertainty information.

Nine criteria and nine alternatives are used in this study to show the best criterion and best alternative.

Sensitivity analysis is conducted to show the stability of the ranks under different cases in the criteria weights.

1.3 Organization of this study

The rest of this study is organized as follows: Section 2 shows the operations of T2NNs with the steps of the two MCDM methods to compute the criteria weights and ranking the alternatives. Section 3 shows the results of the proposed approach under the T2NN. Sensitivity analysis is conducted in section 4 to show the stability of the ranks of the alternatives. Section 5 shows the conclusion of this study.

2. Type-2 Neutrosophic-Entropy- MAIRCA (T2N-Entropy-MAIRCA)

This section shows the steps of the proposed approach under T2N to deal with uncertainty and vague information. We use the Entropy method to compute the criteria weights and the MAIRCA method to rank the alternatives.

Operations of T2NN [12]

Let two type-2 neutrosophic numbers (T2NNs) such as:

$$B_1 = \left(\left(X_{T_{B_1}}(d), X_{I_{B_1}}(d), X_{F_{B_1}}(d)\right), \left(Y_{T_{B_1}}(d), Y_{I_{B_1}}(d), Y_{F_{B_1}}(d)\right), \left(Z_{T_{B_1}}(d), Z_{I_{B_1}}(d), Z_{F_{B_1}}(d)\right)\right) \text{ and}$$

$$B_2 = \left(\left(X_{T_{B_2}}(d), X_{I_{B_2}}(d), X_{F_{B_2}}(d)\right), \left(Y_{T_{B_2}}(d), Y_{I_{B_2}}(d), Y_{F_{B_2}}(d)\right), \left(Z_{T_{B_2}}(d), Z_{I_{B_2}}(d), Z_{F_{B_2}}(d)\right)\right)$$

$$B_1 \oplus B_2 = \begin{pmatrix} \begin{pmatrix} X_{T_{B_1}}(d) + X_{T_{B_2}}(d) - T_{T_{A_1}}(x) X_{T_{B_2}}(d), \\ X_{I_{B_1}}(d) + X_{I_{B_2}}(d) - X_{I_{B_1}}(x) X_{I_{B_2}}(d), \\ X_{F_{B_1}}(d) + X_{F_{B_2}}(d) - X_{F_{B_1}}(x) X_{F_{B_2}}(d) \end{pmatrix}, \\ \left(Y_{T_{B_1}}(d) Y_{T_{B_2}}(d), Y_{I_{B_1}}(d) Y_{I_{B_2}}(d), Y_{F_{B_1}}(d) Y_{F_{B_2}}(d)\right), \\ \left(Z_{T_{B_1}}(d) Z_{T_{B_2}}(d), Z_{I_{B_1}}(d) Z_{I_{B_2}}(d), Z_{F_{B_1}}(d) Z_{F_{B_2}}(d)\right) \end{pmatrix} \quad (1)$$

$$B_1 \otimes B_2 = \begin{pmatrix} \left(X_{T_{B_1}}(d) X_{T_{B_2}}(d), X_{I_{B_1}}(d) X_{I_{B_2}}(d), X_{F_{B_1}}(d) X_{F_{B_2}}(d)\right), \\ \begin{pmatrix} Y_{T_{B_1}}(d) + Y_{T_{B_2}}(d) - Y_{T_{B_1}}(x) Y_{T_{B_2}}(d), \\ Y_{I_{B_1}}(d) + Y_{I_{B_2}}(d) - Y_{I_{B_1}}(x) Y_{I_{B_2}}(d), \\ Y_{F_{B_1}}(d) + Y_{F_{B_2}}(d) - Y_{F_{B_1}}(x) Y_{F_{B_2}}(d) \end{pmatrix}, \\ \begin{pmatrix} Z_{T_{B_1}}(d) + Z_{T_{B_2}}(d) - Z_{T_{B_1}}(x) Z_{T_{B_2}}(d), \\ Z_{I_{B_1}}(d) + Z_{I_{B_2}}(d) - Z_{I_{B_1}}(x) Z_{I_{B_2}}(d), \\ Z_{F_{B_1}}(d) + Z_{F_{B_2}}(d) - Z_{F_{B_1}}(x) Z_{F_{B_2}}(d) \end{pmatrix} \end{pmatrix} \quad (2)$$

$$\curlywedge B_1 = \begin{pmatrix} \begin{pmatrix} \left(1-\left(1-X_{T_{B_1}}(d)\right)^{\curlywedge}\right), \\ \left(1-\left(1-X_{I_{B_1}}(d)\right)^{\curlywedge}\right), \\ \left(1-\left(1-X_{F_{B_1}}(d)\right)^{\curlywedge}\right) \end{pmatrix}, \\ \left(\left(Y_{T_{B_1}}(d)\right)^{\curlywedge}, \left(Y_{I_{B_1}}(d)\right)^{\curlywedge}, \left(Y_{F_{B_1}}(d)\right)^{\curlywedge}\right), \\ \left(\left(Z_{T_{B_1}}(d)\right)^{\curlywedge}, \left(Z_{I_{B_1}}(d)\right)^{\curlywedge}, \left(Z_{F_{B_1}}(d)\right)^{\curlywedge}\right) \end{pmatrix} \tag{3}$$

$$B_1^{\curlywedge} = \begin{pmatrix} \left(\left(X_{T_{B_1}}(d)\right)^{\curlywedge}, \left(X_{I_{B_1}}(d)\right)^{\curlywedge}, \left(X_{F_{B_1}}(d)\right)^{\curlywedge}\right), \\ \begin{pmatrix} \left(1-\left(1-Y_{T_{B_1}}(d)\right)^{\curlywedge}\right), \\ \left(1-\left(1-Y_{I_{B_1}}(d)\right)^{\curlywedge}\right), \\ \left(1-\left(1-Y_{F_{B_1}}(d)\right)^{\curlywedge}\right) \end{pmatrix}, \\ \begin{pmatrix} \left(1-\left(1-Z_{T_{B_1}}(d)\right)^{\curlywedge}\right), \\ \left(1-\left(1-Z_{I_{B_1}}(d)\right)^{\curlywedge}\right), \\ \left(1-\left(1-Z_{F_{B_1}}(d)\right)^{\curlywedge}\right) \end{pmatrix} \end{pmatrix} \tag{4}$$

T2N-Entropy

We use the entropy method to compute the criteria weights[20], [21].

S1-Create the decision matrix

The decision matrix is created

$$F = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix}; i = 1, \dots, m; j = 1, \dots, n \tag{5}$$

S2-Obtain crip values

We apply T2N score function to obtain crisp values.

S3-Combine the decision matrix.

The decision matrixes are combined into one matrix.

S4-Obtain the normalized decision matrix.

The decision matrix is normalized as:

$$k_{ij} = f_{ij} / \sum_{i=1}^{m} f_{ij} \quad (6)$$

S5-Determine the entropy value.

The entropy value is determined as:

$$e_j = -h \sum_{i=1}^{m} k_{ij} \ln k_{ij} \quad (7)$$

$$h = \frac{1}{\ln m} \quad (8)$$

S6-Determine the criteria weights.

The criteria weights are computed as:

$$w_j = \left(1 - e_j\right) / \sum_{j=1}^{n} \left(1 - e_j\right) \quad (9)$$

T2N-MAIRCA

This method is used to rank the alternatives. We use the combined decision matrix from the Entropy method to apply the steps of the MAIRCA method[22], [23].

S7-Determine the components of $t_{p_{ij}}$ of the theoretical rating matrix between the criteria and alternatives.

$$t_{p_{ij}} = f_{ij} * w_j \quad (10)$$

S8-Determine the components of the real rating matrix such as:

$$y_{ij} = \left(f_{ij} - \min_i f_{ij}\right) / (\max_i f_{ij} - \min_i f_{ij})\ positive\ criteria \quad (11)$$

$$y_{ij} = \left(f_{ij} - \max_i f_{ij}\right) / (\min_i f_{ij} - \max_i f_{ij})\ negative\ criteria \quad (12)$$

Then we multiply the results of y_{ij} by the $t_{p_{ij}}$ such as:

$$V_{ij} = t_{p_{ij}} * y_{ij} \quad (13)$$

S9-Dternine the sum of gap matrix.

We compute the sum of the gap matrix such as:

$$r_{ij} = t_{p_{ij}} - V_{ij} \quad (14)$$

S10-Determine the score value of each alternative, then we rank the alternatives based on the highest score value.

$$Q_i = \sum_{i=1}^{m} r_{ij} \tag{15}$$

3. Results and Discussion

This section shows the results of Risk Disclosure in Financial Management. This study uses nine main criteria and nine alternatives to evaluate the Risk Disclosure in Financial Management. These criteria can aid the ranking methodology to rank the alternatives. The criteria of this study are:

- Financial Performance Impact
- Risk Identification and Mitigation
- Market Stability
- Decision-Making for Stakeholders
- Technological Integration in Disclosure
- Corporate Governance
- Transparency and Reporting Quality
- Investor Confidence
- Regulatory Compliance

The alternatives of this study are:

- Real-Time Risk Monitoring Systems
- Investor Education Programs
- Blockchain-Based Transparency Systems
- Third-Party Audits and Independent Risk Assessments
- Voluntary Risk Disclosure Enhancement
- Integrated Reporting Approach
- Mandatory Risk Reporting
- Standardized Risk Disclosure Frameworks
- AI-Driven Risk Analysis

Results of T2N-Entropy

S1-S3-Three decision makers use the T2NNs to create the decision matrix as shown in Tables 1-3. Then we replaced these numbers with crisp values. Then we combine these matrices into a single matrix.

S4-Then we normalized the decision matrix using Eq. (6) as shown in Table 4.

S5-Then we determine the entropy value using Eq. (7).

S6-Then we determine the criteria weights using Eq. (9) as shown in Fig 1.

Table 1. First decision matrix.

	RCFM₁	RCFM₂	RCFM₃	RCFM₄	RCFM₅	RCFM₆	RCFM₇	RCFM₈	RCFM₉
RCFMA₁	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))
RCFMA₂	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))
RCFMA₃	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))
RCFMA₄	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))
RCFMA₅	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))
RCFMA₆	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))
RCFMA₇	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))
RCFMA₈	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))
RCFMA₉	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))

Table 2. Second decision matrix.

	RCFM₁	RCFM₂	RCFM₃	RCFM₄	RCFM₅	RCFM₆	RCFM₇	RCFM₈	RCFM₉
RCFMA₁	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))
RCFMA₂	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))
RCFMA₃	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))
RCFMA₄	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))
RCFMA₅	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))
RCFMA₆	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))
RCFMA₇	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))
RCFMA₈	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))
RCFMA₉	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))

Table 3. Third decision matrix.

	RCFM₁	RCFM₂	RCFM₃	RCFM₄	RCFM₅	RCFM₆	RCFM₇	RCFM₈	RCFM₉
RCFMA₁	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))
RCFMA₂	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))
RCFMA₃	((0.20,0.20,0.10),(0.65,0.80,0.85),(0.45,0.80,0.70))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))
RCFMA₄	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))
RCFMA₅	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))
RCFMA₆	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.60,0.45,0.50),(0.20,0.15,0.25),(0.10,0.25,0.15))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))
RCFMA₇	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.35,0.35,0.10),(0.50,0.75,0.80),(0.50,0.75,0.65))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))
RCFMA₈	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.50,0.30,0.50),(0.50,0.35,0.45),(0.45,0.30,0.60))	((0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20))	((0.95,0.90,0.95),(0.10,0.10,0.05),(0.05,0.05,0.05))

RCFMA9	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.60,0.45,0.50), (0.20,0.15,0.25), (0.10,0.25,0.15))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.40,0.45,0.50), (0.40,0.45,0.50), (0.35,0.40,0.45))	((0.35,0.35,0.10), (0.50,0.75,0.80), (0.50,0.75,0.65))	((0.50,0.30,0.50), (0.50,0.35,0.45), (0.45,0.30,0.60))

Table 4. Normalization decision matrix.

	RCFM1	RCFM2	RCFM3	RCFM4	RCFM5	RCFM6	RCFM7	RCFM8	RCFM9
RCFMA1	0.099187	0.101832	0.116047	0.135602	0.138812	0.101228	0.136655	0.135359	0.110945
RCFMA2	0.145699	0.112877	0.06709	0.100505	0.096216	0.139154	0.120996	0.062922	0.113038
RCFMA3	0.105071	0.101562	0.116047	0.135602	0.127788	0.102865	0.101068	0.128607	0.114833
RCFMA4	0.131129	0.073815	0.127531	0.098644	0.047106	0.115416	0.08968	0.143646	0.152512
RCFMA5	0.119361	0.127155	0.141432	0.111407	0.117013	0.106958	0.123369	0.11725	0.091507
RCFMA6	0.105912	0.15167	0.093986	0.073385	0.119519	0.101501	0.136655	0.100061	0.101675
RCFMA7	0.145699	0.102909	0.096102	0.082159	0.095214	0.115689	0.100356	0.108656	0.091507
RCFMA8	0.085738	0.102371	0.113327	0.138527	0.130544	0.104229	0.090154	0.099141	0.110945
RCFMA9	0.062202	0.125808	0.128438	0.124169	0.127788	0.11296	0.101068	0.104359	0.113038

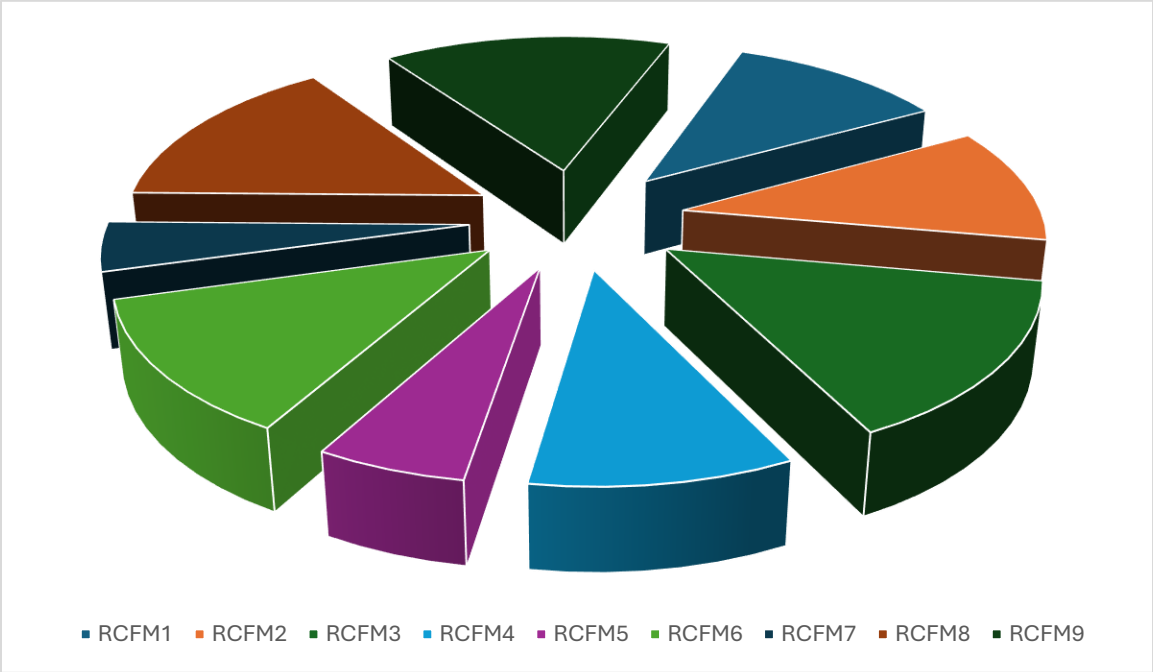

Fig 1. The criteria weights.

T2N-MAIRCA

S7-We determine the components of $t_{p_{ij}}$ of the theoretical rating matrix between the criteria and alternatives using Eq. (10) as shown in Table 5.

S8-We determine the components of the real rating matrix using Eqs. (11 and 12) as shown in Table 6.

Then we multiply the results of y_{ij} by the $t_{p_{ij}}$ using Eq. (13) as shown in Table 7.

S9-Then we determine the sum of gap matrix using Eq. (14) as shown in Table 8.

S10-then we determine the score value of each alternative using Eq. (15) as shown in Fig 2. Then we rank the alternatives as shown in Fig 3.

Table 5. The theoretical rating matrix.

	RCFM$_1$	RCFM$_2$	RCFM$_3$	RCFM$_4$	RCFM$_5$	RCFM$_6$	RCFM$_7$	RCFM$_8$	RCFM$_9$
RCFMA$_1$	0.05610039	0.056913	0.079279	0.069672	0.043236	0.064662	0.036724	0.092195	0.078515
RCFMA$_2$	0.082407353	0.063086	0.045833	0.051639	0.029969	0.088889	0.032516	0.042857	0.079997
RCFMA$_3$	0.059428379	0.056763	0.079279	0.069672	0.039802	0.065708	0.02716	0.087596	0.081267
RCFMA$_4$	0.074166617	0.041254	0.087124	0.050683	0.014672	0.073726	0.0241	0.097839	0.107932
RCFMA$_5$	0.067510639	0.071066	0.096621	0.05724	0.036446	0.068323	0.033153	0.07986	0.064759
RCFMA$_6$	0.059903806	0.084767	0.064208	0.037705	0.037227	0.064837	0.036724	0.068153	0.071955
RCFMA$_7$	0.082407353	0.057515	0.065653	0.042213	0.029656	0.0739	0.026969	0.074007	0.064759
RCFMA$_8$	0.048493558	0.057214	0.077421	0.071174	0.040661	0.06658	0.024227	0.067526	0.078515
RCFMA$_9$	0.035181601	0.070313	0.087744	0.063797	0.039802	0.072157	0.02716	0.07108	0.079997

Table 6. The real rating matrix.

	RCFM$_1$	RCFM$_2$	RCFM$_3$	RCFM$_4$	RCFM$_5$	RCFM$_6$	RCFM$_7$	RCFM$_8$	RCFM$_9$
RCFMA$_1$	0.442953	0.359862	0.658537	0.955102	1	0	1	0.897338	0.318627
RCFMA$_2$	1	0.50173	0	0.416327	0.535519	1	0.666667	0	0.352941
RCFMA$_3$	0.513423	0.356401	0.658537	0.955102	0.879781	0.043165	0.242424	0.813688	0.382353
RCFMA$_4$	0.825503	0	0.813008	0.387755	0	0.374101	0	1	1
RCFMA$_5$	0.684564	0.685121	1	0.583673	0.762295	0.151079	0.717172	0.673004	0
RCFMA$_6$	0.52349	1	0.361789	0	0.789617	0.007194	1	0.460076	0.166667
RCFMA$_7$	1	0.373702	0.390244	0.134694	0.52459	0.381295	0.227273	0.56654	0
RCFMA$_8$	0.281879	0.366782	0.621951	1	0.909836	0.079137	0.010101	0.448669	0.318627
RCFMA$_9$	0	0.66782	0.825203	0.779592	0.879781	0.309353	0.242424	0.513308	0.352941

Table 7. The multiplication of the results of y_{ij} by the $t_{p_{ij}}$.

	RCFM$_1$	RCFM$_2$	RCFM$_3$	RCFM$_4$	RCFM$_5$	RCFM$_6$	RCFM$_7$	RCFM$_8$	RCFM$_9$
RCFMA$_1$	0.02485	0.020481	0.052208	0.066543	0.043236	0	0.036724	0.08273	0.025017
RCFMA$_2$	0.082407	0.031652	0	0.021499	0.016049	0.088889	0.021677	0	0.028234
RCFMA$_3$	0.030512	0.02023	0.052208	0.066543	0.035017	0.002836	0.006584	0.071275	0.031073
RCFMA$_4$	0.061225	0	0.070833	0.019652	0	0.027581	0	0.097839	0.107932
RCFMA$_5$	0.046215	0.048689	0.096621	0.033409	0.027783	0.010322	0.023777	0.053746	0
RCFMA$_6$	0.031359	0.084767	0.02323	0	0.029395	0.000466	0.036724	0.031356	0.011992
RCFMA$_7$	0.082407	0.021494	0.025621	0.005686	0.015557	0.028178	0.006129	0.041928	0
RCFMA$_8$	0.013669	0.020985	0.048152	0.071174	0.036994	0.005269	0.000245	0.030297	0.025017
RCFMA$_9$	0	0.046957	0.072406	0.049736	0.035017	0.022322	0.006584	0.036486	0.028234

Table 8. The gap matrix.

	RCFM$_1$	RCFM$_2$	RCFM$_3$	RCFM$_4$	RCFM$_5$	RCFM$_6$	RCFM$_7$	RCFM$_8$	RCFM$_9$
RCFMA$_1$	0.031250553	0.036432	0.027071	0.003128	0	0.064662	0	0.009465	0.053498
RCFMA$_2$	0	0.031434	0.045833	0.03014	0.01392	0	0.010839	0.042857	0.051763
RCFMA$_3$	0.028916493	0.036532	0.027071	0.003128	0.004785	0.062872	0.020576	0.01632	0.050194
RCFMA$_4$	0.012941826	0.041254	0.016292	0.03103	0.014672	0.046145	0.0241	0	0
RCFMA$_5$	0.021295302	0.022377	0	0.023831	0.008663	0.058001	0.009377	0.026114	0.064759
RCFMA$_6$	0.028544767	0	0.040978	0.037705	0.007832	0.06437	0	0.036797	0.059962
RCFMA$_7$	0	0.036022	0.040032	0.036527	0.014099	0.045722	0.02084	0.032079	0.064759
RCFMA$_8$	0.034824233	0.036229	0.029269	0	0.003666	0.061311	0.023983	0.037229	0.053498
RCFMA$_9$	0.035181601	0.023357	0.015337	0.014061	0.004785	0.049835	0.020576	0.034594	0.051763

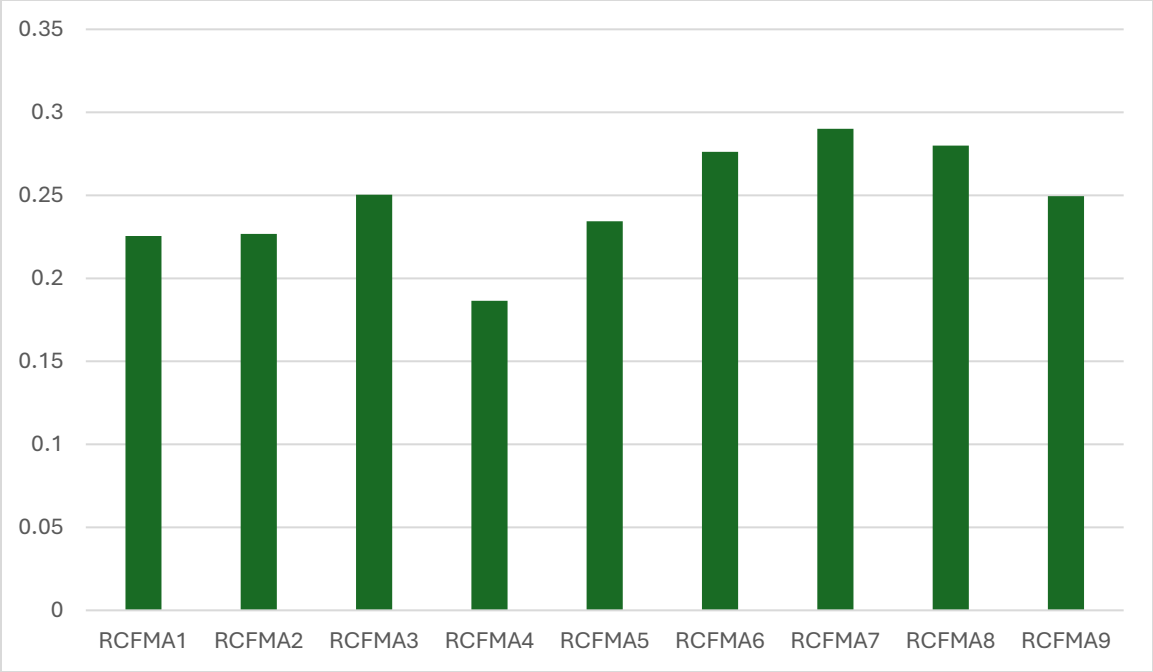

Fig 2. The score value of each alternative.

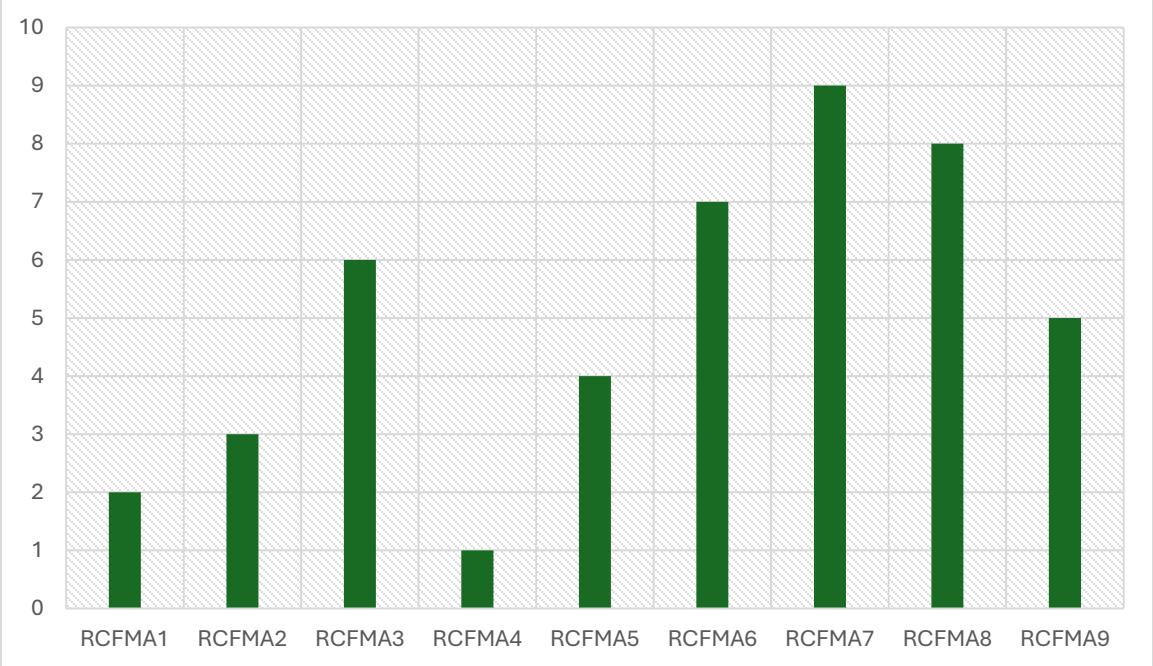

Fig 3. The rank of the alternatives.

4. Analysis

This section shows the sensitivity analysis of the ranks of the alternatives. We conducted this analysis to show the stability of the ranks under different cases. First, we change the criteria weights under ten alternatives.

- First case, the criteria have the same weights. Each criterion has 0.11 weight.
- Second case, first criterion has increased 15% and other criteria have the same weight.
- Third case, second criterion has increased 15% and other criteria have the same weight, and so one to the ten cases. Fig 4 shows the different criteria weights.

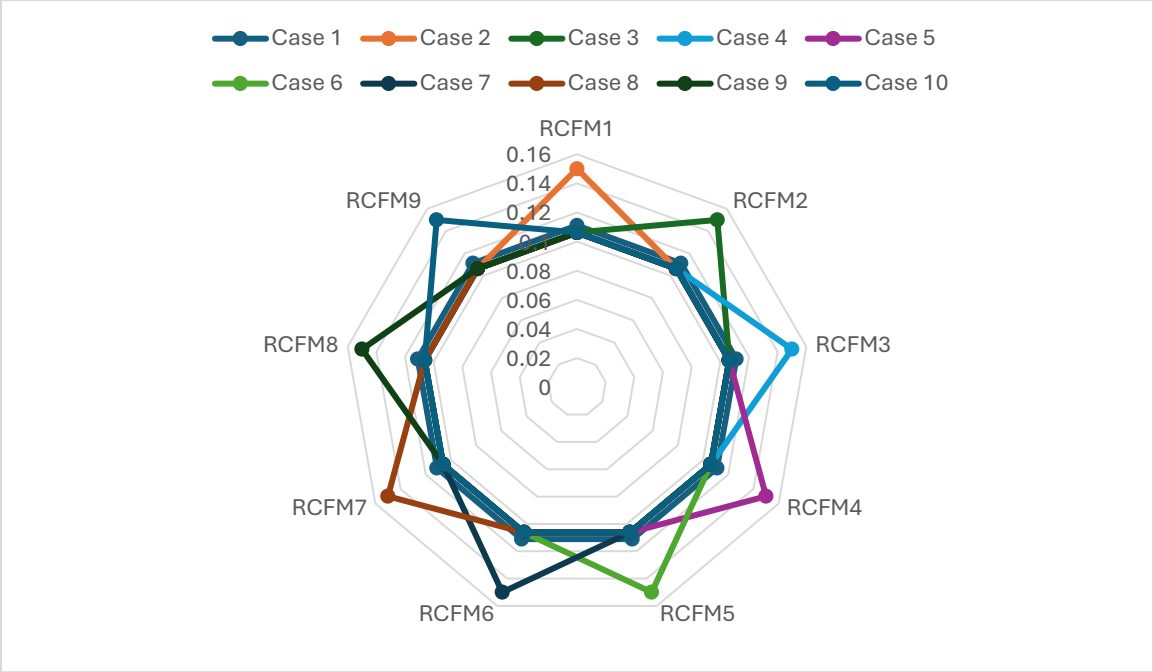

Fig 4 Different ten cases.

Then we apply the ranking method to obtain the score value of each alternative and ranking the alternatives. Fig 5 shows the score values of ten cases. Fig 6 shows the ten ranks of the alternatives under ten cases. We show the ranks of the alternatives as:

- In the first case, we show alternative 7 is the best followed by alternative 8 and alternative 3. We show alternative 1 is the worst.
- In the second case, we show alternative 7 is the best, followed by alternative 8 and alternative 3. We show alternative 1 is the worst.
- In the first case, we show alternative 7 is the best, followed by alternative 8 and alternative 3. We show alternative 1 is the worst.

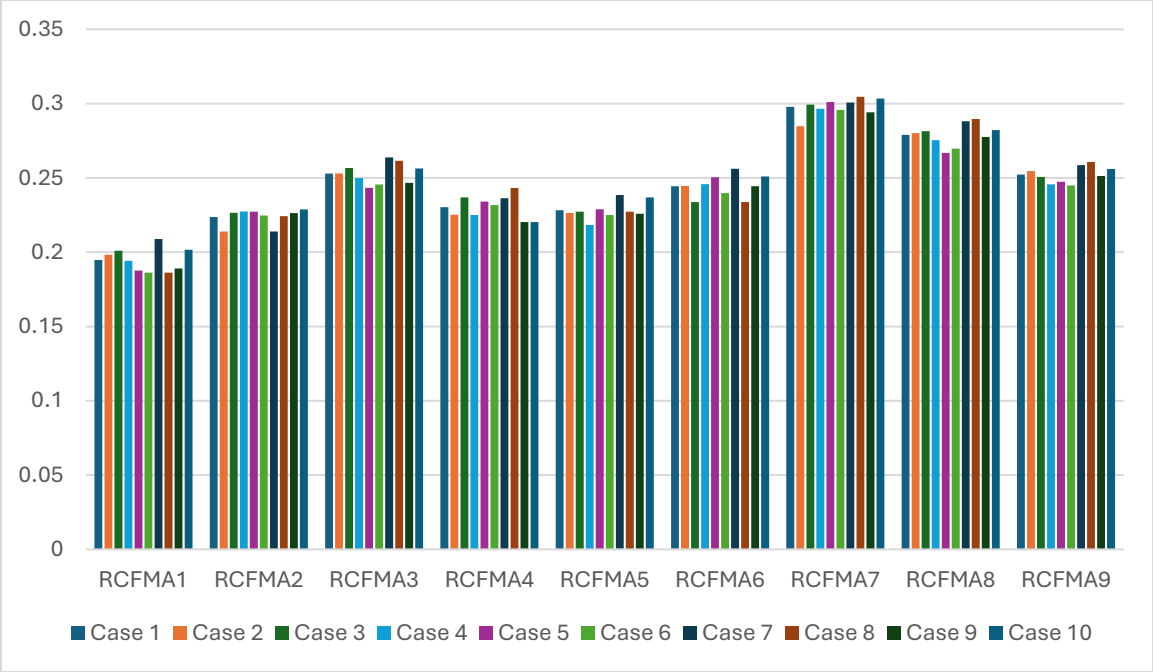

Fig 5. Ten cases of score values.

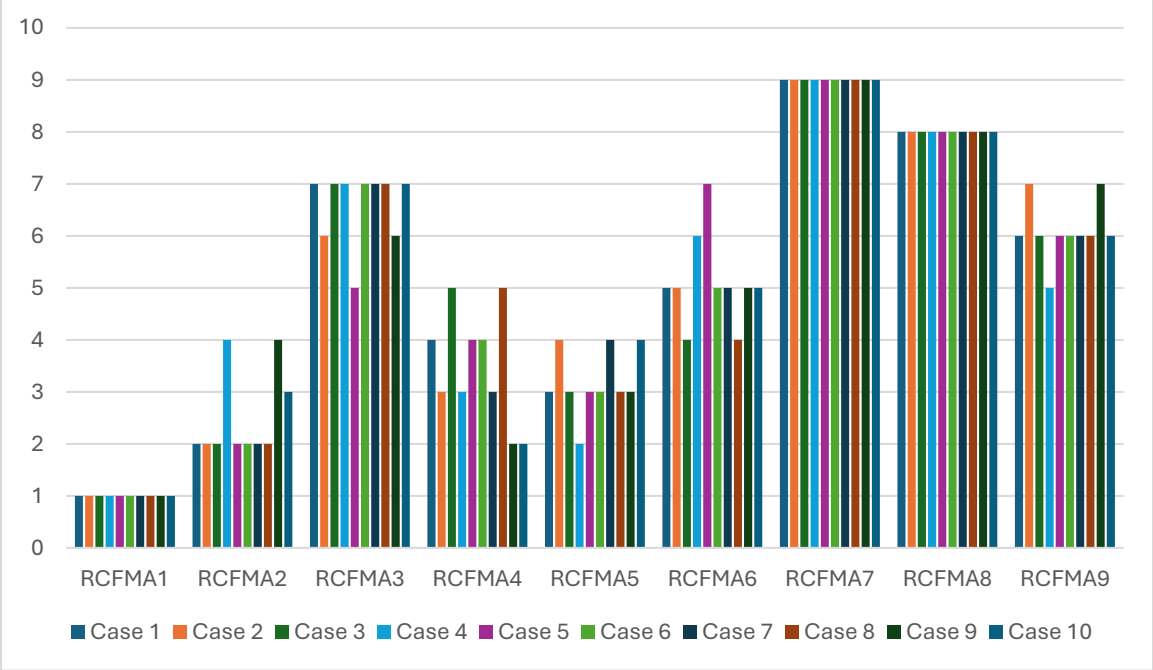

Fig 6. Ten cases of ranking of the alternatives.

5. Conclusions and Future Study

We proposed two MCDM methods such as Entropy method to compute the criteria weights and the MAIRCA method to rank the alternatives. The proposed is used for evaluating the importance of risk disclosure in financial management. We used the type-2 neutrosophic set to deal with uncertainty and vague information. Nine criteria and nine alternatives are used in this study. The results show alternative 7 and alternative 4. The sensitivity analysis was conducted to show the stability of the ranks.

In the future work, the proposed approach can be used to evaluate the MCDM issues to compute the criteria weights and ranking the alternatives. Different MCDM methods can be used in evaluating the importance of risk disclosure in financial management such as AHP, TOPSIS, and VIKOR. Different neutrosophic extensions can be used to overcome the uncertainty and vague information.

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Uncertainty-Driven Entrepreneurial Career Education Decision-Making in College Incubators: A Neutrosophic Vague N-Soft Set Approach

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Abstract: Entrepreneurship education has been demonstrating an important role in shaping the career trajectories of university students. Nevertheless, deciding the entrepreneurial potential leftovers as a complicated procedure because of inherent uncertainty in student characteristics. In this study, we propose a Neutrosophic Vague N-Soft Set-based Gradient Boosting (NVNSS-GB) approach to offer an uncertainty-driven decision about the likeliness of students to become entrepreneurs. Our approach introduces novel application NVNSS theory to transform student attributes into truth, falsity, and indeterminacy membership intervals to provide effective handling of vagueness and hesitancy in their responses. Then, we introduce an N-Soft Set classification method to assign a multi-level graded evaluation to university students. Next, we fed the generated enriched feature representation to GB to decide on if a student is likely to be an entrepreneur. Comprehensive analyses on a real case study is conducted using sensitivity analysis, feature importance ranking, ROC-AUC evaluation, and complexity analysis, demonstrated the efficiency, effectiveness, and interpretability of NVNSS-GB. Our findings implied that NVNSS-GB has promising power in improving decision-making within entrepreneurship education, offering an insightful tool for career guidance that can allow universities to design targeted interventions to foster entrepreneurial skills among students.

Keywords: Neutrosophic Sets (NSs), Soft Sets, Neutrosophic Vague N-Soft Sets, Multi-valued Logic, Entrepreneurship Education, College Incubators.

1. Introduction

Entrepreneurship has become key driver of economic growth and innovation, particularly in the context of local college incubators that support student-led startups [1]. These incubators provide a ecosystem for aspiring entrepreneurs to experiment with business ideas, develop prototypes, and seek funding [2]. Nevertheless, the entrepreneurial process is characteristically uncertain, since decision-makers have to navigate unpredictable market conditions, evolving consumer demands, and financial constraints. The previous decision-making (DM) frameworks often fall short in addressing the complexity and vagueness inherent in early-stage entrepreneurship. To better support student entrepreneurs, there is a pressing need

for advanced computational models that could handle indeterminate, and conflicting information in DM processes [3], [4].

One of the primary challenges in entrepreneurial ecosystems is the inability to effectively quantify uncertainty in DM. Many business models rely in conventional probability-based approaches that assume complete or well-defined information [5]. Nevertheless, real-world entrepreneurial decisions are inclined with multiple conflicting factors, like market dynamics, funding availability, and competitor strategies—many of which are uncertain and partially acknowledged [6]. Also, decision-makers in college incubators, who are first-time entrepreneurs, might struggle with limited access to reliable data, subjective risk perceptions, and cognitive biases. These challenges necessitate a more flexible, intelligent DM framework that could process vague data more excellently.

Neutrosophic logic, by Florentin Smarandache, extended the classical logic systems with inclusion of concept of truth, falsity, and indeterminacy [7], [8]. Unlike earlier systems relied on fixed degrees of truth and falsity, neutrosophic logic allowed for the representation of uncertainty, contradiction, and incomplete information in DM. This makes it particularly suitable for complex domains like entrepreneurship, where real-world decisions often contain ambiguous or conflicting elements.

In the context of entrepreneurship, neutrosophic logic [7] can model uncertain investment decisions, ambiguous customer responses, and unpredictable market shifts, making it an ideal tool for risk assessment and strategic planning in college incubators. However, while neutrosophic logic provides a powerful foundation, it needs to be integrated with structured DM models to be effectively applied in business settings. Building upon the principles of neutrosophic logic, Neutrosophic Vague N-Soft Sets (NVNSS) [9] is a promising solution for making decision analysis based on the evaluation of alternatives under vague, uncertain, and incomplete conditions [9].

Motivated by that, this study aims to explore novel application of NVNS in addressing uncertainty-driven decision-making challenges faced by student entrepreneurs in college incubators. Specifically, we propose a Neutrosophic Vague N-Soft Set-based Gradient Boosting (NVNSS-GB) approach to offer an uncertainty-driven decision about the likeliness of students to become entrepreneurs. Our approach introduces novel application NVNSS theory to transform student attributes into truth, falsity, and indeterminacy membership intervals to provide effective handling of vagueness and hesitancy in their responses. Then, we introduce an N-Soft Set classification method to assign a multi-level graded evaluation to university students. We introduce a real case study on student entrepreneurs in college incubators, and provide in depth exploration of the data to get useful infights about features and factors affecting the entrepreneurial DM**.** Next, using this case study, we drive novel application of our hybrid NVNSS approach to demonstrate the effectiveness in handling uncertainty through real-world scenarios of college incubators. Proof of concept experimental comparisons NVNSS-GB approach decision models with traditional against set of competing methods highlight the advantages of our solutions and their promise to entrepreneurial settings.

The left part of this paper is structured as follows. Section 2 discusses basic concepts and related literature. Section 3 presents the proposed NVNSS-based DM framework and its mathematical implementation of

NVNSS in entrepreneurial DM. Section 4 discussed the experimental results and comparative analysis. Section 5 presents key findings.

2. Background and Basics

This section provides the fundamental concepts and theoretical background necessary to understand the proposed methodology. This introducing mathematical foundations of Neutrosophic Sets, Vague Sets, Soft Sets, and N-Soft Sets, leading to the formulation of Neutrosophic Vague N-Soft Sets (NVNSS).

Definition 1 ([8]**):** Given universe set $U = \{x_1, x_2, \cdots, x_n\}$, A neutrosophic set, A_{NS}, can be defined with of triplet membership functions namely Truth: $T_{A_N}(x)$, Indeterminacy: $I_{A_N}(x)$, and Falsity: $F_{A_N}(x)$.

$$A_{NS} = \left\{\langle x, \left(T_{A_N}(x), I_{A_N}(x), F_{A_N}(x)\right)\rangle : x \in U\right\} \quad (1)$$

such that

$$^{-}0 \leq \sup T_{A_N}(x) + \sup I_{A_N}(x) + \sup F_{A_N}(x) \leq 3^{+}.$$

Each membership function is subset of $]0^{-}, 1^{+}[$, but, for real-world application, the following interval [0,1] is used instead.

Definition 2 ([10]**) :** Given universe set $U = \{x_1, x_2, \cdots, x_n\}$, A vague set, A_V, is defined with two membership functions namely one for truth, $T_{VS}(x)$, and other for falseness $F_{VS}(x)$.

$$A_{VS} = \{\langle x, \left(T_{VS}(x), F_{VS}(x)\right)\rangle : x \in U\} \quad (2)$$

Definition 3: Given two vague sets $A = \{x_i, [t_A(x_i), 1 - f_A(x_i)] \mid x_i \in U\}$ and $B = \{x_i, [t_B(x_i), 1 - f_B(x_i)] \mid x_i \in U\}$, then, the following relations can be computed

The union between A and B is defined as follows

$$A \cup B = \{(x_i, [\max(t_A(x_i), t_B(x_i)), \max(1 - f_A(x_i), 1 - f_B(x_i))] \mid x_i \in U)\}, \quad (3)$$

The intersection between A and B is defined as follows

$$A \cap B = \{(x_i, [\min(t_A(x_i), t_B(x_i)), \min(1 - f_A(x_i), 1 - f_B(x_i))] \mid x_i \in U)\}, \quad (4)$$

The complement of A is defined as follows:

$$A^c = \{(x_i, [f_A(x_i), 1 - t_A(x_i)] \mid x_i \in U)\}. \quad (5)$$

A is subset of if the following condition apply:

$$A \subseteq B \text{ iff } t_A(x_i) \leq t_B(x_i), \text{and } 1 - f_A(x_i) \leq 1 - f_B(x_i) \quad (6)$$

Definition 4 ([9], [11]**):** Given universe set $U = \{x_1, x_2, \cdots, x_n\}$, A neutrosophic vague set, A_{NV}, is defined with three membership functions valued as follows:

$$\begin{aligned} A_{NV} &= \{\langle x, (T_{NV}(x), I_{NV}(x), F_{NV}(x))\rangle : x \in U\} \\ &= \left\{\langle x, \begin{pmatrix} [T_{NV}(x)^-, T_{NV}(x)^+], [I_{NV}(x)^-, I_{NV}(x)^+], \\ [F_{NV}(x)^-, F_{NV}(x)^+] \end{pmatrix}\rangle : x \in U\right\} \\ &= \left\{\langle x, \begin{pmatrix} [T_{NV}(x)^-, 1 - F_{NV}(x)^-], [I_{NV}(x)^-, I_{NV}(x)^+], \\ [F_{NV}(x)^-, 1 - T_{NV}(x)^-] \end{pmatrix}\rangle : x \in U\right\} \end{aligned} \quad (7)$$

Such that

$$^-0 \le T^-_{A_{NV}} + I^-_{A_{NV}} + F^-_{A_{NV}} \le 2^+$$

Definition 5: Given two neutrosophic vague set $A_{NV} = \{\langle x, ([T_{A_{NV}}(x)^-, T_{A_{NV}}(x)^+], [I_{A_{NV}}(x)^-, I_{A_{NV}}(x)^+], [F_{A_{NV}}(x)^-, F_{A_{NV}}(x)^+])\rangle : x \in U\}$ and $B_{NV} = \{\langle x, ([T_{B_{NV}}(x)^-, T_{B_{NV}}(x)^+], [I_{B_{NV}}(x)^-, I_{B_{NV}}(x)^+], [F_{B_{NV}}(x)^-, F_{B_{NV}}(x)^+])\rangle : x \in U\}$, then, the following relations can be computed

The union between A_{NV} and B_{NV} is defined as follows

$$A_{NV} \cup B_{NV} = \left\{\langle x, \begin{pmatrix} [\min(T_{A_{NV}}(x)^-, T_{B_{NV}}(x)^-), \min(T_{A_{NV}}(x)^+, T_{B_{NV}}(x)^+)], \\ [\max(I_{A_{NV}}(x)^-, I_{B_{NV}}(x)^-), \max(I_{A_{NV}}(x)^+, I_{B_{NV}}(x)^+)], \\ [\max(F_{A_{NV}}(x)^-, F_{B_{NV}}(x)^-), \max(F_{A_{NV}}(x)^+, F_{B_{NV}}(x)^+)] \end{pmatrix}\rangle : x \in U\right\}, \quad (8)$$

The intersection between A and B is defined as follows

$$A_{NV} \cup B_{NV} = \left\{\langle x, \begin{pmatrix} [\max(T_{A_{NV}}(x)^-, T_{B_{NV}}(x)^-), \max(T_{A_{NV}}(x)^+, T_{B_{NV}}(x)^+)], \\ [\min(I_{A_{NV}}(x)^-, I_{B_{NV}}(x)^-), \min(I_{A_{NV}}(x)^+, I_{B_{NV}}(x)^+)], \\ [\min(F_{A_{NV}}(x)^-, F_{B_{NV}}(x)^-), \min(F_{A_{NV}}(x)^+, F_{B_{NV}}(x)^+)] \end{pmatrix}\rangle : x \in U\right\}, \quad (9)$$

The complement of A_{NV} is defined as follows:

$$\begin{aligned} {A_{NV}}^c = \{\langle x, ([1 - T_{A_{NV}}(x)^+, 1 - T_{A_{NV}}(x)^-], [1 - I_{A_{NV}}(x)^+, 1 - I_{A_{NV}}(x)^-], [1 - F_{A_{NV}}(x)^+, 1 \\ - F_{A_{NV}}(x)^-])\rangle : x \in U\} \end{aligned} \quad (10)$$

A is subset of if the following condition apply:

$$A \subseteq B_{NV} \text{ iff } t_{A_{NV}}(x_i) \le t_{B_{NV}}(x_i), \text{and } 1 - f_{A_{NV}}(x_i) \le 1 - f_{B_{NV}}(x_i) \quad (11)$$

Definition 6 ([12]**):** Given universe set $U = \{x_1, x_2, \cdots, x_n\}$, and P be a set of parameters that describe properties, attributes, or criteria relevant to elements of U, then a soft set (F, S) over U is defined as a pair:

$$(F, S): F: S \to P(U)\ \forall\ e \in S \text{ such that } S \subseteq P \quad (12)$$

where S is a finite set of parameters. F is a mapping function.

Example 1: Given $U = \{x_1, x_2, x_3, x_4, x_5\}$ denote a set of students, and $P = \{e_1: \text{Good in math}, e_2: \text{Good in programming}, e_3: \text{Good in communication skills}\}$ is a set of parameters, then, we can drive soft set (S, A) with $A = \{e_1, e_2\}$ could be defined as:

$$F(e_1) = \{x_1, x_3, x_5\}$$
$$F(e_2) = \{x_2, x_4\}$$

This means that the students x_1, x_3, x_5 are good in mathematics, while students x_2, x_4 are good in programming.

Definition 7: Given universe set $U = \{x_1, x_2, \cdots, x_n\}$, and P be a set of parameters that describe attributes, or criteria relevant to elements of U, then a neutrosophic vague soft set (NVSS) (F, T) over U is defined as a pair:

$$(F, T), \quad F: T \to NV(U)\ \forall\ e \in T \text{ such that } T \subseteq P \tag{13}$$

Definition 8: Given a group of universe sets $U_1, U_2, \ldots, U_n$, and P as a set of parameters, an N-Soft Set, an extension of soft set theory, is defined over this group

$$(F, S): F: S \to P(U_1) \times P(U_2) \times \ldots \times P(U_n)\ \forall\ e \in S \text{ such that } S \subseteq P \tag{14}$$

where F is a multi-set-valued mapping function

Example 2: Given three different domains of members: students denoted as $U1 = \{x_1, x_2, x_3\}$, mentors denoted as $U_2 = \{y_1, y_2, y_3, y4\}$, and investment firms denoted as $U_3 = \{z_1, z_2, z_3\}$. Also, we have parameters $P = \{e_1: \text{Entrepreneurial skills}, e_2: \text{Technical proficiency}\}$. N-soft set (F, S) over these domains could be defined as:

$$F(e_1) = (\{x_1, x_3\}, \{y_2, y_3\}, \{z_1, z_3\})$$

$$F(e_2) = (\{x_2, x_3\}, \{y_1, y_4\}, \{z_2\})$$

This means that the members that have entrepreneurial skills are students: x_1, x_3, mentors: y_2, y_3, investment firms: z_1, z_3, meanwhile the members that have Technical Proficiency are students x_2, and x_3, mentors: $y_1$and y_4, and the investment Firm z_2.

Definition 9 ([9]): Given universe set $U = \{x_1, x_2, \cdots, x_n\}$ with set of parameters $P = \{e_1, e_2, \ldots, e_m\}$, subset of parameters $T \subseteq E$ under evaluation, and set of ordered grades $G = \{0,1, \ldots, N-1\}$, then Neutrosophic Vague N-Soft Set (NVNSS), denoted as (NV, K), is defined as a mapping:

$$NV: T \to \bigcup_{e_j \in T} \mathcal{NV}\left(NV(e_j)\right) \tag{15}$$

where $\mathcal{NV}\left(NV(e_j)\right)$is a NVSS of parameter $e_j \in T$, and for each object $x_i \in U$ and parameter $e_j \in T$, there is a distinct pair $(x_i, g_{ij}) \in U \times G$. Thus, the definition of NVNSS is expressed as:

$$\mathcal{NV}\left(NV(e_j)\right) = \left\{\left((x_1, g_{1j}), T_{NV}(x_1), I_{NV}(x_1), F_{NV(x_1)}\right), \ldots, \left((x_n, g_{nj}), T_{NV}(x_n), I_{NV}(x_n), F_{NV}(x_n)\right)\right\}$$

Example 2: Consider having three students $U = \{x_1, x_2, x_3\}$, set of attributes E (e_1 =Mathematics Skill, e_2 =Programming Skill, e_3 =Communication Skill), subset of attributes $T = \{e_1, e_2\}$, and set of grades $G = \{0 = \text{Poor}, 1 = \text{Fair}, 2 = \text{Good}, 3 = \text{Excellent}\}$. If we define the NVNSS for mathematics skill criteria as follows.

$$\mathcal{NV}(NV(e_1)) = \left\{\begin{array}{c}((x_1, 3), [0.7,0.9], [0.1,0.3], [0.0,0.2]), ((x_2, 2), [0.5,0.8], [0.2,0.4], [0.1,0.3]), \\ ((x3,1), [0.3,0.6], [0.3,0.5], [0.2,0.4])\end{array}\right\}$$

This means that student x_1 has excellent mathematics skills, with truth-membership [0.7, 0.9], meaning they are highly skilled. Student x_2 has good mathematics skills, but with some uncertainty (I = [0.2, 0.4]). Student x_3 has fair mathematics skills, with more uncertainty (I = [0.3, 0.5]).

Definition 10 ([9]): Given two NVNSS (NV_1, K_1) and (NV_2, K_2) on the universe discourse $U = \{x_1, x_2, \cdots, x_n\}$, then $(NV_1, K_1) \subseteq (NV_2, K_2)$ if and only if

$$\begin{gathered} T^-_{NV_1}(x) \le T^-_{NV_2}(x), T^+_{NV_1}(x) \le T^+_{NV_2}(x); \\ (NV_1, K_1) \subseteq (NV_2, K_2) \text{iff } I^-_{NV_1}(x) \ge I^+_{NV_2}(x), I^+_{NV_1}(x) \ge I^-_{NV_2}(x); \\ F^-_{NV_1}(x) \ge F^-_{NV_2}(x), F^+_{NV_1}(x) \ge F^+_{NV_2}(x). \end{gathered} \tag{16}$$

Definition 11 ([9]): Given two NVNSS (NV_1, K_1) and (NV_2, K_2) on the universe discourse $U = \{x_1, x_2, \cdots, x_n\}$, then (NV_1, K_1) is said to be equal (NV_2, K_2) iff

$$\begin{gathered} (NV_1, K_1) = (NV_2, K_2) \\ \text{iff} \\ (NV_1, K_1) \subseteq (NV_2, K_2) \ \& \ (NV_2, K_2) \subseteq (NV_1, K_1) \end{gathered} \tag{17}$$

Definition 12 ([9]): Given an NVNSS (NV_1, K_1) on the universe discourse $U = \{x_1, x_2, \cdots, x_n\}$, then, it can be called neutrosophic absolute vague N-soft set iff.

$$\begin{gathered} A_{NV}(e_j) = \{< (o_1, N-1), [1,1], [0,0], [0,0] >, \cdots, < (o_n, N-1), [1,1], [0,0], [0,0] >\} \\ \forall e_j \in T \subseteq E. \end{gathered} \tag{18}$$

Definition 13 ([9]): Given two NVNSS (NV_1, K_1) and (NV_2, K_2) on the universe discourse $U = \{x_1, x_2, \cdots, x_n\}$, then, the following relations can be computed:

The restricted intersection between $(NV_1, K_1) \ \widetilde{\cap}_R \ (NV_2, K_2)$ is defined as follows

$$\begin{gathered} K_1 \cap_{\mathcal{R}} K_2 = \big(E, T \cap S, \min(N_1, N_2)\big) \forall e_j \in T \cap S \ x_i \in U \\ < \big(x_i, g_{ij}\big), \hat{T}_{NV}(x_i), \hat{I}_{NV}(x_i), \hat{F}_{NV}(x_i) > \in \eta_{NV}\big(e_j\big) \\ \Leftrightarrow \\ g_{ij} = \min(g^1_{ij}, g^2_{ij}), \\ \hat{T}_{NV}(x_i) = \left[\min\left(T^-_{NV_1}(x_i), T^-_{NV_2}(x_i)\right), \min\left(T^+_{NV_1}(x_i), T^+_{NV_2}(x_i)\right)\right], \\ \hat{I}_{NV}(x_i) = \left[\max\left(I^-_{NV_1}(x_i), I^-_{NV_2}(x_i)\right), \max\left(I^+_{NV_1}(x_i), I^+_{NV_2}(x_i)\right)\right], \text{and} \\ \hat{F}_{NV}(x_i) = \left[\max\left(F^-_{NV_1}(x_i), F^-_{NV_2}(x_i)\right), \max\left(F^+_{NV_1}(x_i), F^+_{NV_2}(x_i)\right)\right]. \end{gathered} \tag{19}$$

The restricted union $(NV_1, K_1) \cup (NV_2, K_2)$ is defined as follows

$$\begin{gathered} K_1 \cup_{\mathcal{R}} K_2 = \big(E, T \cap S, \min(N_1, N_2)\big) \forall e_j \in T \cap S \ x_i \in U \\ g_{ij} = \max(g^1_{ij}, g^2_{ij}), \\ \hat{T}_{NV}(x_i) = [\max(T^-_{NV_1}(x_i), T^-_{NV_2}(x_i)), \max(T^+_{NV_1}(x_i), T^+_{NV_2}(x_i))], \\ \hat{I}_{NV}(x_i) = [\min(I^-_{NV_1}(x_i), I^-_{NV_2}(x_i)), \min(I^+_{NV_1}(x_i), I^+_{NV_2}(x_i))], \text{and} \\ \hat{F}_{NV}(x_i) = [\min(F^-_{NV_1}(x_i), F^-_{NV_2}(x_i)), \min(F^+_{NV_1}(x_i), F^+_{NV_2}(x_i))]. \end{gathered} \tag{20}$$

The complement of (NV_1, K_1) is defined as $(NV_1, K_1)^c \ N_1 \ \widetilde{\cap} \ N_1^c = \phi$, where NV_1^c

$$NV_1^c = \begin{matrix} \widehat{T}^c_{NV_1}(x_i) = [1 - T^+_{NV_1}(x_i), 1 - T^-_{NV_1}(x_i)] = [F^-_{NV_1}(x_i), F^+_{NV_1}(x_i)], \\ \hat{I}^c_{NV_1}(x_i) = [1 - I^+_{NV_1}(x_i), 1 - I^-_{NV_1}(x_i)], \\ \widehat{F}^c_{NV_1}(x_i) = [1 - F^+_{NV_1}(x_i), 1 - F^-_{NV_1}(x_i)] = [T^-_{NV_1}(x_i), T^+_{NV_1}(x_i)]. \end{matrix} \quad (21)$$

3. Proposed Methods

In this section, we introduce the methodology of the proposed NVNSS-integrated approach for modeling entrepreneurial competency based on uncertainty and vagueness in student characteristics while utilizing ML models for accurate classification.

Given $U = \{x_1, x_2, \dots, x_n\}$ be the set of students, and $E = \{e_1, e_2, \dots, e_m\}$ be the set of evaluation attributes (features). Given a dataset D consisting of $|E| = 16$ features, we perform the following preprocessing steps:

In step 1, we replace missing values with "Unknown", while for numerical features, we impute missing values with median. Then, we apply a function f: $C \to \mathbb{Z}$ where each categorical variable is assigned a unique integer label such that $C = \{c_1, c_2, \dots, c_k\}$ be the set of categorical variables.

In step 2, we use a numerical feature $X = \{x_1, x_2, \dots, x_n\}$, we apply Min-Max normalization:

$$X' = \frac{X - \min(X)}{\max(X) - \min(X)} \quad (22)$$

This encodes all numerical values into interval [0,1].

In step 3, for each numerical feature X, we compute the NS memberships as follows

$$\begin{matrix} T^-(x_i) = x_i \\ T^+(x_i) = 1 - (1 - x_i) \times 0.5 \end{matrix} \quad (23)$$

Similarly, we measure falsity membership to signify the degree to which a student lacks a characteristic:

$$\begin{matrix} F^-(x_i) = 1 - x_i \\ F^+(x_i) = 1 - T^-(x_i) \end{matrix} \quad (24)$$

The indeterminacy membership function models uncertainty due to conflicting evidence:

$$\begin{matrix} I^-(x_i) = 1 - \frac{T^-(x_i) + F^-(x_i)}{2} \\ I^+(x_i) = 1 - \frac{T^+(x_i) + F^+(x_i)}{2} \end{matrix} \quad (25)$$

In step 3, we incorporate N -Soft Set arrangement, in which each feature X is discretized into N stages:

$$G(x_i) = \arg\min_k \left\{ x_i \in \left[\frac{k}{N}, \frac{k+1}{N} \right] \right\}, \; k \in \{0,1,2, \dots, N-1\} \quad (26)$$

This way, we assign soft classification level $G(x_i)$ according to N categories.

In step 4, we divide data into features X and target variable y (entrepreneurship potential). The data is split into 80% training and 20% testing:

$$(X_{train}, X_{test}, y_{train}, y_{test}) = \text{split}\,(X, y, \text{test\_size} = 0.2) \quad (27)$$

In step 5, we start building our ML model

$$\hat{y} = f(X_{train}) \tag{28}$$

where f is trained using Gradient Boosting [13]. The main algorithmic steps is as follows
In step 5.1, we begin with an initial constant predictor with loss $L(y_i, F(x))$:

$$F_0(x) = \arg\min_{\gamma} \sum_{i=1}^{n} L(y_i, \gamma)$$

where (29)

$$L(y_i, F(x)) = -\left[y_i \log \sigma\big(F(x)\big) + (1 - y_i) \log\left(1 - \sigma\big(F(x)\big)\right)\right]$$

such that $\sigma(F(x)) = \frac{1}{1+e^{-F(x)}}$

In step 5.2, we calculate pseudo residuals, which indicate the direction of improvement

$$r_i^{(m)} = -\frac{\partial L\big(y_i, F(x_i)\big)}{\partial F(x_i)} \rightarrow m = 1, \ldots, M \tag{30}$$

For log loss, the gradient is:

$$r_i^{(m)} = y_i - \sigma\big(F_{m-1}(x_i)\big)$$

In step 5.3, we fit a weak learner (decision tree) $h_m(x)$ to the residuals:

$$h_m = \arg\min_{h} \sum_{i=1}^{n} \left(r_i^{(m)} - h(x_i)\right)^2 \tag{31}$$

In step 5.4, we compute the optimal shrinkage factor γ_m :

$$\gamma_m = \arg\min_{\gamma} \sum_{i=1}^{n} L\big(y_i, F_{m-1}(x_i) + \gamma h_m(x_i)\big) \tag{32}$$

For log loss, we do the following computation:

$$\gamma_m = \arg\min_{\gamma} \sum_{i=1}^{n} -\left[y_i \log\ \sigma\big(F_{m-1}(x_i) + \gamma h_m(x_i)\big) + (1 - y_i) \log\left(1 - \sigma\big(F_{m-1}(x_i) + \gamma h_m(x_i)\big)\right)\right] \tag{33}$$

In step 5.5, we bring up to date the boosted model, while learning rate $\eta(0 < \eta \leq 1)$ is used to slow learning

$$F_m(x) = F_{m-1}(x) + \gamma_m h_m(x)$$

$$F_m(x) = F_{m-1}(x) + \eta\gamma_m h_m(x) \tag{34}$$

We continue repeating steps 5.2 to 5.5 until reaching maximum iterations M or convergence.

In step 6, we evaluate the performance measured using the following metrics:

$$\text{Accuracy } = \frac{\sum \mathbf{1}(y_i = \hat{y}_i)}{|y|}, \text{ Precision } = \frac{TP}{TP+FP}, \text{ Recall } = \frac{TP}{TP+FN} \tag{35}$$

4. Case Study & Exploratory Analysis

This study focuses on analyzing entrepreneurial competency among university students in India. The dataset, collected in 2019 for research purposes, serves as the foundation for this case study. The primary objective is to apply our approach to predict whether a student is likely to become an entrepreneur based on their personal traits, experiences, and psychological attributes [14]. The dataset consists of 219 entries and 17 columns (16 input features + 1 target variable). It captures various factors influencing students' entrepreneurial competency, including Age, Gender, Education Sector, City, Personality & Psychological Traits, Health & Mental Well-being, External Influence, Barriers to Entrepreneurship, and Individual Project Experience. Target Variable (y) indicates whether a student is likely to become an entrepreneur (1: Yes, 0: No).

	Age	Perseverance	DesireToTakeInitiative	Competitiveness	SelfReliance	StrongNeedToAchieve	SelfConfidence	GoodPhysicalHealth	y
count	219.0000	219.0000	219.0000	219.0000	219.0000	219.0000	219.0000	219.0000	219.0000
mean	19.7534	3.3516	3.6210	3.5890	3.7215	3.9087	3.5753	3.5616	0.4155
std	1.2898	0.9952	1.1525	1.1109	1.0536	1.0231	1.1201	1.1003	0.4939
min	17.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000
25%	19.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	0.0000
50%	20.0000	3.0000	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000	0.0000
75%	20.0000	4.0000	5.0000	4.5000	5.0000	5.0000	4.0000	4.0000	1.0000
max	26.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	5.0000	1.0000

To better get the relationship between educational background and entrepreneurial inclination, we analyze the distribution of students who are likely to become entrepreneurs versus those who are not across various educational sectors. Figure 1 presenting a bar plot illustrating these distributions, highlighting variations in entrepreneurial interest among students from diverse academic disciplines.

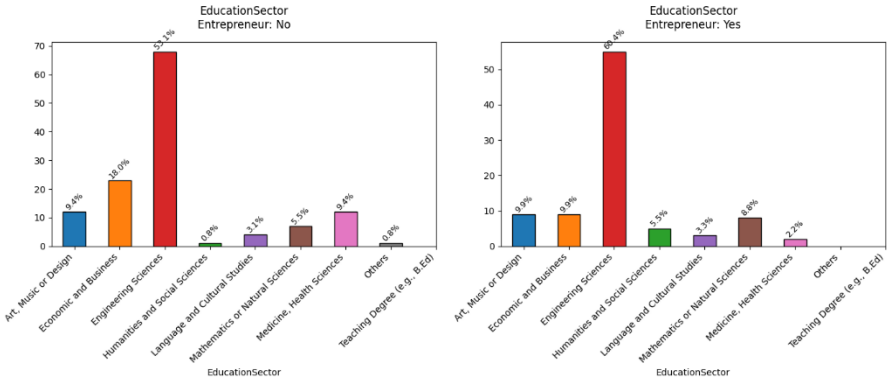


Figure 1. Percentage distribution of entrepreneurial and non-entrepreneurial students based on key personality traits.

To gain deep insight about demographic characteristics of entrepreneurial and non-entrepreneurial students, we conducted a demographic analysis focusing on factors such as gender, age groups, and city of residence**.** Figure 2 presented pie charts to visualize the proportion of entrepreneurs and non-entrepreneurs within these demographic classes, providing a clearer understanding of how demographic factors influence entrepreneurial inclination.

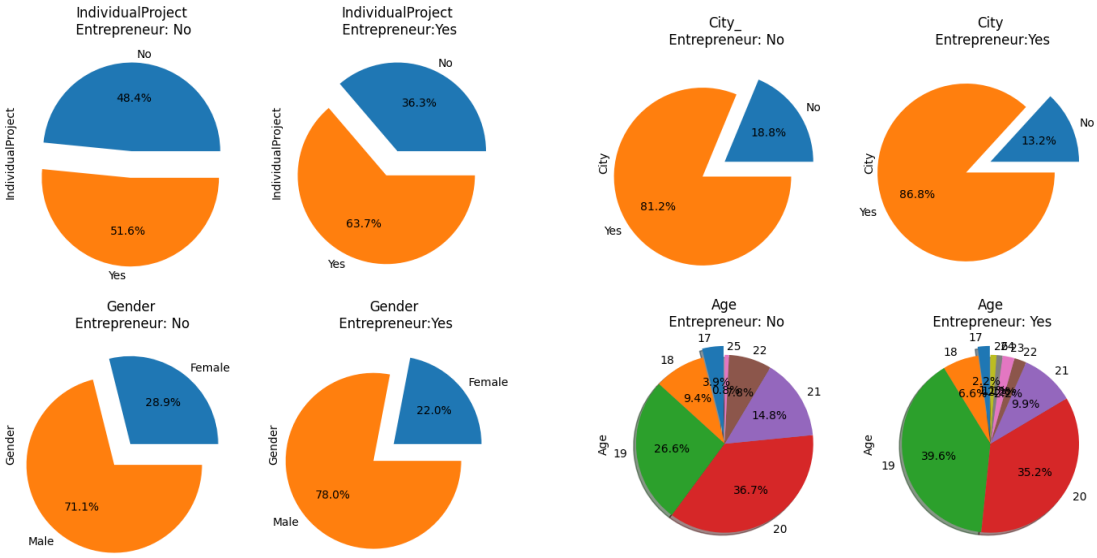


Figure 2. Distribution of key psychological traits among entrepreneurial and non-entrepreneurial students.

Soft skills always play a crucial role in shaping entrepreneurial competency. As a result, we analyze the impact of different soft skills on entrepreneurial potential, we inspected the relationship of entrepreneurial and non-entrepreneurial students w.r.t different soft skills. Figure 3 presented charts that visualize these distributions, providing insights into which soft skills are more prevalent among students likely to turn out to be entrepreneurs.

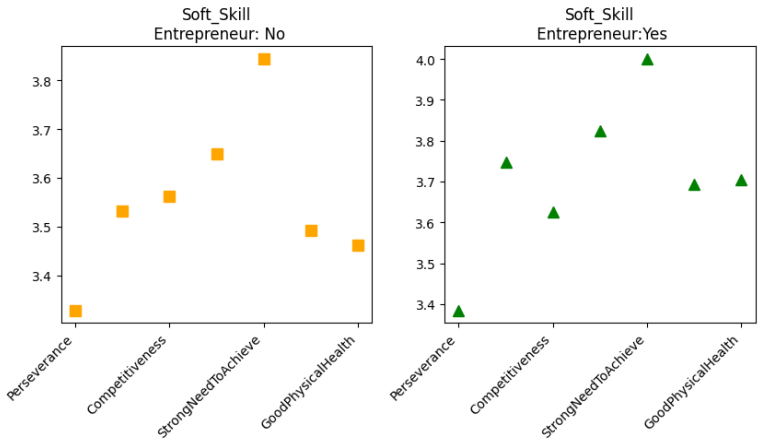


Figure 3. Proportion of entrepreneurial and non-entrepreneurial students based on key soft skills.

Key personality traits always influence an individual's entrepreneurial potential. Hence, we analyzed the distribution of entrepreneurial and non-entrepreneurial students based on various key traits. Figure 4 show these distributions, which highlight the differences in key traits between students who may be entrepreneurs and those who may not.

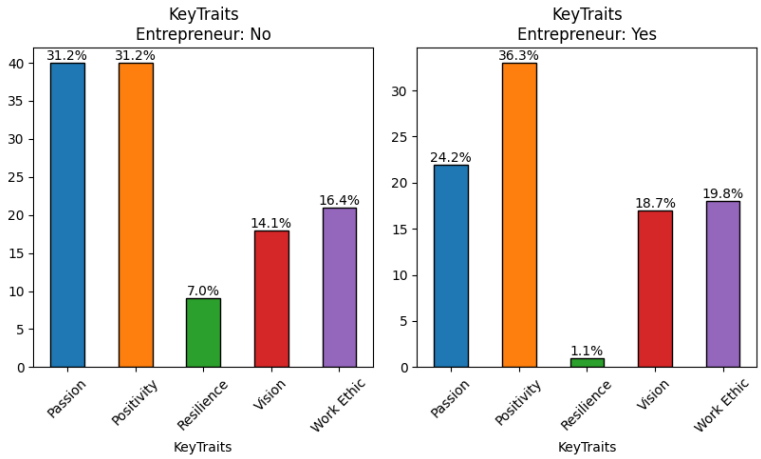
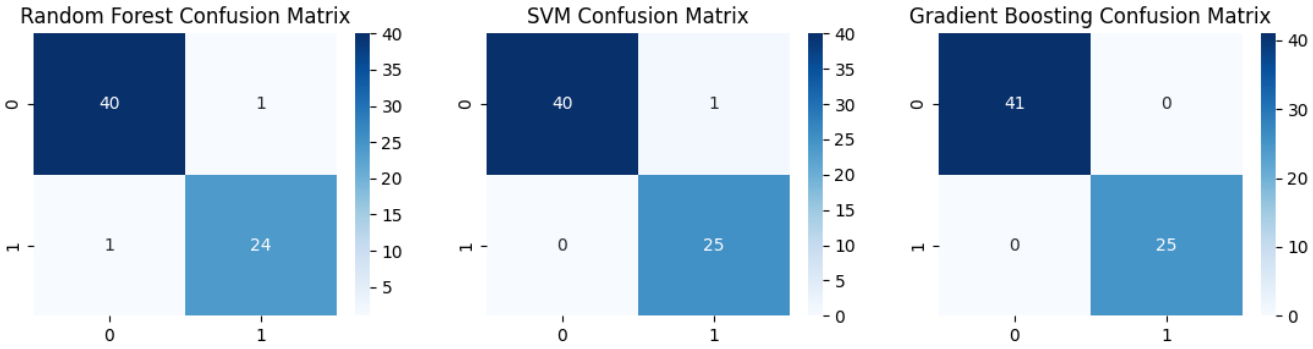


Figure 4. Distribution of key personality traits among entrepreneurial and non-entrepreneurial students.

5. Results & Discussions

In this section we present and discuss different types of analysis to validate the proposed approach. To begin, we compare the performance of NVNSS-GB against other NVNSS-based classifiers in Figure 5 which display their confusion matrices. As shown, the confusion matrix of NVNSS-GB shows a higher number of true positives and true negatives, indicating that it correctly classifies more students with entrepreneurial potential. On the other hand, the alternative NVNSS-based classifiers demonstrated a higher failure rate, (i.e., positive and false negatives). This observed behavior can be attributed to its gradient-based optimization, which iteratively refines weak learners and minimizes classification errors by reducing the impact of indeterminacy and vague memberships in student characteristics. In addition, NVNSS-GB successfully captures nonlinear patterns in the dataset.



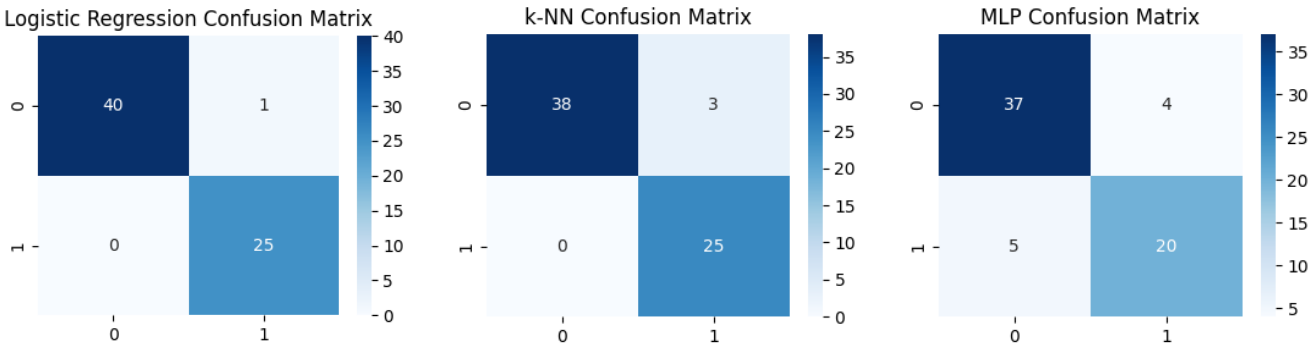


Figure 5. Comparison of Confusion Matrices: NVNSS-GB vs. NVNSS-Classifiers"

In Figure 6, we presented a sensitivity analysis heatmap to illustrate the influence of changing the learning rate and the amount of estimators included. The results indicate that lower learning rates (0.01–0.1) deliver stable performance, while increasing the number of estimators improves accuracy up to a certain threshold, beyond which overfitting occurs.

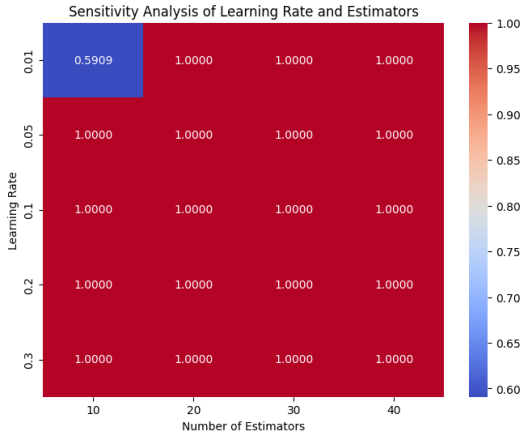


Figure 6. Impact of Learning Rate and Number of Estimators on NVNSS-GB Performance.

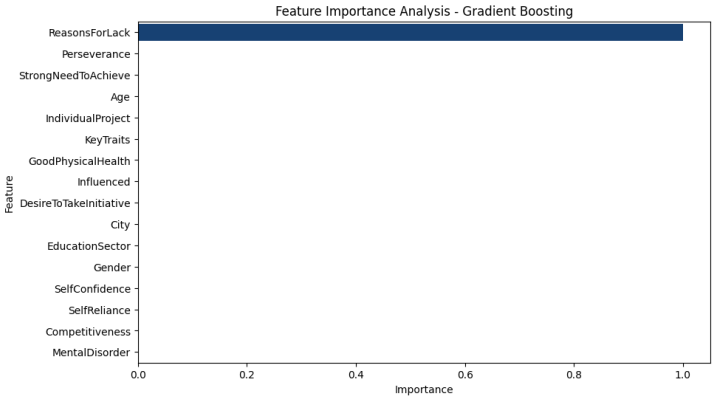


Figure 7. Feature Importance Ranking Using Gradient Boosting and SHAP Values

In addition, we introduce Figure 7 to communicate the feature importance ranking gotten from inner Gradient Boosting importance metrics. Key entrepreneurial traits is indicated as reasons for lack show the highest influence on classification decisions.

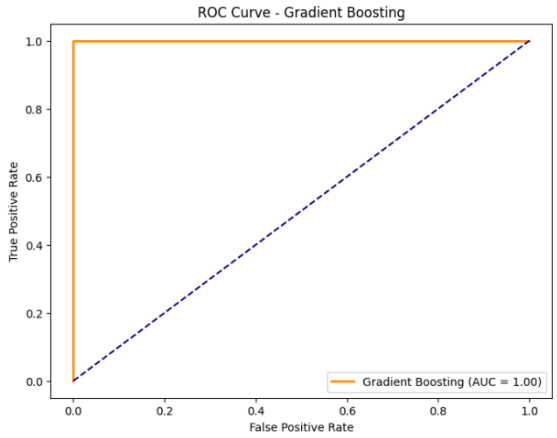


Figure 8. ROC Curve and AUC Score for NVNSS-GB Model.

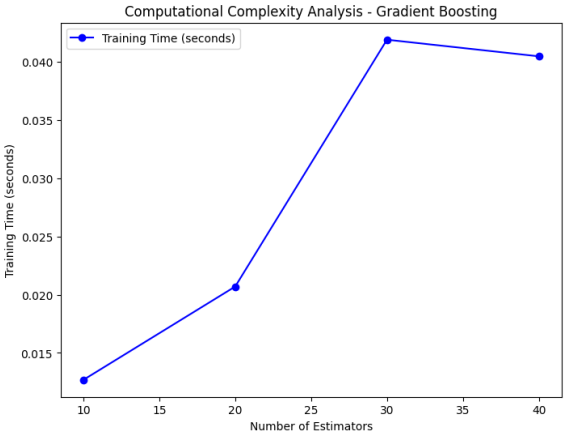


Figure 9. Training Time vs. Number of Estimators in NVNSS-GB.

Besides, Figure 8 presented NVNSS-GB's Receiver Operating Characteristic (ROC), which show that AUC score is 1.00, which indicates the model's strong discrimination ability, effectively distinguishing between students with entrepreneurial potential and those without. In Figure 9, we visualized the relation between training time of NVNSS-GB and number of estimators. As anticipated, increasing the number of estimators causes long training times, with high accuracy improvements.

To investigate instance-level interpretability of the NVNSS-GB, Figure 10 provided SHAP force plot to explain how individual features contribute to a specific prediction, showing whether they push the classification decision towards entrepreneural or non-entrepreneural potential. This effectively demonstrating that ReasonsforLack positively influences predictions.

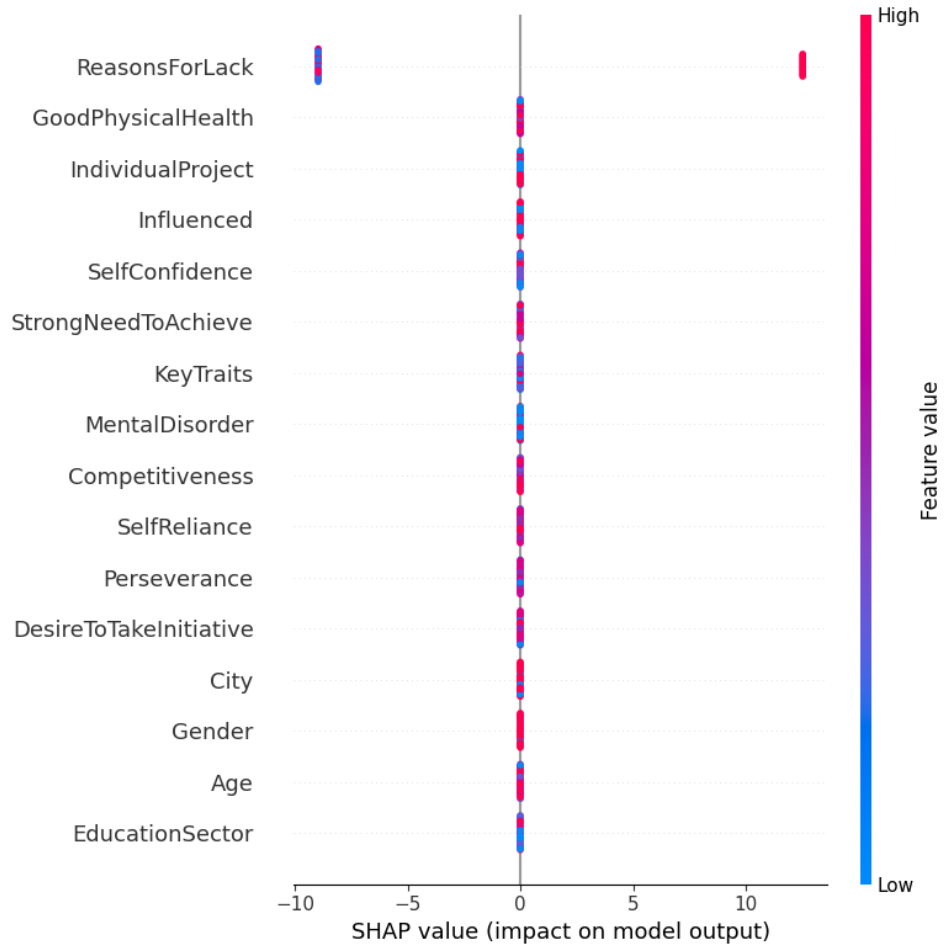


Figure 11. SHAP Force Plot for instance-level Impacts on Entrepreneurial Classification.

6. Conclusion

In this study, we proposed an innovative hybrid approach that integrate Gradient Boosting and NVNSS in a unified framework for enhancing the modeling and optimization of entrepreneurial potential among university students. The NVNSS-based transformation effectively captured uncertainty, vagueness, and indeterminacy in student characteristics, enabling a more robust feature representation. The experimental results on public dataset demonstrate that NVNSS-GB outperforms other NVNSS-based classifiers, as apparent in the confusion matrix. The achieved results imply that the gradient-based optimization can enhances decision boundaries as well as improves the handling of vague membership functions in complex DM scenarios. Future research will emphasize investigating the integration of NVNSS with advanced deep-learning solutions [15], [16].

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Risk Management of Import and Export Products Based on Big Data Analysis: Assessment Model with IndetermSoft Set

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Abstract: Risk management in the import and export sector has become increasingly complex due to globalization, technological advancements, and evolving regulatory landscapes. The integration of big data analytics into risk evaluation processes has revolutionized the ability to detect, assess, and mitigate potential threats across international trade operations. This study explores the application of multi-criteria decision-making methods methodologies for assessing risks associated with import and export products, focusing on quality compliance, supply chain vulnerabilities, cybersecurity threats, and regulatory adherence. By leveraging data analytics, businesses and regulatory bodies can enhance decision-making processes, reduce financial losses, and maintain high-quality standards in cross-border trade. We use the IndetermSoft set to deal with indeterminacy in the criteria values. Two MCDM methods are used in this study such as SWARA method to compute the criteria weights and the COPRAS method to rank the alternatives.

Keywords: Risk Management; Import and Export Products; Big Data Analysis; IndetermSoft Set.

1. Introduction

The global trade landscape is characterized by dynamic shifts in market demands, regulatory policies, and supply chain networks, making risk management a critical component of import and export operations. Traditional risk assessment methods often struggle to keep pace with the rapid flow of international transactions, leading to increased exposure to financial, legal, and operational uncertainties. With the advancement of big data technologies, businesses can now analyze vast amounts of structured and unstructured data to improve risk identification and mitigation strategies. Big data analytics offers a systematic approach to evaluating trade risks, enabling real-time monitoring, predictive insights, and strategic decision-making to enhance product safety and regulatory compliance[1], [2].

One of the most pressing concerns in international trade is the quality assurance of imported and exported products. Variations in manufacturing standards, counterfeit goods, and discrepancies in supply chain documentation pose significant risks to businesses and consumers alike. Big data-driven risk management allows for continuous monitoring of product quality by analyzing historical trends, inspection records, and real-time sensor data. This proactive approach helps detect anomalies and irregularities, preventing the circulation of substandard or hazardous goods in the global market. Ensuring product compliance with international standards minimizes financial losses, legal disputes, and reputational damage for stakeholders[3], [4].

Supply chain transparency and cybersecurity are also integral aspects of risk evaluation in import and export operations. Global trade relies on complex networks of suppliers, distributors, and logistics providers, each presenting potential vulnerabilities. Digital threats such as data breaches, fraud, and cyber-attacks can compromise sensitive trade information, leading to financial losses and regulatory non-compliance. Through big data analytics, companies can track supply chain movements, identify weak links, and implement cybersecurity measures to safeguard trade transactions. Real-time data processing enhances the accuracy of risk assessments, allowing businesses to anticipate disruptions and strengthen supply chain resilience.

Regulatory compliance remains a significant challenge for importers and exporters due to the variations in trade policies across different countries. Non-compliance with customs regulations, environmental laws, and safety standards can result in shipment delays, financial penalties, and legal consequences. Big data analytics helps organizations stay ahead of regulatory changes by analyzing historical compliance patterns, automating document verification, and predicting potential red flags in trade documentation. By streamlining regulatory adherence, businesses can ensure smooth trade operations while reducing the likelihood of customs-related disputes[5], [6].

The integration of big data into risk management not only enhances operational efficiency but also fosters a more sustainable and secure global trade ecosystem. As artificial intelligence (AI) and machine learning continue to evolve, risk evaluation models can become more refined, enabling businesses to detect emerging threats with greater accuracy. This research underscores the importance of adopting a data-driven approach to risk assessment in import and export processes, advocating advanced analytics to mitigate risks, optimize trade efficiency, and enhance consumer trust in international markets[7], [8].

By offering an organized way for assessing and contrasting various options according to several criteria, multi-criteria approaches can aid in the resolution of selection issues. The factors taken into consideration and the Risk Management of Import and Export Products application field determine the selection[9], [10]. Due to the increasing complexity and significance of decision-making in Risk Management of Import and Export Products, research on multi-criteria selection in Risk Management of Import and Export Products situations has changed dramatically in recent years[11], [12].

The structure of the paper is as follows. Steps of MCDM methods for Risk Management of Import and Export Products are conducted in Section 2. Section 3 presents a case study of Risk Management of Import and Export Products by dealing with indeterminacy values by the IndetermSoft set. The paper is concluded in Section 4.

2. MCDM Methods

The three main steps of this study are organized as follows:

1. The SWARA method is used to compute the criteria weights.
2. The COPRAS method is used to rank the alternatives.
3. The IndetermSoft Set is used to deal with different values in the main criteria.

Step 1. Computing the criteria weights.

This step we show the steps of the SWARA methodology to compute the criteria weights to rank the alternatives.

Initial sorting the criteria

First, the criteria are sorted based on the relative importance by the opinions of experts and decision makers.

Determine the coefficient of the criteria

$$K_j = \begin{cases} 1 & if\ j = 1 \\ X_j + 1 & if\ j > 1 \end{cases}; j = 1,2,3, \dots, n \quad (1)$$

Determine the initial weight of the criteria as:

$$q_j = \begin{cases} 1 & if\ j = 1 \\ \dfrac{q_j}{K_j} & if\ j > 1 \end{cases}; j = 1,2,3, \dots, n \quad (2)$$

Determine the relative weight

$$w_j = q_j / \sum_{j=1}^{n} q_j \quad (3)$$

Step 2. Sorting the alternatives.

Then, in this step we apply the steps of the COPRAS method to rank the alternatives[13], [14].

Create the decision matrix.

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}_{mn}; i = 1, \dots, m; j = 1, \dots, n \quad (3)$$

Determine the normalized decision matrix.

$$d_{ij} = x_{ij} \sum_{i=1}^{m} x_{ij} \; ; j = 1, .., n \tag{4}$$

Determine the weighted normalized matrix.

$$y_{ij} = w_j d_{ij} \tag{5}$$

Determine max and min indexes for positive and cost criteria

$$R_{+i} = \sum_{j=1}^{g} y_{ij}; i = 1, \dots, m \tag{6}$$

$$R_{-i} = \sum_{j=g+1}^{n} y_{ij}; i = 1, \dots, m \tag{7}$$

Determine the significance of value of each alternative

$$S_i = R_{+i} + \frac{\sum_{i=1}^{m} R_{-i}}{R_{-i} \sum_{i=1}^{m} \frac{1}{R_{-i}}} \tag{8}$$

Step 3. IndetermSoft Set

Let U be a universe of discourse and H be non-empty subset of U, and the powerset is a P(H). Let set of criteria such as c and C is the set of criteria values[15], [16].

A function F: $C \to P(H)$ is called IndetermSoft set if:

The set C and P(H) have some indeterminacy or at least one criterion has an indeterminacy.

3. Case Study

This section shows the case study of the MCDM method. We can show the validation of the proposed approach by computing the criteria weights and ranking the alternatives. This study has 12 main criteria and seven alternatives. The criteria of this study are organized as follows:

- Product Compliance with International Standards (High compliance, Moderate compliance, Low compliance)
- Supply Chain Transparency (Fully transparent)
- Manufacturing Quality Control (Moderate quality control)
- Logistics and Transportation Risks (Low transportation risk)
- Product Safety and Consumer Protection (Highly safe with certifications)
- Regulatory and Customs Compliance (Fully compliant)
- Market Reputation and Consumer Feedback (Excellent market reputation)
- Cybersecurity and Data Protection in Trade (Strong Cybersecurity Measures)
- Environmental and Sustainability Impact (Sustainable and eco-friendly)

- Counterfeit and Fraud Risks (Low risk of counterfeiting)
- Big Data Predictive Analytics Accuracy (Highly accurate predictions)
- Crisis Management and Risk Response (Proactive and well-prepared)

The alternatives of this study are:

- Chemical and Industrial Materials
- Medical Equipment and Devices
- Textile and Apparel
- Automobile Components and Machinery
- Pharmaceutical Products
- Electronics and Consumer Goods
- Food and Agricultural Products

From the previous criteria, we have indeterminacy in one criterion. So, we use the IndetermSoft set to deal with this indeterminacy.

Step 1. Computing the criteria weights.

Initial sorting the criteria.

Eq. (1) is used to determine the coefficient of the criteria.

Eq. (2) is used to determine the initial weight of the criteria.

Eq. (3) is used to determine the relative weight as shown in Fig 1.

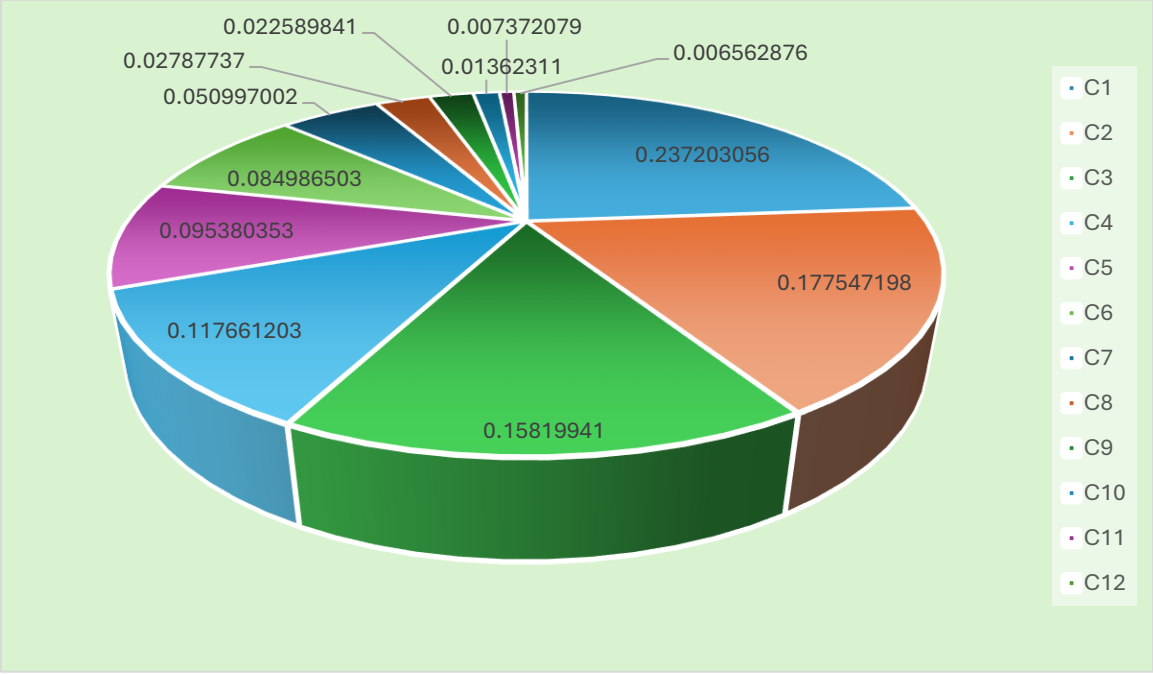

Fig 1. The criteria weights.

Step 2. Sorting the alternatives.

Then we apply the COPRAS method to deal with indeterminacy in the first criterion. So, we apply the COPRAS method seven cases in each time we obtain the rank of the alternatives. These values (High compliance, Moderate compliance, Low compliance) are used seven times.

First Case

Eq. (3) is used to create the decision matrix. Three experts evaluate 12 criteria and 7 alternatives by crisp values.

Eq. (4) is used to determine the normalized decision matrix as shown in Table 1.

Eq. (5) is used to determine the weighted normalized matrix as shown in Table 2.

Eq. (6 and 7) is used to determine max and min indexes for positive and cost criteria

Then we determine the significance of value of each alternative using Eq. (8). Then we ranked the alternatives.

Table 1. Normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.161073	0.129953	0.185611	0.146509	0.12788	0.120073	0.128902
C_2	0.203544	0.163455	0.127091	0.1023	0.140788	0.140526	0.122296
C_3	0.098326	0.177664	0.107934	0.190211	0.130089	0.187842	0.107934
C_4	0.150631	0.131078	0.141559	0.133701	0.170395	0.131078	0.141559
C_5	0.130567	0.151871	0.153435	0.142877	0.153435	0.121817	0.145998
C_6	0.14774	0.160826	0.132992	0.107109	0.191741	0.169951	0.089642
C_7	0.184986	0.107779	0.126501	0.14178	0.139929	0.152836	0.146189
C_8	0.061719	0.146608	0.126146	0.164679	0.239152	0.1911	0.070597
C_9	0.143138	0.13599	0.169147	0.105081	0.15839	0.143312	0.144943
C_{10}	0.187926	0.191708	0.153958	0.160798	0.106027	0.091281	0.108302
C_{11}	0.120195	0.144841	0.1204	0.18099	0.195707	0.153122	0.084744
C_{12}	0.142495	0.168782	0.127265	0.137902	0.127672	0.168782	0.127102

Table 2. Weighted normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.038207	0.030825	0.044028	0.034752	0.030334	0.028482	0.030576
C_2	0.036139	0.029021	0.022565	0.018163	0.024997	0.02495	0.021713
C_3	0.015555	0.028106	0.017075	0.030091	0.02058	0.029717	0.017075
C_4	0.017723	0.015423	0.016656	0.015731	0.020049	0.015423	0.016656
C_5	0.012454	0.014486	0.014635	0.013628	0.014635	0.011619	0.013925
C_6	0.012556	0.013668	0.011303	0.009103	0.016295	0.014444	0.007618
C_7	0.009434	0.005496	0.006451	0.00723	0.007136	0.007794	0.007455
C_8	0.001721	0.004087	0.003517	0.004591	0.006667	0.005327	0.001968

C_9	0.003233	0.003072	0.003821	0.002374	0.003578	0.003237	0.003274
C_{10}	0.00256	0.002612	0.002097	0.002191	0.001444	0.001244	0.001475
C_{11}	0.000886	0.001068	0.000888	0.001334	0.001443	0.001129	0.000625
C_{12}	0.000935	0.001108	0.000835	0.000905	0.000838	0.001108	0.000834

Second Case

Eq. (3) is used to create the decision matrix. Three experts evaluate 12 criteria and 7 alternatives by crisp values.

Eq. (4) is used to determine the normalized decision matrix as shown in Table 3.

Eq. (5) is used to determine the weighted normalized matrix as shown in Table 4.

Eq. (6 and 7) is used to determine max and min indexes for positive and cost criteria

Then we determine the significance of value of each alternative using Eq. (8). Then we ranked the alternatives.

Table 3. Normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.099764	0.036889	0.090552	0.15674	0.210402	0.229901	0.175751
C_2	0.203544	0.163455	0.127091	0.1023	0.140788	0.140526	0.122296
C_3	0.098326	0.177664	0.107934	0.190211	0.130089	0.187842	0.107934
C_4	0.150631	0.131078	0.141559	0.133701	0.170395	0.131078	0.141559
C_5	0.130567	0.151871	0.153435	0.142877	0.153435	0.121817	0.145998
C_6	0.14774	0.160826	0.132992	0.107109	0.191741	0.169951	0.089642
C_7	0.184986	0.107779	0.126501	0.14178	0.139929	0.152836	0.146189
C_8	0.061719	0.146608	0.126146	0.164679	0.239152	0.1911	0.070597
C_9	0.143138	0.13599	0.169147	0.105081	0.15839	0.143312	0.144943
C_{10}	0.187926	0.191708	0.153958	0.160798	0.106027	0.091281	0.108302
C_{11}	0.120195	0.144841	0.1204	0.18099	0.195707	0.153122	0.084744
C_{12}	0.142495	0.168782	0.127265	0.137902	0.127672	0.168782	0.127102

Table 4. Weighted normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.023664	0.00875	0.021479	0.037179	0.049908	0.054533	0.041689
C_2	0.036139	0.029021	0.022565	0.018163	0.024997	0.02495	0.021713
C_3	0.015555	0.028106	0.017075	0.030091	0.02058	0.029717	0.017075
C_4	0.017723	0.015423	0.016656	0.015731	0.020049	0.015423	0.016656
C_5	0.012454	0.014486	0.014635	0.013628	0.014635	0.011619	0.013925
C_6	0.012556	0.013668	0.011303	0.009103	0.016295	0.014444	0.007618
C_7	0.009434	0.005496	0.006451	0.00723	0.007136	0.007794	0.007455
C_8	0.001721	0.004087	0.003517	0.004591	0.006667	0.005327	0.001968

C_9	0.003233	0.003072	0.003821	0.002374	0.003578	0.003237	0.003274
C_{10}	0.00256	0.002612	0.002097	0.002191	0.001444	0.001244	0.001475
C_{11}	0.000886	0.001068	0.000888	0.001334	0.001443	0.001129	0.000625
C_{12}	0.000935	0.001108	0.000835	0.000905	0.000838	0.001108	0.000834

Third Case

Eq. (3) is used to create the decision matrix. Three experts evaluate 12 criteria and 7 alternatives by crisp values.

Eq. (4) is used to determine the normalized decision matrix as shown in Table 5.

Eq. (5) is used to determine the weighted normalized matrix as shown in Table 6.

Eq. (6 and 7) is used to determine max and min indexes for positive and cost criteria

Then we determine the significance of value of each alternative using Eq. (8). Then we ranked the alternatives.

Table 5. Normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.033682	0.151512	0.033682	0.151512	0.205904	0.263239	0.160469
C_2	0.203544	0.163455	0.127091	0.1023	0.140788	0.140526	0.122296
C_3	0.098326	0.177664	0.107934	0.190211	0.130089	0.187842	0.107934
C_4	0.150631	0.131078	0.141559	0.133701	0.170395	0.131078	0.141559
C_5	0.130567	0.151871	0.153435	0.142877	0.153435	0.121817	0.145998
C_6	0.14774	0.160826	0.132992	0.107109	0.191741	0.169951	0.089642
C_7	0.184986	0.107779	0.126501	0.14178	0.139929	0.152836	0.146189
C_8	0.061719	0.146608	0.126146	0.164679	0.239152	0.1911	0.070597
C_9	0.143138	0.13599	0.169147	0.105081	0.15839	0.143312	0.144943
C_{10}	0.187926	0.191708	0.153958	0.160798	0.106027	0.091281	0.108302
C_{11}	0.120195	0.144841	0.1204	0.18099	0.195707	0.153122	0.084744
C_{12}	0.142495	0.168782	0.127265	0.137902	0.127672	0.168782	0.127102

Table 6. Weighted normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.007989	0.035939	0.007989	0.035939	0.048841	0.062441	0.038064
C_2	0.036139	0.029021	0.022565	0.018163	0.024997	0.02495	0.021713
C_3	0.015555	0.028106	0.017075	0.030091	0.02058	0.029717	0.017075
C_4	0.017723	0.015423	0.016656	0.015731	0.020049	0.015423	0.016656
C_5	0.012454	0.014486	0.014635	0.013628	0.014635	0.011619	0.013925
C_6	0.012556	0.013668	0.011303	0.009103	0.016295	0.014444	0.007618
C_7	0.009434	0.005496	0.006451	0.00723	0.007136	0.007794	0.007455
C_8	0.001721	0.004087	0.003517	0.004591	0.006667	0.005327	0.001968

C_9	0.003233	0.003072	0.003821	0.002374	0.003578	0.003237	0.003274
C_{10}	0.00256	0.002612	0.002097	0.002191	0.001444	0.001244	0.001475
C_{11}	0.000886	0.001068	0.000888	0.001334	0.001443	0.001129	0.000625
C_{12}	0.000935	0.001108	0.000835	0.000905	0.000838	0.001108	0.000834

Fourth Case

Eq. (3) is used to create the decision matrix. Three experts evaluate 12 criteria and 7 alternatives by crisp values.

Eq. (4) is used to determine the normalized decision matrix as shown in Table 7.

Eq. (5) is used to determine the weighted normalized matrix as shown in Table 8.

Eq. (6 and 7) is used to determine max and min indexes for positive and cost criteria

Then we determine the significance of value of each alternative using Eq. (8). Then we ranked the alternatives.

Table 7. Normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.16369	0.134863	0.149277	0.134863	0.154086	0.13565	0.12757
C_2	0.203544	0.163455	0.127091	0.1023	0.140788	0.140526	0.122296
C_3	0.098326	0.177664	0.107934	0.190211	0.130089	0.187842	0.107934
C_4	0.150631	0.131078	0.141559	0.133701	0.170395	0.131078	0.141559
C_5	0.130567	0.151871	0.153435	0.142877	0.153435	0.121817	0.145998
C_6	0.14774	0.160826	0.132992	0.107109	0.191741	0.169951	0.089642
C_7	0.184986	0.107779	0.126501	0.14178	0.139929	0.152836	0.146189
C_8	0.061719	0.146608	0.126146	0.164679	0.239152	0.1911	0.070597
C_9	0.143138	0.13599	0.169147	0.105081	0.15839	0.143312	0.144943
C_{10}	0.187926	0.191708	0.153958	0.160798	0.106027	0.091281	0.108302
C_{11}	0.120195	0.144841	0.1204	0.18099	0.195707	0.153122	0.084744
C_{12}	0.142495	0.168782	0.127265	0.137902	0.127672	0.168782	0.127102

Table 8. Weighted normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.038828	0.03199	0.035409	0.03199	0.03655	0.032177	0.03026
C_2	0.036139	0.029021	0.022565	0.018163	0.024997	0.02495	0.021713
C_3	0.015555	0.028106	0.017075	0.030091	0.02058	0.029717	0.017075
C_4	0.017723	0.015423	0.016656	0.015731	0.020049	0.015423	0.016656
C_5	0.012454	0.014486	0.014635	0.013628	0.014635	0.011619	0.013925
C_6	0.012556	0.013668	0.011303	0.009103	0.016295	0.014444	0.007618
C_7	0.009434	0.005496	0.006451	0.00723	0.007136	0.007794	0.007455
C_8	0.001721	0.004087	0.003517	0.004591	0.006667	0.005327	0.001968

C_9	0.003233	0.003072	0.003821	0.002374	0.003578	0.003237	0.003274
C_{10}	0.00256	0.002612	0.002097	0.002191	0.001444	0.001244	0.001475
C_{11}	0.000886	0.001068	0.000888	0.001334	0.001443	0.001129	0.000625
C_{12}	0.000935	0.001108	0.000835	0.000905	0.000838	0.001108	0.000834

Fifth Case

Eq. (3) is used to create the decision matrix. Three experts evaluate 12 criteria and 7 alternatives by crisp values.

Eq. (4) is used to determine the normalized decision matrix as shown in Table 9.

Eq. (5) is used to determine the weighted normalized matrix as shown in Table 10.

Eq. (6 and 7) is used to determine max and min indexes for positive and cost criteria

Then we determine the significance of value of each alternative using Eq. (8). Then we ranked the alternatives.

Table 9. Normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.168801	0.080826	0.184527	0.092006	0.080826	0.203531	0.189482
C_2	0.203544	0.163455	0.127091	0.1023	0.140788	0.140526	0.122296
C_3	0.098326	0.177664	0.107934	0.190211	0.130089	0.187842	0.107934
C_4	0.150631	0.131078	0.141559	0.133701	0.170395	0.131078	0.141559
C_5	0.130567	0.151871	0.153435	0.142877	0.153435	0.121817	0.145998
C_6	0.14774	0.160826	0.132992	0.107109	0.191741	0.169951	0.089642
C_7	0.184986	0.107779	0.126501	0.14178	0.139929	0.152836	0.146189
C_8	0.061719	0.146608	0.126146	0.164679	0.239152	0.1911	0.070597
C_9	0.143138	0.13599	0.169147	0.105081	0.15839	0.143312	0.144943
C_{10}	0.187926	0.191708	0.153958	0.160798	0.106027	0.091281	0.108302
C_{11}	0.120195	0.144841	0.1204	0.18099	0.195707	0.153122	0.084744
C_{12}	0.142495	0.168782	0.127265	0.137902	0.127672	0.168782	0.127102

Table 10. Weighted normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.04004	0.019172	0.04377	0.021824	0.019172	0.048278	0.044946
C_2	0.036139	0.029021	0.022565	0.018163	0.024997	0.02495	0.021713
C_3	0.015555	0.028106	0.017075	0.030091	0.02058	0.029717	0.017075
C_4	0.017723	0.015423	0.016656	0.015731	0.020049	0.015423	0.016656
C_5	0.012454	0.014486	0.014635	0.013628	0.014635	0.011619	0.013925
C_6	0.012556	0.013668	0.011303	0.009103	0.016295	0.014444	0.007618
C_7	0.009434	0.005496	0.006451	0.00723	0.007136	0.007794	0.007455
C_8	0.001721	0.004087	0.003517	0.004591	0.006667	0.005327	0.001968

C_9	0.003233	0.003072	0.003821	0.002374	0.003578	0.003237	0.003274
C_{10}	0.00256	0.002612	0.002097	0.002191	0.001444	0.001244	0.001475
C_{11}	0.000886	0.001068	0.000888	0.001334	0.001443	0.001129	0.000625
C_{12}	0.000935	0.001108	0.000835	0.000905	0.000838	0.001108	0.000834

Sixth Case

Eq. (3) is used to create the decision matrix. Three experts evaluate 12 criteria and 7 alternatives by crisp values.

Eq. (4) is used to determine the normalized decision matrix as shown in Table 11.

Eq. (5) is used to determine the weighted normalized matrix as shown in Table 12.

Eq. (6 and 7) is used to determine max and min indexes for positive and cost criteria

Then we determine the significance of value of each alternative using Eq. (8). Then we ranked the alternatives.

Table 11. Normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.099031	0.171705	0.144303	0.210251	0.127152	0.119389	0.128168
C_2	0.203544	0.163455	0.127091	0.1023	0.140788	0.140526	0.122296
C_3	0.098326	0.177664	0.107934	0.190211	0.130089	0.187842	0.107934
C_4	0.150631	0.131078	0.141559	0.133701	0.170395	0.131078	0.141559
C_5	0.130567	0.151871	0.153435	0.142877	0.153435	0.121817	0.145998
C_6	0.14774	0.160826	0.132992	0.107109	0.191741	0.169951	0.089642
C_7	0.184986	0.107779	0.126501	0.14178	0.139929	0.152836	0.146189
C_8	0.061719	0.146608	0.126146	0.164679	0.239152	0.1911	0.070597
C_9	0.143138	0.13599	0.169147	0.105081	0.15839	0.143312	0.144943
C_{10}	0.187926	0.191708	0.153958	0.160798	0.106027	0.091281	0.108302
C_{11}	0.120195	0.144841	0.1204	0.18099	0.195707	0.153122	0.084744
C_{12}	0.142495	0.168782	0.127265	0.137902	0.127672	0.168782	0.127102

Table 12. Weighted normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.023491	0.040729	0.034229	0.049872	0.030161	0.028319	0.030402
C_2	0.036139	0.029021	0.022565	0.018163	0.024997	0.02495	0.021713
C_3	0.015555	0.028106	0.017075	0.030091	0.02058	0.029717	0.017075
C_4	0.017723	0.015423	0.016656	0.015731	0.020049	0.015423	0.016656
C_5	0.012454	0.014486	0.014635	0.013628	0.014635	0.011619	0.013925
C_6	0.012556	0.013668	0.011303	0.009103	0.016295	0.014444	0.007618
C_7	0.009434	0.005496	0.006451	0.00723	0.007136	0.007794	0.007455
C_8	0.001721	0.004087	0.003517	0.004591	0.006667	0.005327	0.001968

C_9	0.003233	0.003072	0.003821	0.002374	0.003578	0.003237	0.003274
C_{10}	0.00256	0.002612	0.002097	0.002191	0.001444	0.001244	0.001475
C_{11}	0.000886	0.001068	0.000888	0.001334	0.001443	0.001129	0.000625
C_{12}	0.000935	0.001108	0.000835	0.000905	0.000838	0.001108	0.000834

Seventh Case

Eq. (3) is used to create the decision matrix. Three experts evaluate 12 criteria and 7 alternatives by crisp values.

Eq. (4) is used to determine the normalized decision matrix as shown in Table 13.

Eq. (5) is used to determine the weighted normalized matrix as shown in Table 14.

Eq. (6 and 7) is used to determine max and min indexes for positive and cost criteria

Then we determine the significance of value of each alternative using Eq. (8). Then we ranked the alternatives.

Table 13. Normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.151058	0.179099	0.152737	0.172929	0.116792	0.109661	0.117725
C_2	0.203544	0.163455	0.127091	0.1023	0.140788	0.140526	0.122296
C_3	0.098326	0.177664	0.107934	0.190211	0.130089	0.187842	0.107934
C_4	0.150631	0.131078	0.141559	0.133701	0.170395	0.131078	0.141559
C_5	0.130567	0.151871	0.153435	0.142877	0.153435	0.121817	0.145998
C_6	0.14774	0.160826	0.132992	0.107109	0.191741	0.169951	0.089642
C_7	0.184986	0.107779	0.126501	0.14178	0.139929	0.152836	0.146189
C_8	0.061719	0.146608	0.126146	0.164679	0.239152	0.1911	0.070597
C_9	0.143138	0.13599	0.169147	0.105081	0.15839	0.143312	0.144943
C_{10}	0.187926	0.191708	0.153958	0.160798	0.106027	0.091281	0.108302
C_{11}	0.120195	0.144841	0.1204	0.18099	0.195707	0.153122	0.084744
C_{12}	0.142495	0.168782	0.127265	0.137902	0.127672	0.168782	0.127102

Table 14. Weighted normalization matrix.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
C_1	0.035831	0.042483	0.03623	0.041019	0.027703	0.026012	0.027925
C_2	0.036139	0.029021	0.022565	0.018163	0.024997	0.02495	0.021713
C_3	0.015555	0.028106	0.017075	0.030091	0.02058	0.029717	0.017075
C_4	0.017723	0.015423	0.016656	0.015731	0.020049	0.015423	0.016656
C_5	0.012454	0.014486	0.014635	0.013628	0.014635	0.011619	0.013925
C_6	0.012556	0.013668	0.011303	0.009103	0.016295	0.014444	0.007618
C_7	0.009434	0.005496	0.006451	0.00723	0.007136	0.007794	0.007455
C_8	0.001721	0.004087	0.003517	0.004591	0.006667	0.005327	0.001968

C_9	0.003233	0.003072	0.003821	0.002374	0.003578	0.003237	0.003274
C_{10}	0.00256	0.002612	0.002097	0.002191	0.001444	0.001244	0.001475
C_{11}	0.000886	0.001068	0.000888	0.001334	0.001443	0.001129	0.000625
C_{12}	0.000935	0.001108	0.000835	0.000905	0.000838	0.001108	0.000834

Then we obtained the final rank of the alternatives as shown in Fig 2. We show the alternative 5 is the best and alternative 2 is the worst.

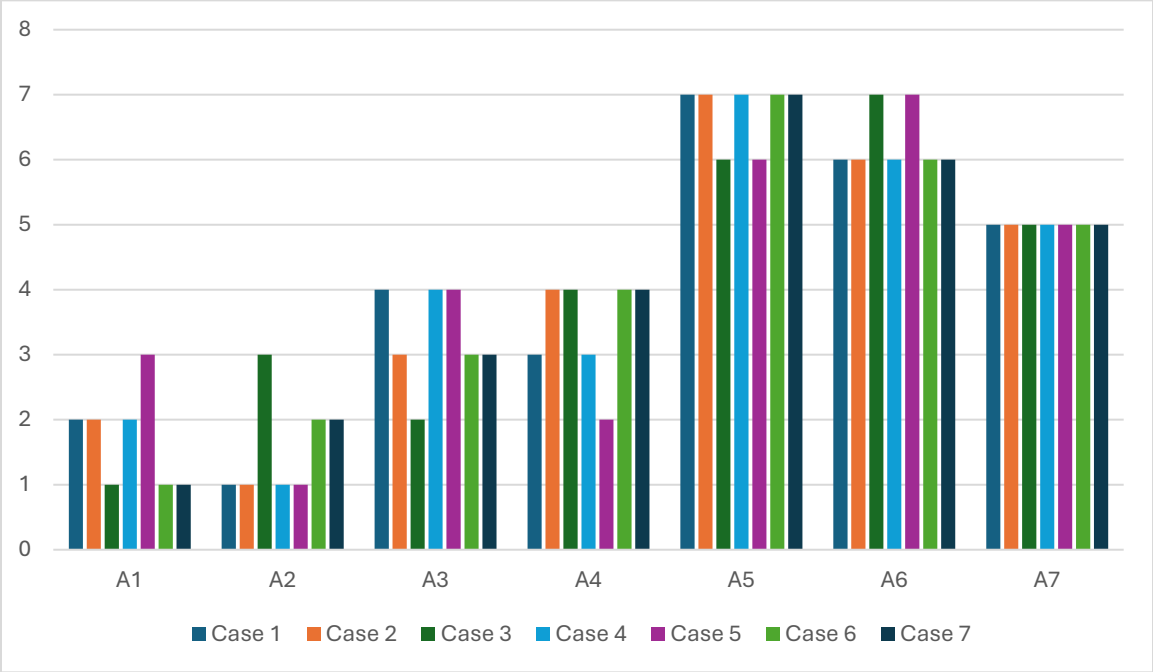

Fig 2. Final rank of alternatives.

4. Conclusions

Risk management in the import and export trade has been significantly improved by the adoption of big data analytics, allowing businesses to identify, assess, and respond to potential threats in real time. The ability to monitor product quality, enhance supply chain transparency, mitigate cybersecurity risks, and ensure regulatory compliance makes big data an invaluable tool for global trade. As trade networks continue to expand, data-driven methodologies will play a crucial role in minimizing uncertainties and safeguarding trade transactions. We used the IndetermSoft set to deal with the indeterminacy values of criteria. We had the one indeterminacy in the first criterion. So, we applied the COPRAS method into seven cases to deal with this indeterminacy. The results show alternative 5 is the best and alternative 2 is the worst.

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Neutrosophic CRITIC MCDM for Prosthesis Rehabilitation, its Applications and Technologies

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Abstract: Prosthesis rehabilitation plays a crucial role in supporting amputees to return to their normal life through increasing mobility, utility, and comfort. Prostheses design during production in VR- Metaverse depends on clients' requirements and many other factors, such as comfort, use, and operation. There are also sub-factors within these factors as illustrated in Figure (1). Several overlapping criteria must be considered when making decisions in this field. Thus, an efficient multi-criteria decision-making (MCDM) technique needs to be employed. One promising technique is the Neutrosophic CRITIC (Criteria Importance Through Intercriteria Correlation) MCDM. In this paper, the fundamentals of the Neutrosophic CRITIC MCDM are discussed, focusing on its applications in prosthesis rehabilitation and the associated technologies that enhance the decision-making process.

Keywords: CRITIC, MCDM, Prosthesis Rehabilitation, Uncertain Data.

1. Introduction

Prosthetic rehabilitation manufacturing involves a complicated decision-making process due to the several factors affecting its work. This includes cost, durability, biomechanical efficiency, patient comfort, and material selection [1]. Unfortunately, due to the inherent uncertainty and subjectivity of these factors, traditional decision-making techniques are often incapable of dealing with these issues.

If neutrosophic logic is combined with the CRITIC method, a more effective solution can be harvested to model and process uncertainty, leading to more intelligent and adaptive rehabilitation strategies [2,3].

The motivation behind using neutrosophic logic for prosthetic systems is that such systems suffer from:

- Noisy sensor signals.
- Uncertainty of the interaction process between the amputees and the device.

These data uncertainties can be managed through the structured manner of neutrosophic theory tools, producing prosthetics with better flexibility and responsiveness.

Figure (1) presents the factors in the prosthetic production design.

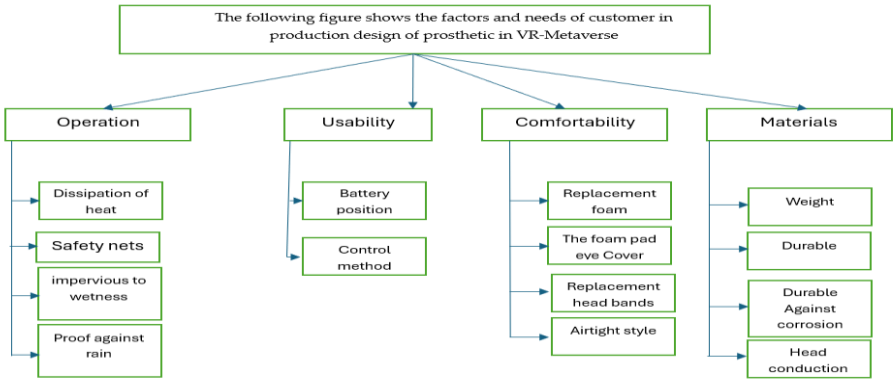

Figure (1): The factors in the prosthetic design

2. Prosthetic Design using Neutrosophic CRITIC MCDM

Adopting Neutrosophic CRITIC MCDM in prosthetic design can offer different merits which include:

1) Prosthetic devices can adapt to the changes in environmental conditions and user inputs by ranking and assessing control factors.

2) Neutrosophic theory can control uncertain inputs from sensors, thus the sensory feedback mechanisms can be improved, which in turn enhances the user's sense of control and natural movement.

3) Neutrosophic CRITIC MCDM can also offer another advantage by prioritizing many factors, such as heat dissipation, weight distribution, and control mechanisms, resulting in more user-centered prosthetic designs.

3. Case Studies

2.1. Case Study 1 [4]: Adaptive Bionic Limb with Neutrosophic-Based Control

A neutrosophic-based adaptive control system is installed on a prosthetic limb. Utilizing the CRITIC MCDM technique helps to prioritize control signals according to environmental factors, sensor data, and user feedback, allowing more accurate and timely adjustments.

2.2. Case Study 2 [5]: Supporting Myoelectric Devices with Sensory Feedback

Neutrosophic sensory feedback can be integrated with the myoelectric prosthetics to enhance user reaction as a result of improved handling of indeterminate signals. Patients reported that their prosthetic was easier to operate and it adapts better to incomplete and unclear muscle signals.

3. Key Findings [6]

3.1. Utilizing neutrosophic based control systems techniques in prosthetics leads to better user adaptability.

3.2. Adapting neutrosophic logic assists prosthetic systems to handle uncertain data more effectively.

3.3. By prioritizing design factors like weight, thermal management, and control mechanisms, the CRITIC MCDM technique can allow more comfortable and functional prosthetics.

4. The Challenges and Limitations [6,7]

- Often, the implementation of neutrosophic systems in control algorithms to serve prosthetics requires considerable computational capabilities.
- The calibration of the parameters for different prosthetics users can be complicated because of the variations in physical conditions and user preferences.

- There are some concerns about data privacy arising when collecting usage data from users.
- There is a requirement that the decision-making algorithms need to be objective and unbiased for equitable user experience.

5. Future Research Directions

5.1. There is a possibility of combining neutrosophic logic with AI technology to create hybrid models such as machine learning or neural networks that could lead to more accurate prosthetic systems with higher adaptivity.

5.2. Future research can focus on personalizing the usage of prosthetic devices through adopting user-specific neutrosophic models, thus enabling better adaptivity to individual preferences.

5.3. The optimization of neutrosophic CRITIC MCDM algorithms for real-time requirements could extend their use in daily prosthetic activities.

6. Conclusion

The Neutrosophic CRITIC MCDM methodology provides a promising possibility to develop prosthetic systems through dealing with uncertainty, indeterminacy, and contradicted information. By manipulating more responsive, adaptive and user-centered prosthetics, this method represents an important step forward in introducing sophisticated decision-making frameworks into medical utilities.

Acknowledgement

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Decision Making Model for Analysis of Culinary Intangible Cultural Heritage Tourism Resources: Forest HyperSoft Set and Results

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Abstract: Culinary intangible cultural heritage plays a pivotal role in preserving the authenticity and uniqueness of a region's traditions while serving as a driving force for tourism development. The evaluation of culinary intangible cultural heritage tourism resources is essential to ensure that such traditions are not only recognized but also sustained for future generations. This study presents a comprehensive framework for assessing the value of culinary heritage through multiple criteria, including cultural significance, economic impact, gastronomic appeal, sustainability, and policy support. By utilizing a multi-criteria decision-making (MCDM) approach, this evaluation aims to provide a structured methodology for ranking and analyzing different culinary traditions worldwide. We use the Forest HyperSoft set to divide each criterion as a TreeSoft set. Then in each TreeSoft set we can compute the criteria weights by MEREC method and rank the alternatives by the MARCOS method. This study uses four main criteria and six sub criteria with values. We have four TreeSoft sets. The results are shown in each TreeSoft set.

Keywords: Decision making; Analysis; Culinary Intangible Cultural Heritage Tourism Resources; Forest HyperSoft Set.

1. Introduction

Culinary intangible cultural heritage represents a crucial element of cultural identity, embodying the history, traditions, and social practices of a particular region. Unlike tangible heritage, which includes monuments and artifacts, intangible heritage focuses on traditions, practices, and skills passed down through generations. As globalization accelerates, many unique culinary traditions face the risk of being diluted or forgotten, necessitating strategic evaluation and preservation efforts. Tourism, as a major global industry, has increasingly integrated culinary experiences, making the evaluation of these resources vital for their sustainability[1], [2].

The growing interest in gastronomic tourism has positioned culinary heritage as an asset that contributes to economic development and cross-cultural exchange. Culinary tourism encourages local economies by creating employment opportunities, supporting small businesses, and attracting international recognition. However, despite its potential, not all culinary traditions receive equal attention, leading to disparities in their protection and promotion. A systematic evaluation framework can help identify key factors that determine the sustainability and competitiveness of different culinary traditions[3], [4].

Assessing the value of culinary intangible cultural heritage requires a multidimensional approach that considers both qualitative and quantitative aspects. Cultural authenticity, economic viability, gastronomic appeal, accessibility, and sustainability all play critical roles in determining the long-term success of a culinary tourism resource. Additionally, the role of policies, local community involvement, and global recognition in shaping culinary tourism destinations must be factored into the evaluation. This study proposes a methodological framework to assess these elements and rank culinary traditions based on their overall effectiveness as tourism resources[5], [6].

Incorporating advanced evaluation techniques, such as multi-criteria decision-making (MCDM), enables a structured assessment of culinary heritage by weighing different factors and ranking alternatives based on their performance. MCDM approaches, such as MEREC and MARCOS methods, allow for a data-driven comparison of culinary heritage resources. By applying these methods, this study seeks to bridge the gap between cultural heritage preservation and modern tourism development.

Furthermore, sustainability remains a key concern in the evaluation of culinary intangible cultural heritage. The increasing commercial appeal of traditional foods can lead to over-commercialization, loss of authenticity, and environmental degradation. Thus, a balanced approach that promotes culinary tourism while ensuring responsible consumption and environmental sustainability is essential. This study also considers how factors such as food security, climate change adaptation, and ethical sourcing of ingredients impact the long-term sustainability of culinary heritage resources[7], [8].

Expert weights assigned subjectively have the potential to be very biased and burdensome. As a result, we advise using an objective weighting method for the evaluation values. Usually, several criteria are used by objective weighing systems to calculate weights. The MEREC approach was developed by Keshavarz-Ghorabaee et al.[9] to determine objective weights of qualities by evaluating the effect of removing a criterion on the alternative's overall performance. A criterion has a larger portion of the weight when its removal has a bigger effect on the overall performance.

To solve different MCDM situations, the MEREC technique has been expanded to several uncertainties. The relative weights rise by increasing distance from the common perspective and fall with decreasing distance from the common perception. The individual determination criterion weights and the performance of alternatives to the deletion criteria are causally related. For this MCDM problem, no researchers have used MEREC[10], [11].

2. MEREC-MARCOS Model

We evaluate the alternatives of this study by applying the decision-making model. This decision-making model has two main steps. In the first step, we compute the criteria weights by the MEREC method and ranking the alternatives by the MARCOS method.

The steps of the MEREC method are organized as follows:

Establish the evaluating matrix.

Different experts and decision makers are evaluated a set of criteria and alternatives.

Obtain the overall evaluation matrix.

We obtain the overall evaluation matrix by combining the decision matrices into a single matrix between criteria and alternatives.

Standardize the overall evaluation matrix.

We can standardize the overall evaluation matrix such as:

$$y_{ij} = \frac{x_{ij}}{\min_i x_{ij}}; i = 1, \dots m; j = 1, \dots n \tag{1}$$

Where x_{ij} refers to the value in the overall evaluation matrix.

Determine the overall performance of each alternative.

We can compute the overall performance by non-linear function such as:

$$Q_i = \ln\left(1 + \left(\frac{1}{n}\sum_{j=1}^{n}\left|\ln y_{ij}\right|\right)\right) \tag{2}$$

Calculate the performance of the program after deleting each criterion

$$U_i = \ln 1 + \left(\frac{1}{n}\sum_{l=1, l\neq j}^{n}\left|\ln y_{ij}\right|\right) \tag{3}$$

Assess the combined of deviations from absolute values.

$$T_j = \sum_{i=1}^{m}|U_i - Q_i| \tag{4}$$

Compute the criteria weights.

$$W_j = \frac{T_j}{\sum_{j=1}^{n} T_j} \tag{5}$$

Sorting the alternatives.

Then we apply the steps of the MARCOS method[12], [13] such as:

Standardize the decision matrix

The decision matrix is standardized for the benefit and cost criteria such as:

$$h_{ij} = \frac{x_{ij}}{\max_i x_{ij}} \tag{6}$$

$$h_{ij} = \frac{x_{ij}}{\min_i x_{ij}} \tag{7}$$

Form the weighted standardized decision matrix

$$k_{ij} = W_j h_{ij} \tag{8}$$

Calculate the utility digress of alternatives

$$V_i^+ = \frac{S_i}{\max_i S_i} \tag{9}$$

$$V_i^- = \frac{S_i}{\min_i S_i} \tag{10}$$

$$S_i = \sum_{j=1}^{n} k_{ij} \tag{11}$$

Compute the utility function

$$f(V_i) = \frac{V_i^+ + V_i^-}{1 + \frac{1 - f(V_i^+)}{f(V_i^+)} + \frac{1 - f(V_i^-)}{f(V_i^-)}} \tag{12}$$

$$f(V_i^+) = \frac{V_i^-}{V_i^+ + V_i^-} \tag{13}$$

$$f(V_i^-) = \frac{V_i^+}{V_i^+ + V_i^-} \tag{14}$$

Forest HyperSoft Set

Let U be a universe of discourse and H is a non-empty subset of U, Q be a set of criteria. Each of the criteria has various levels.

Level 1 be the sub criteria values.

Level 2 be the sub-sub criteria values.

Level 3 be the sub-sub-sub criteria values.

Level 4 be the sub-sub-sub-sub criteria values.

Level n be the n-sub criteria values.

Each of these criteria can form the TreeSoft set and these TreeSoft are combined to form the Forest HyperSoft set[14].

We can define the Forest HyperSoft as:

$$G: P\big(Forest(Q)\big) \to P(H) \tag{15}$$

3. Application of the Proposed Framework

This section shows the results of the proposed approach. In this study we have four main criteria and nine alternatives. So, we divide the four main criteria into four TreeSoft sets. In each TreeSoft we can compute the criteria weights and ranking the alternatives. Fig 1 shows the classification of the criteria.

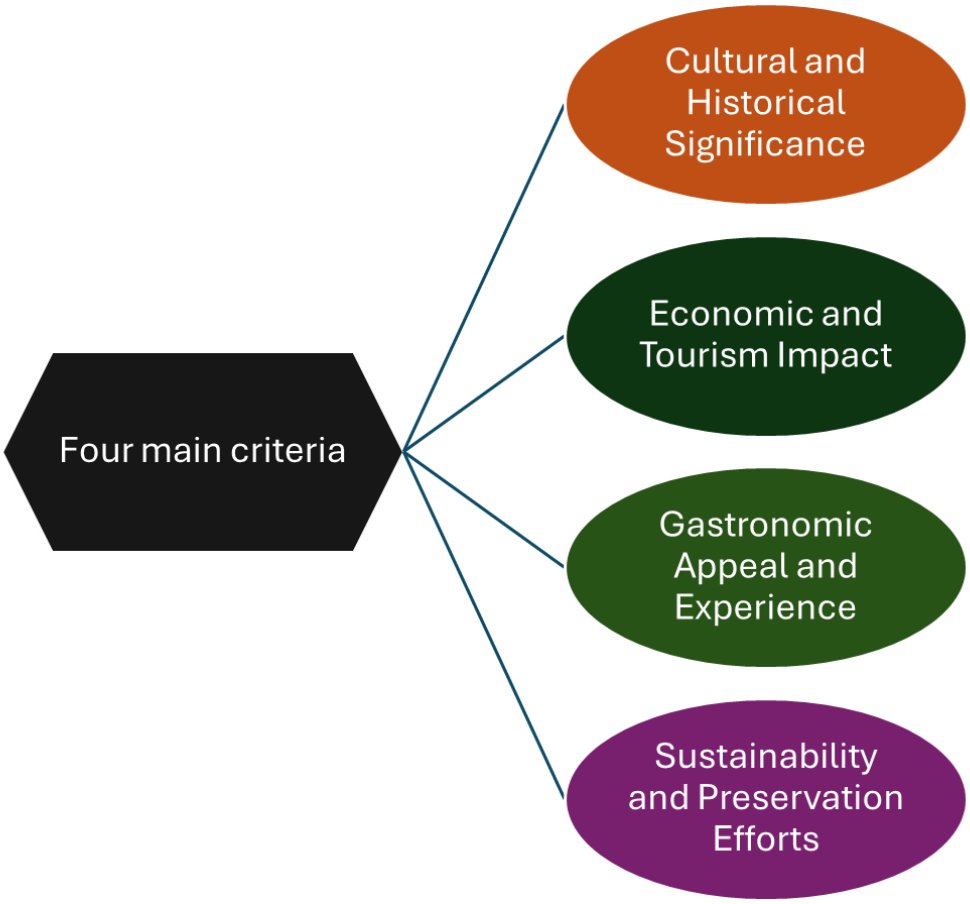

Fig 1. Classification of the main criteria.

Then we apply the MEREC and MARCOS method in each criterion.

First criterion

We established the evaluating matrix. Three experts evaluate the criteria and alternatives using a scale between 0.1 to 0.9. This criterion has three sub-criteria as shown in Fig 2.

We obtained the overall evaluation matrix using the average method.

We standardized the overall evaluation matrix using Eq. (1) as shown in Table 1.

We determined the overall performance of each alternative using Eq. (2).

Then we calculate the performance of the program after deleting each criterion using Eq. (3).

Then we assessed the combined of deviations from absolute values using Eq. (4).

Then we computed the criteria weights using Eq. (5) as shown in Fig 3.

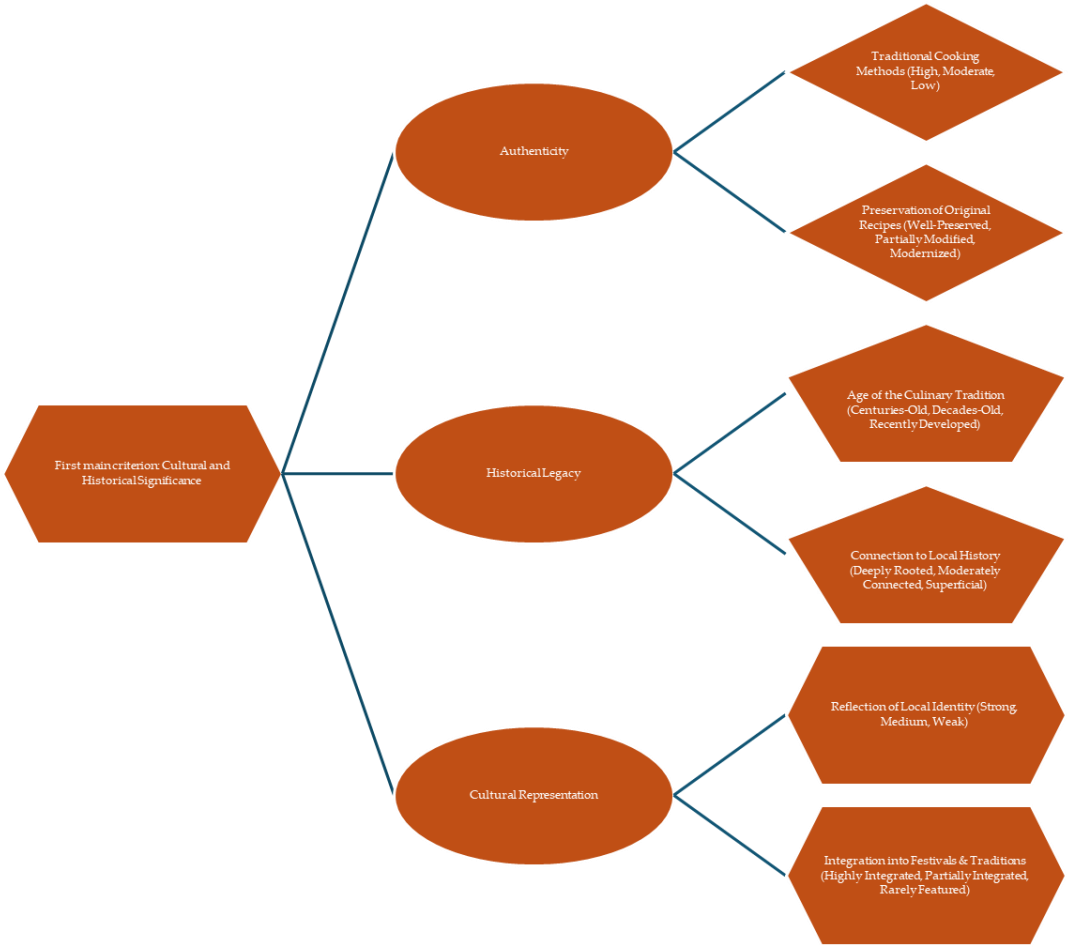
First main criterion: Cultural and Historical Significance
Authenticity
Historical Legacy
Cultural Representation
Traditional Cooking Methods (High, Moderate, Low)
Preservation of Original Recipes (Well-Preserved, Partially Modified, Modernized)
Age of the Culinary Tradition (Centuries-Old, Decades-Old, Recently Developed)
Connection to Local History (Deeply Rooted, Moderately Connected, Superficial)
Reflection of Local Identity (Strong, Medium, Weak)
Integration into Festivals & Traditions (Highly Integrated, Partially Integrated, Rarely Featured)

Fig 2. Classification of first criterion.

Table 1. Scandalized matrix.

	C_{1111}	C_{1121}	C_{1211}	C_{1221}	C_{1311}	C_{1321}
A_1	1.521523	1.290475	1.149151	1	1.258466	1
A_2	1.596351	1	1.484866	1.393295	1.404326	1.292269
A_3	1.120481	1.074638	1.149151	1	1.202264	1
A_4	1	1.420902	1.168068	1.664759	1.404326	1.201508
A_5	1.074575	1.242427	1	1.079677	1.258466	1.301808
A_6	1.28939	1.16863	1.288717	1.483601	1.561003	1.02989
A_7	1.596351	1.196903	1.298386	1.443082	1	1
A_8	1.166555	1.243689	1.120481	1.342328	1.437784	1.241755
A_9	1.484866	1.355268	1.428704	1.463251	1.112605	1.131098

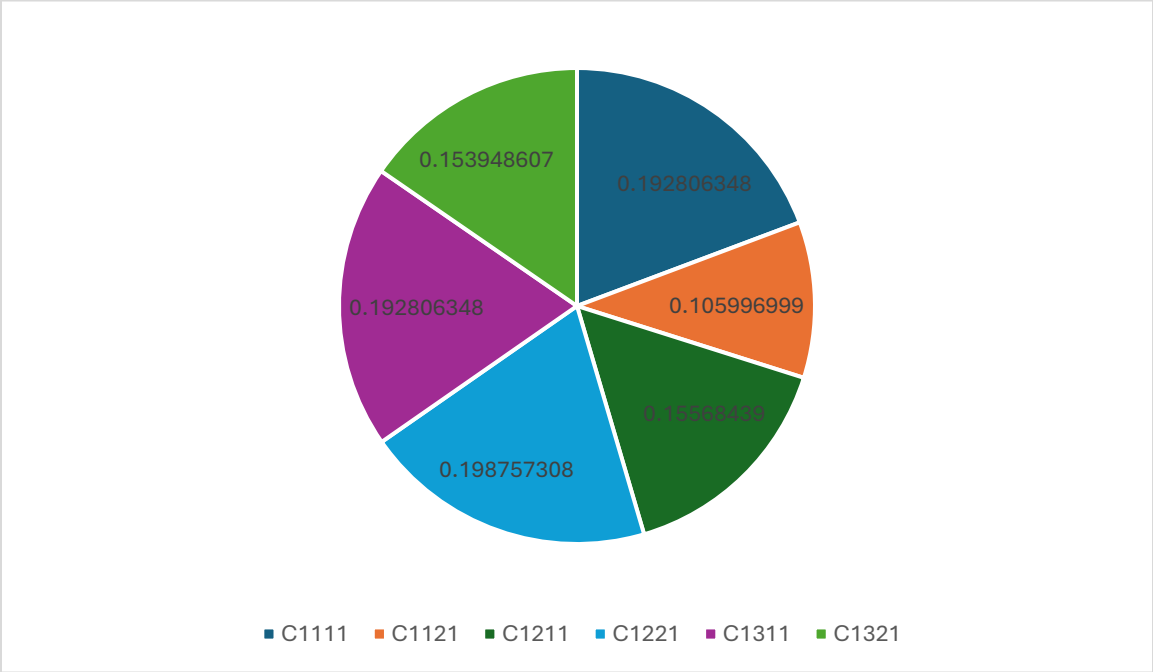

Fig 3. Weights of the first criterion.

We applied the MARCOS method to obtain Standardize the decision matrix using Eq. (6) as shown in Table 2.

Then we formed the weighted standardized decision matrix using Eq. (8) as shown in Table 3.

Then we calculated the utility digress of alternatives using Eqs. (9-11).

Then we computed the utility function using Eqs. (12-14). Then we ranked the alternatives in Fig 4.

Table 2. Scandalized matrix by MARCOS.

	C_{1111}	C_{1121}	C_{1211}	C_{1221}	C_{1311}	C_{1321}
A_1	0.953126	0.908208	0.773909	0.600688	1.240402	0.768162
A_2	1	0.703778	1	0.836935	1.111567	0.992672
A_3	0.701901	0.756307	0.773909	0.600688	1.298386	0.768162
A_4	0.626429	1	0.786649	1	1.111567	0.922953
A_5	0.673145	0.874393	0.673461	0.648548	1.240402	1
A_6	0.807711	0.822456	0.867901	0.891181	1	0.791123
A_7	1	0.842355	0.874413	0.866841	1.561003	0.768162
A_8	0.730763	0.875281	0.754601	0.80632	1.0857	0.95387
A_9	0.930163	0.953808	0.962177	0.878957	1.403016	0.868867

Table 3. Weighted scandalized matrix.

	C_{1111}	C_{1121}	C_{1211}	C_{1221}	C_{1311}	C_{1321}
A_1	0.183769	0.096267	0.120485	0.119391	0.239157	0.118258
A_2	0.192806	0.074598	0.155684	0.166347	0.214317	0.152821
A_3	0.135331	0.080166	0.120485	0.119391	0.250337	0.118258
A_4	0.120779	0.105997	0.122469	0.198757	0.214317	0.142087
A_5	0.129787	0.092683	0.104847	0.128904	0.239157	0.153949
A_6	0.155732	0.087178	0.135119	0.177129	0.192806	0.121792
A_7	0.192806	0.089287	0.136132	0.172291	0.300971	0.118258
A_8	0.140896	0.092777	0.11748	0.160262	0.20933	0.146847
A_9	0.179341	0.101101	0.149796	0.174699	0.27051	0.133761

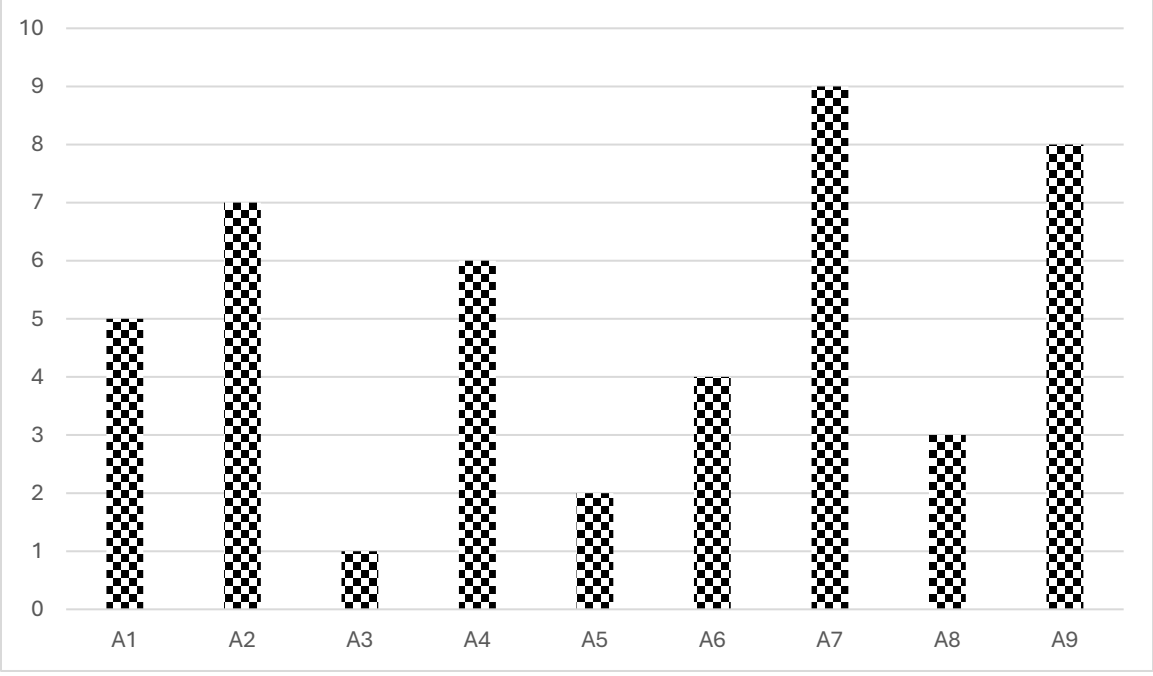

Fig 4. The ranks of the alternatives.

Second criterion

We established the evaluating matrix. Three experts evaluate the criteria and alternatives using a scale between 0.1 to 0.9. This criterion has three sub-criteria as shown in Fig 5.

We obtained the overall evaluation matrix using the average method.

We standardized the overall evaluation matrix using Eq. (1) as shown in Table 4.

We determined the overall performance of each alternative using Eq. (2).

Then we calculate the performance of the program after deleting each criterion using Eq. (3).

Then we assessed the combined of deviations from absolute values using Eq. (4).

Then we computed the criteria weights using Eq. (5) as shown in Fig 6.

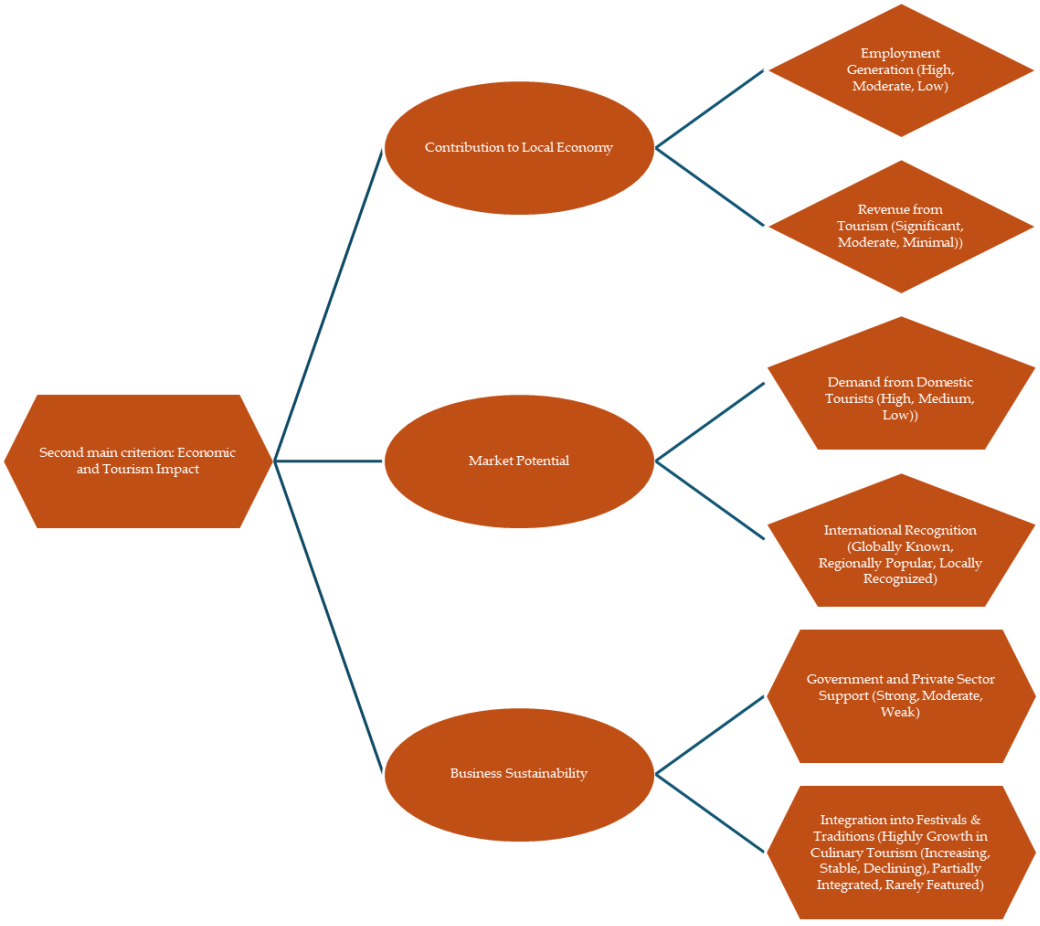
Second main criterion: Economic and Tourism Impact
Contribution to Local Economy
Market Potential
Business Sustainability
Employment Generation (High, Moderate, Low)
Revenue from Tourism (Significant, Moderate, Minimal))
Demand from Domestic Tourists (High, Medium, Low))
International Recognition (Globally Known, Regionally Popular, Locally Recognized)
Government and Private Sector Support (Strong, Moderate, Weak)
Integration into Festivals & Traditions (Highly Growth in Culinary Tourism (Increasing, Stable, Declining), Partially Integrated, Rarely Featured)

Fig 5. Classification of second criterion.

Table 4. Scandalized matrix.

	C_{2111}	C_{2121}	C_{2211}	C_{2221}	C_{2311}	C_{2321}
A_1	1.416706	1.315266	1.78317	1.000909	1.418157	1.201508
A_2	1.486379	1.228902	2.30411	1.53551	1.835175	1.543836
A_3	1.173634	1.628388	2.30411	1.52487	2.087254	1.160534
A_4	1	1	2.563601	1	1.783233	1.1003
A_5	1.104744	1.038679	1	1.26289	1	1.160534
A_6	1.313136	1.191081	1.725375	1.302992	1.506436	1.1003
A_7	1.225536	1.048285	2.101239	1.313176	1.581957	1.261743
A_8	1.173399	1.143825	1.738682	1.343548	1.328967	1.059962
A_9	1.382574	1.381304	1.927593	1.464581	1.253787	1

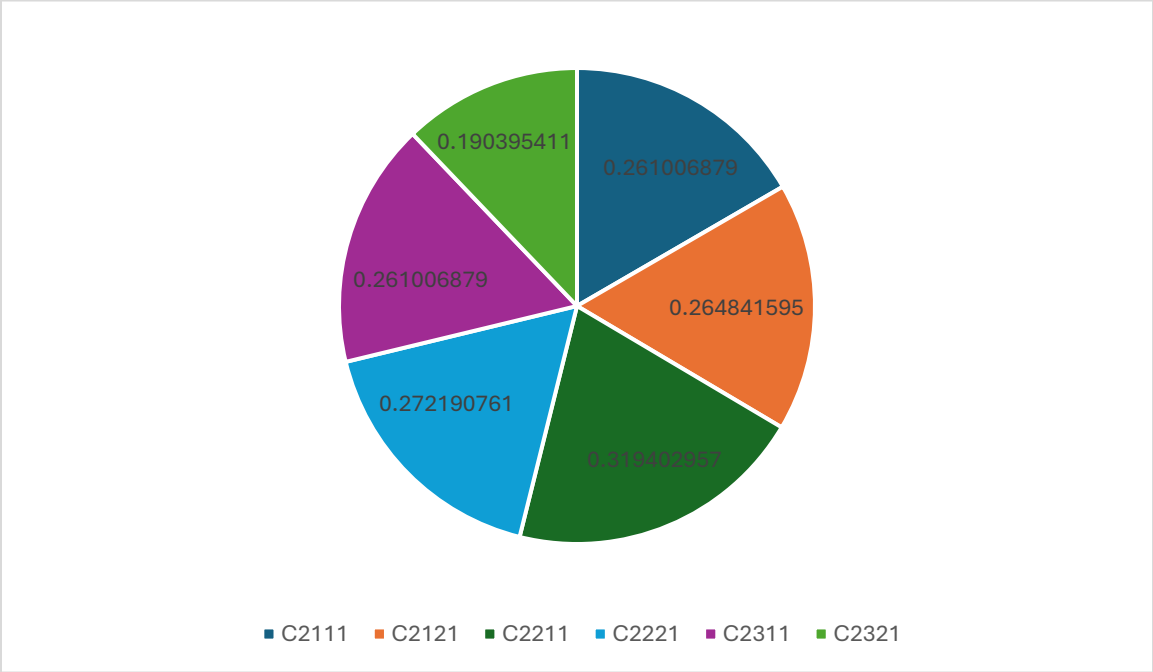

Fig 6. Weights of the second criterion.

We applied the MARCOS method to obtain Standardize the decision matrix using Eq. (6) as shown in Table 5.

Then we formed the weighted standardized decision matrix using Eq. (8) as shown in Table 6.

Then we calculated the utility digress of alternatives using Eqs. (9-11).

Then we computed the utility function using Eqs. (12-14). Then we ranked the alternatives in Fig 7.

Table 5. Scandalized matrix by MARCOS.

	C_{2111}	C_{2121}	C_{2211}	C_{2221}	C_{2311}	C_{2321}
A_1	0.953126	0.807711	0.695573	0.651842	1.471807	0.778262
A_2	1	0.754674	0.898779	1	1.13736	1
A_3	0.789593	1	0.898779	0.993071	1	0.751721
A_4	0.672776	0.614104	1	0.65125	1.170489	0.712705
A_5	0.743245	0.637857	0.390076	0.822456	2.087254	0.751721
A_6	0.883447	0.731448	0.673028	0.848573	1.385558	0.712705
A_7	0.824512	0.643756	0.819644	0.855205	1.319412	0.817278
A_8	0.789435	0.702428	0.678219	0.874985	1.570584	0.686577
A_9	0.930163	0.848265	0.751908	0.953808	1.664759	0.647737

Table 6. Weighted scandalized matrix.

	C_{2111}	C_{2121}	C_{2211}	C_{2221}	C_{2311}	C_{2321}
A_1	0.248772	0.213915	0.222168	0.177425	0.384152	0.148177
A_2	0.261007	0.199869	0.287073	0.272191	0.296859	0.190395
A_3	0.206089	0.264842	0.287073	0.270305	0.261007	0.143124
A_4	0.175599	0.16264	0.319403	0.177264	0.305506	0.135696
A_5	0.193992	0.168931	0.124592	0.223865	0.544788	0.143124
A_6	0.230586	0.193718	0.214967	0.230974	0.36164	0.135696
A_7	0.215203	0.170493	0.261797	0.232779	0.344376	0.155606
A_8	0.206048	0.186032	0.216625	0.238163	0.409933	0.130721
A_9	0.242779	0.224656	0.240162	0.259618	0.434513	0.123326

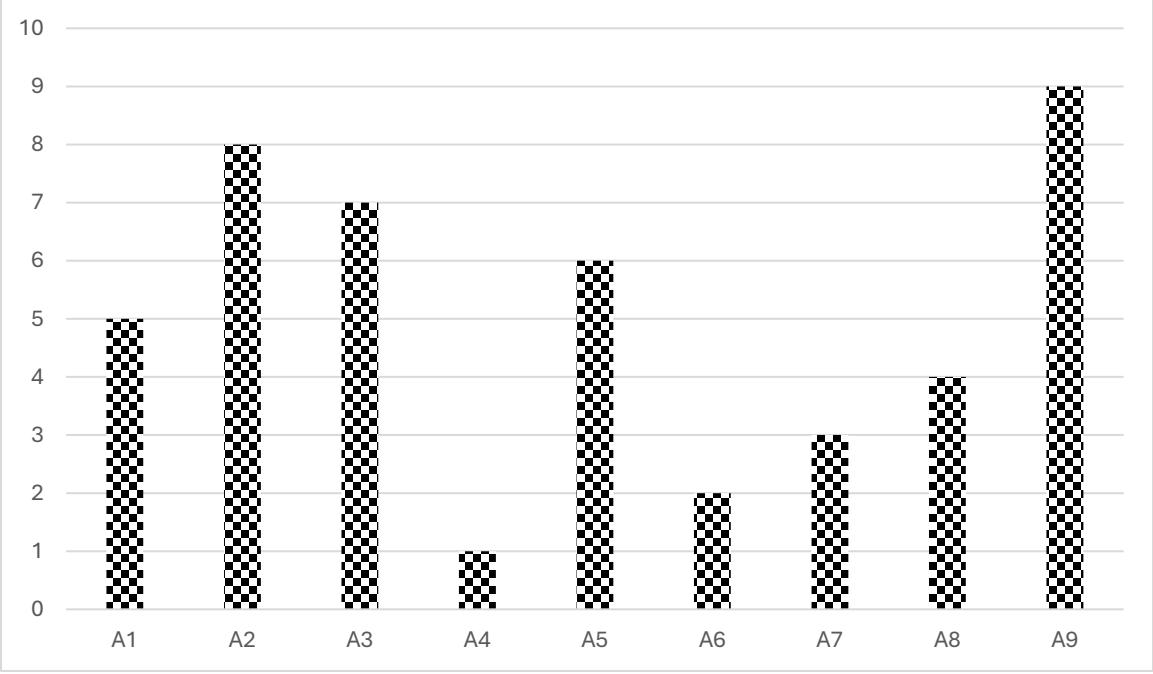

Fig 7. The ranks of the alternatives.

Third criterion

We established the evaluating matrix. Three experts evaluate the criteria and alternatives using a scale between 0.1 to 0.9. This criterion has three sub-criteria as shown in Fig 8.

We obtained the overall evaluation matrix using the average method.

We standardized the overall evaluation matrix using Eq. (1) as shown in Table 7.

We determined the overall performance of each alternative using Eq. (2).

Then we calculate the performance of the program after deleting each criterion using Eq. (3).

Then we assessed the combined of deviations from absolute values using Eq. (4).

Then we computed the criteria weights using Eq. (5) as shown in Fig 9.

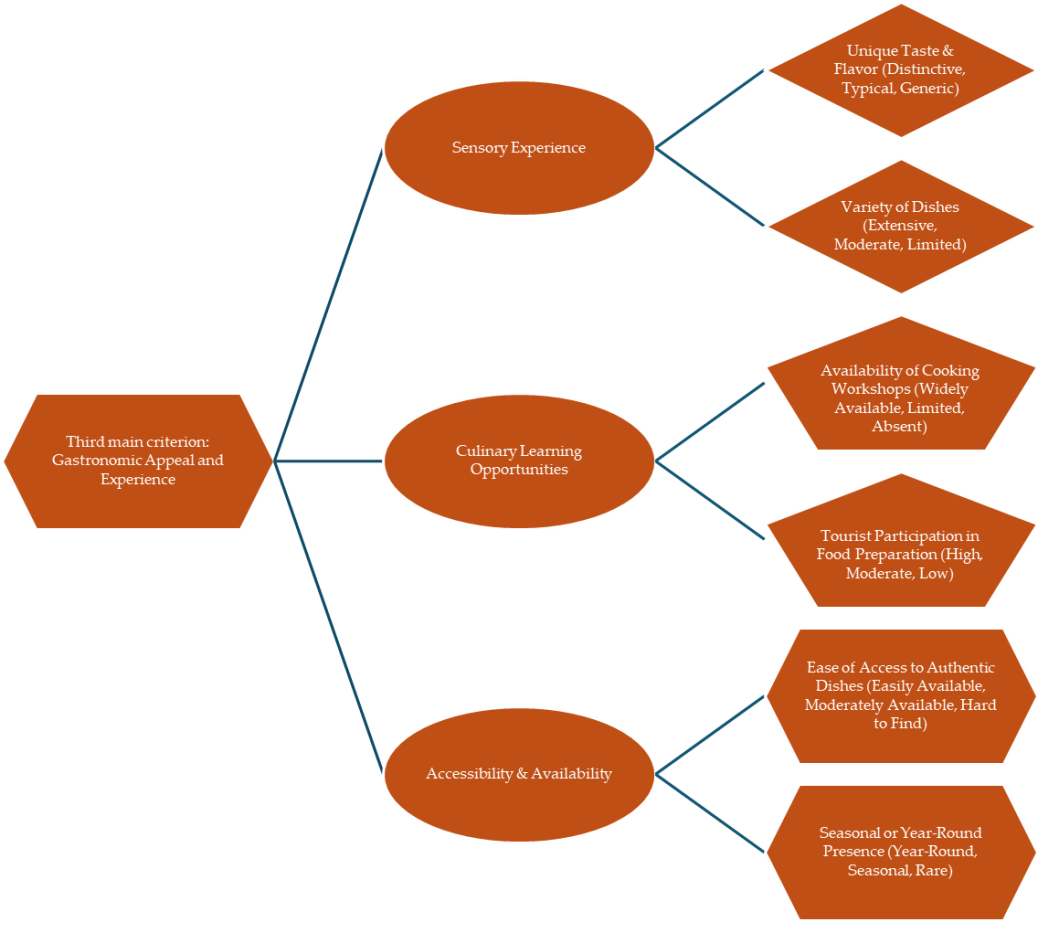
Third main criterion: Gastronomic Appeal and Experience
Sensory Experience
Culinary Learning Opportunities
Accessibility & Availability
Unique Taste & Flavor (Distinctive, Typical, Generic)
Variety of Dishes (Extensive, Moderate, Limited)
Availability of Cooking Workshops (Widely Available, Limited, Absent)
Tourist Participation in Food Preparation (High, Moderate, Low)
Ease of Access to Authentic Dishes (Easily Available, Moderately Available, Hard to Find)
Seasonal or Year-Round Presence (Year-Round, Seasonal, Rare)

Fig 8. Classification of third criterion.

Table 7. Scandalized matrix.

	C_{3111}	C_{3121}	C_{3211}	C_{3221}	C_{3311}	C_{3321}
A_1	1.044016	1.550187	1.241755	1	1.131098	1
A_2	1.115904	1	1.604524	1.262197	1.131098	1.262197
A_3	1	1.291721	1	1.131098	1.131098	1.131098
A_4	1.346908	1.112605	1.393295	1.261743	1.262197	1.392841
A_5	1.435341	1.840898	1	1.573726	1.261743	1.442627
A_6	1.400321	1.515112	1.785228	1.382393	1.70437	1.644136
A_7	1.471807	1.852219	1.542655	1.594349	1.563278	1.382393
A_8	1.114458	1.60558	1.604524	1.523213	1.483601	1.362678
A_9	1.418554	1.852219	1.543836	1.403016	1	1.131098

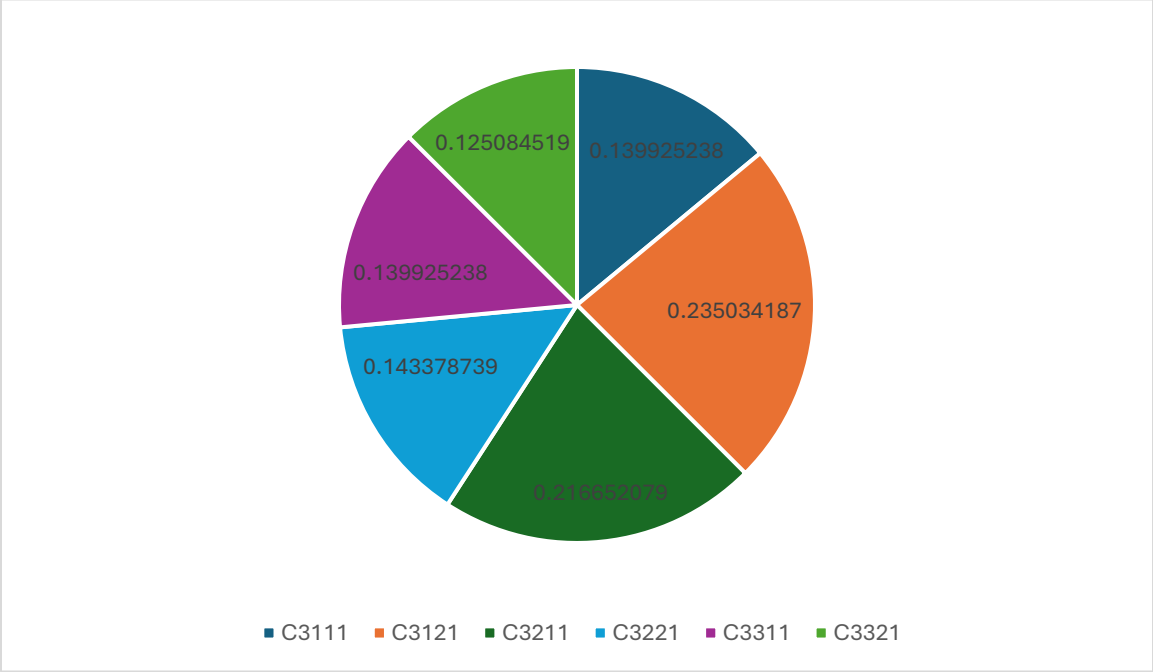

Fig 9. Weights of the third criterion.

We applied the MARCOS method to obtain Standardize the decision matrix using Eq. (6) as shown in Table 8.

Then we formed the weighted standardized decision matrix using Eq. (8) as shown in Table 9.

Then we calculated the utility digress of alternatives using Eqs. (9-11).

Then we computed the utility function using Eqs. (12-14). Then we ranked the alternatives in Fig 10.

Table 8. Scandalized matrix by MARCOS.

	C_{3111}	C_{3121}	C_{3211}	C_{3221}	C_{3311}	C_{3321}

A_1	0.709343	0.836935	0.695573	0.627215	1.506827	0.608222
A_2	0.758186	0.539893	0.898779	0.791669	1.506827	0.767696
A_3	0.679437	0.697391	0.560153	0.709442	1.506827	0.687959
A_4	0.915139	0.600688	0.780458	0.791384	1.35032	0.847157
A_5	0.975224	0.993888	0.560153	0.987065	1.350806	0.877438
A_6	0.95143	0.817998	1	0.867058	1	1
A_7	1	1	0.864122	1	1.090254	0.840802
A_8	0.757204	0.866841	0.898779	0.955382	1.148806	0.828811
A_9	0.963818	1	0.864784	0.879993	1.70437	0.687959

Table 9. Weighted scandalized matrix.

	C_{3111}	C_{3121}	C_{3211}	C_{3221}	C_{3311}	C_{3321}
A_1	0.099255	0.196708	0.150697	0.089929	0.210843	0.076079
A_2	0.106089	0.126893	0.194722	0.113509	0.210843	0.096027
A_3	0.09507	0.163911	0.121358	0.101719	0.210843	0.086053
A_4	0.128051	0.141182	0.169088	0.113468	0.188944	0.105966
A_5	0.136458	0.233598	0.121358	0.141524	0.189012	0.109754
A_6	0.133129	0.192258	0.216652	0.124318	0.139925	0.125085
A_7	0.139925	0.235034	0.187214	0.143379	0.152554	0.105171
A_8	0.105952	0.203737	0.194722	0.136981	0.160747	0.103671
A_9	0.134862	0.235034	0.187357	0.126172	0.238484	0.086053

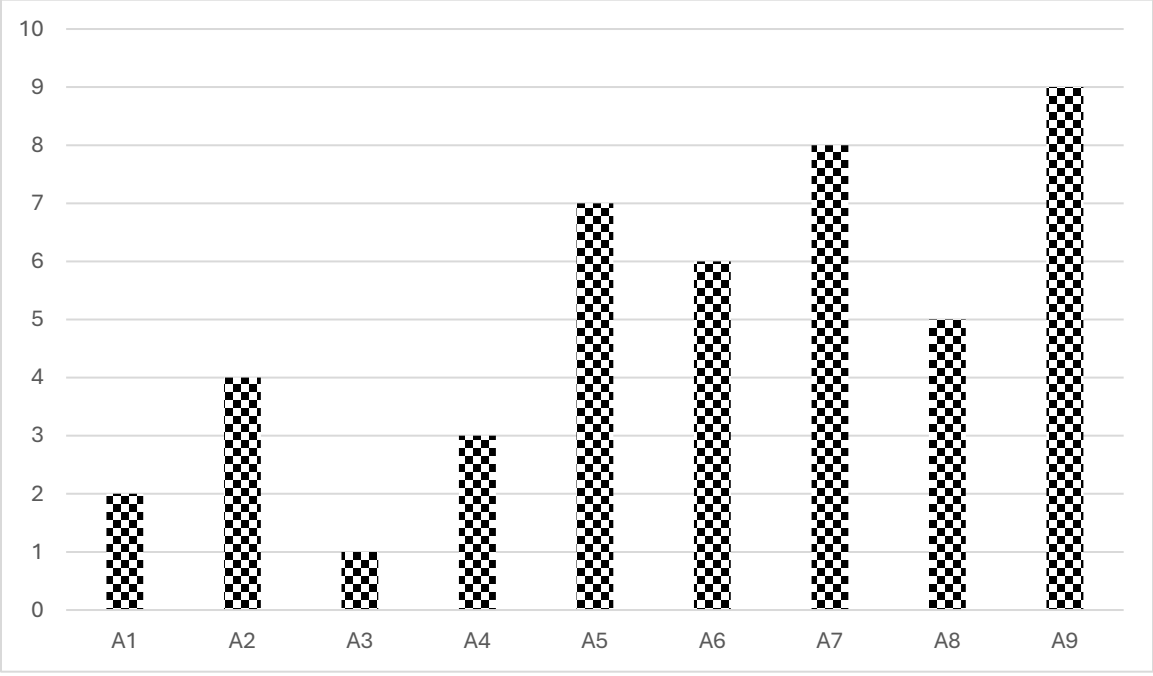

Fig 10. The ranks of the alternatives.

Fourth criterion

We established the evaluating matrix. Three experts evaluate the criteria and alternatives using a scale between 0.1 to 0.9. This criterion has three sub-criteria as shown in Fig 11.

We obtained the overall evaluation matrix using the average method.

We standardized the overall evaluation matrix using Eq. (1) as shown in Table 10.

We determined the overall performance of each alternative using Eq. (2).

Then we calculate the performance of the program after deleting each criterion using Eq. (3).

Then we assessed the combined of deviations from absolute values using Eq. (4).

Then we computed the criteria weights using Eq. (5) as shown in Fig 12.

Fourth main criterion: Sustainability and Preservation Efforts
Environmental Sustainability
Knowledge Transmission & Education
Government & Policy Support
Use of Local and Organic Ingredients (High, Medium, Low)
Eco-Friendly Packaging & Serving Methods (Sustainable, Partially Sustainable, Unsustainable)
Involvement of Young Generations (Strong, Moderate, Weak)
Role of Local Communities in Preservation (Active, Moderate, Minimal)
Cultural Heritage Protection Policies (Strong, Moderate, Weak)
Support from Tourism Boards (High, Medium, Low)

Fig 11. Classification of fourth criterion.

Table 10. Scandalized matrix.

	C_{4111}	C_{4121}	C_{4211}	C_{4221}	C_{4311}	C_{4321}
A_1	1.329575	1.199906	1.070237	1.016978	1.046746	1.180885

A_2	1.347349	1.085596	1.347349	1.242782	1.260215	1.392568
A_3	1.365124	1.328535	1.329575	1.346295	1.334791	1.382393
A_4	1.329575	1.364056	1.365124	1.328535	1.465781	1.563278
A_5	1.035549	1.346295	1.347349	1.208278	1.427611	1.1003
A_6	1.156683	1.016978	1.017775	1.242156	1.111905	1.392568
A_7	1.261139	1	1.000783	1.016978	1.241971	1
A_8	1	1.156404	1	1	1.074575	1.241755
A_9	1.382899	1.260152	1.330593	1.260152	1	1.131098

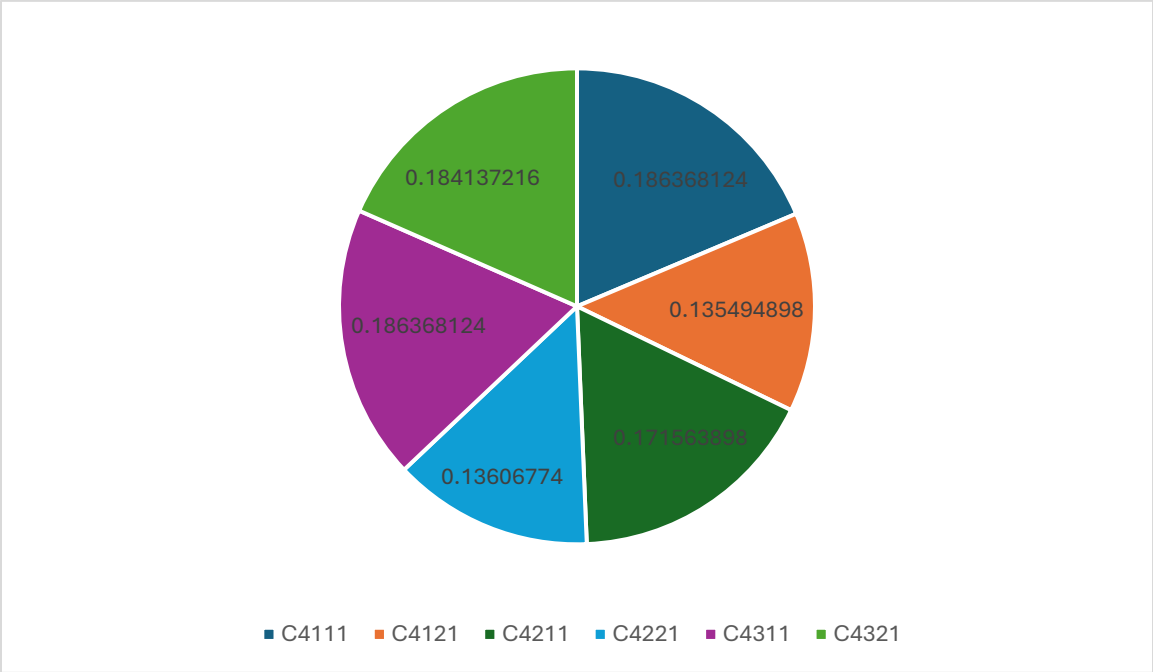

Fig 12. Weights of the fourth criterion.

We applied the MARCOS method to obtain Standardize the decision matrix using Eq. (6) as shown in Table 11.

Then we formed the weighted standardized decision matrix using Eq. (8) as shown in Table 12.

Then we calculated the utility digress of alternatives using Eqs. (9-11).

Then we computed the utility function using Eqs. (12-14). Then we ranked the alternatives in Fig 13.

Table 11. Scandalized matrix by MARCOS.

	C_{4111}	C_{4121}	C_{4211}	C_{4221}	C_{4311}	C_{4321}
A_1	0.96144	0.87966	0.783985	0.75539	1.400321	0.75539
A_2	0.974294	0.795859	0.986979	0.923113	1.16312	0.8908

A_3	0.987147	0.973959	0.973959	1	1.098136	0.884291
A_4	0.96144	1	1	0.986808	1	1
A_5	0.748825	0.986979	0.986979	0.897484	1.026737	0.703841
A_6	0.836419	0.745555	0.745555	0.922648	1.318261	0.8908
A_7	0.911953	0.733108	0.733108	0.75539	1.180206	0.639682
A_8	0.723119	0.847769	0.732534	0.742779	1.364056	0.794328
A_9	1	0.923827	0.974705	0.936014	1.465781	0.723543

Table 12. Weighted scandalized matrix.

	C_{4111}	C_{4121}	C_{4211}	C_{4221}	C_{4311}	C_{4321}
A_1	0.179182	0.11919	0.134504	0.102784	0.260975	0.139095
A_2	0.181577	0.107835	0.16933	0.125606	0.216768	0.164029
A_3	0.183973	0.131966	0.167096	0.136068	0.204657	0.162831
A_4	0.179182	0.135495	0.171564	0.134273	0.186368	0.184137
A_5	0.139557	0.133731	0.16933	0.122119	0.191351	0.129603
A_6	0.155882	0.101019	0.12791	0.125543	0.245682	0.164029
A_7	0.169959	0.099332	0.125775	0.102784	0.219953	0.117789
A_8	0.134766	0.114868	0.125676	0.101068	0.254217	0.146265
A_9	0.186368	0.125174	0.167224	0.127361	0.273175	0.133231

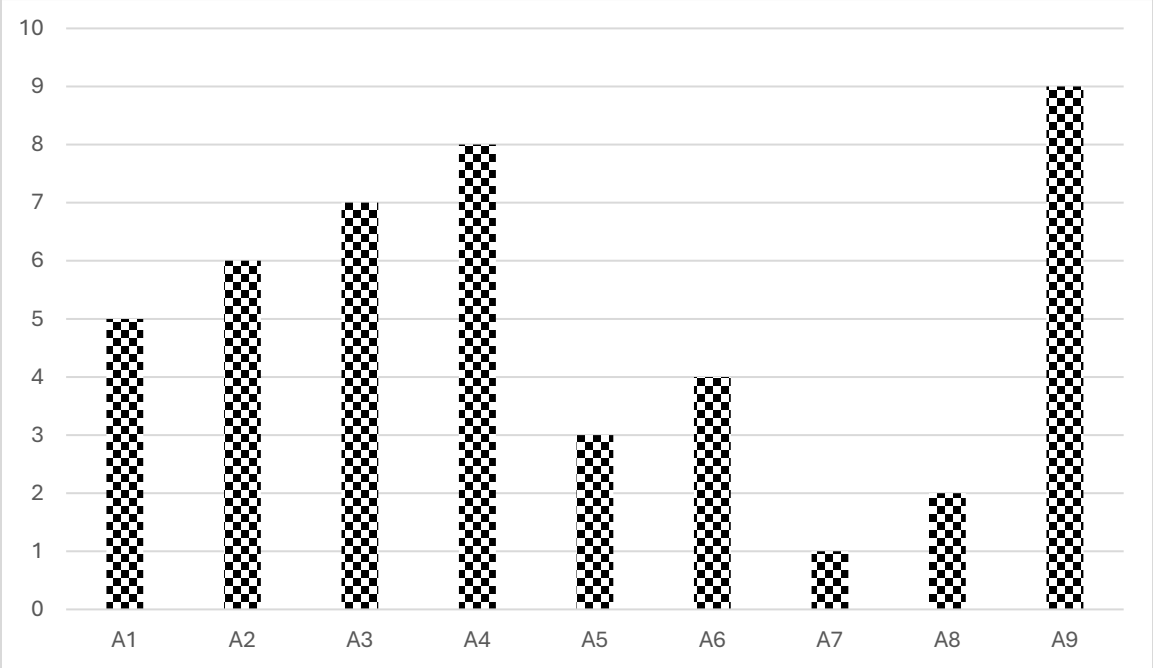

Fig 13. The ranks of the alternatives.

4. Conclusions

The evaluation of culinary intangible cultural heritage tourism resources is a necessary step in preserving and promoting the diversity of global food traditions. A structured, multi-criteria framework ensures that key aspects such as cultural authenticity, economic impact, gastronomic appeal, sustainability, and policy support are carefully analyzed. By leveraging advanced decision-making techniques, stakeholders can make informed decisions on how best to enhance, protect, and market culinary heritage for future generations. We used the Forest HyperSoft set to divide each criterion in main criteria as a TreeSoft set. In each TreeSoft set, we can compute the criteria weights by the MEREC method and ranking the alternatives.

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University of New Mexico

Enhanced Federated Learning Framework based on Deep Learning and Neutrosophic Set for Android Malware Classification

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Abstract: Malware detection is one of the critical tasks of cybersecurity, especially considering the growing popularity of mobile devices. The integrity and security of mobile ecosystems rely on the capacity to identify malware quickly and precisely, as security hacks against these devices become increasingly frequent. Due to the requirement for massive volumes of data aggregation, traditional centralized machine learning (ML) techniques for malware detection face challenges with data sharing, computational complexity, and privacy. This research addresses these challenges by proposing a novel model, called "CNN-Fed", which is based on Convolutional Neural Networks (CNN) with Federated Learning (Fed). The main goal of this work is to create a global classifier for Android malware detection that is highly accurate and does not require centralised data aggregation. Using four benchmark datasets—Drebin, Malgenome, Kronodroid, and Tuandromd—the CNN-Fed model is trained across many clients in a federated setting under the suggested architecture. Then we use multi-criteria decision-making (MCDM) methodology to evaluate the number of clients and number of rounds. We use two MCDM methods such as Entropy methodology to compute the criteria weights and the CoCoSo methodology to rank the alternatives. We use the interval-valued neutrosophic sets (IVNSs) to deal with uncertainty and vague information. We use five criteria such as accuracy, F1-score, precision, recall, and FP and six of alternatives refer to the number of clients and number of rounds. The results show the best alternative has a larger number of clients and rounds.

Keywords: Malware detection; Deep Learning; Federated Learning; Interval Valued Neutrosophic Sets.

1. Introduction

Data security and user privacy are becoming increasingly of a problem with the proliferation of personal and large computing systems. Deep learning and conventional machine learning methods become less useful

as malware threats rise. Malware is a type of software that targets and steals important data from computer systems, digital systems, and mobile devices. The journey of malware began in 1986 with the development of the first malware, the brain[1], [2].

1.1 Malware

For cyber security services, malware consultation is an essential topic as one of malicious software, requiring a large investment of time and human resources. Still, there is a gap between recent research and practical implementations. Depending on the type, target, and attacker's target, malware can affect individuals, groups, companies, governments, and society. Common effects of malware include [3]:

- Data Theft and Breaches.
- Financial Loss.
- System Disruption.
- Data Destruction.
- Privacy Violations.
- Propagation.
- Reputation Damage.
- Resource Consumption.
- Supply Chain Attack.

The primary reason malware detection is crucial is that it shields users, networks, and computer systems from the damaging effects of malicious software. Here are some key arguments supporting the importance of malware detection:

- Security.
- Data Protection.
- System Integrity.
- Financial Impact.
- Privacy.
- Mitigating Reputation Damage.
- Preventing Secondary Infections.
- Protection against Evolving Threats.

There are three common techniques employed in malware analysis [3]:
Static Analysis: This approach involves recognizing malware signatures and extracting important information from codes or files. It includes establishing the identity of the malware, associated libraries, URLs and the programming language.
Dynamic Analysis: The method looks at how a certain type of malware behaves when it is active in the real world under different environmental conditions. This helps in understanding what such software is designed for, how it propagates and potential harm that may be caused.
Hybrid Malware Analysis: Combining both static and dynamic analyses, this method offers a more complete and exact assessment of the behaviour of malware with respect to its behaviour, capabilities, effects.

1.2 Interval valued neutrosophic sets (IVNSs)

Well-known techniques for handling complicated situations include multicriteria decision-making (MCDM) models, which rank, prioritize, and sort various options while considering several criteria. Decision-makers can use MCDM models to convey ambiguous viewpoints using a variety of uncertainty sets, such as fuzzy logic[4], [5].

In Boolean logic, a proposition can be true or false, but nothing in between; in optimization, a solution is either feasible or not; and in conventional set theory, an element can either belong to a set or not. However, practically everything in actual life is imprecise, a question of degree, and outside the scope of conventional reasoning.

Zadeh developed the fuzzy sets theory to address this type of ambiguity. It was initially presented in 1965 and has since been expanded to include several fuzzy set variations. Zadeh created the type-n fuzzy set to deal with the membership function's uncertainty in fuzzy set theory. Zadeh invented interval-valued fuzzy sets (IVFSs)[6], [7].

To properly characterize a membership function with its membership and non-membership degrees, including the hesitation of decision-makers, Atanassov introduced intuitionistic fuzzy sets (IFSs). Torra first defined hesitant fuzzy sets (HFSs), which are expansions of conventional fuzzy sets in which a single element may belong to a set of values. Even with all these additions, the description of membership functions could not accurately represent the sentiments of decision-makers. As a result, certain extensions are still needed.

To overcome this shortcoming, Smarandache extended intuitionistic fuzzy sets to create neutrosophic logic and neutrosophic sets (NSs)[8], [9]. Every element of the cosmos has a degree of truthiness, indeterminacy, and falsehood inside the nonstandard unit interval, which is known as the neutrosophic set. The degree of belongingness or non-belongingness and the indeterminacy value were the factors integrated as relativity or absoluteness in the neutrosophic sets, where uncertainty is represented as truth and falsity values. In addition to managing system uncertainty, neutrosophic sets also lessen the indecision of contradicting data using this notation[10]. As a result, we may independently think of the indeterminacy value as the hesitation degree and the truth and falsehood values as membership and non-membership degrees[11], [12].

1.3 Learning Techniques

1.3.1 Deep learning for malware detection

Convolutional Neural Networks, sometimes known as CNNs, have recently been suggested as a DL technique for detecting malware. These ideas most certainly do not, however, uphold the DL principle that is essential for detecting malware. Instead, a sequence of linear transformations is frequently suggested, which makes them difficult to learn efficiently because they may result in limited performance from DL or increase classification efficiency in the absence of training data. The challenges prompted a development in Malware detection techniques. Malware detection has used a variety of DL models. On the other hand, few deep-learning options are available for real-time virus detection in an operational system [13], [14].

1.3.2 Federated Learning

Google introduced Federated Learning (FL) in 2016, providing a workaround for data privacy concerns while facilitating cross-device collaborative learning. With this novel approach, devices can work together to train ML algorithms without directly exchanging sensitive raw data. Particularly around cybersecurity, FL has demonstrated potential as a malware detection technique. FL distributed architecture not only safeguards security and privacy on distributed networks but also effectively thwarts malware threats. Here, FL protects sensitive information by allowing local clients on various devices to build models using their data. These models are then combined by a global server, enabling the construction of an all-encompassing defense system against dynamic malware attacks[14].

This paper has the following contributions:

- CNN-FED, a new model: This model combines Deep Learning (DL) and Federated Learning (FL) algorithms to detect malware.
- Global classifier of high accuracy: CNN-FED has been designed in such a way that it can generate highly accurate global classifiers without having access to distributed data.
- Results from experiments: Four well-known datasets, Drebin, Malgenome, Kronodroid, and Tuandromd were used for experiments. The experimental findings show high levels of accuracy for CNN-FED.
- We use interval-valued neutrosophic sets (IVNSs) methodology to evaluate the number of clients with the number of rounds after obtaining the results of the DL models.

- We use two MCDM methods such as the Entropy method to compute the criteria weights and the CoCoSo method to rank the alternatives.

2. Methodology

This section shows the steps of the proposed approach. We show some definitions of the IVNSs[15], [16].

Definition 1.

The neutrosophic sets has three membership functions such as Truth $T_C(y)$, Indeterminacy $I_C(y)$, and Falsity $F_C(y)$.

$$0-\leq \sup T_C(y) + \sup T_C(y) + \sup T_C(y) \leq 3 + \tag{1}$$

Definition 2.

There are two intervals valued neutrosophic numbers (IVNNs) and in this part, we introduce their operations such as:

$$C + D = \begin{pmatrix} \begin{bmatrix} T_C^L(y) + T_D^L(y) - T_C^L(y)T_D^L(y), \\ T_C^U(y) + T_D^U(y) - T_C^U(y)T_D^U(y) \end{bmatrix}, \\ [I_C^L(y)I_D^L(y), I_C^U(y)I_D^U(y)], \\ [F_C^L(y)F_D^L(y), F_C^U(y)F_D^U(y)] \end{pmatrix} \tag{2}$$

$$CD = \begin{pmatrix} [T_C^L(y)T_D^L(y), T_C^U(y)T_D^U(y)], \\ \begin{bmatrix} I_C^L(y) + I_D^L(y) - I_C^L(y)I_D^L(y) \\ , I_C^U(y) + I_D^U(y) - I_C^U(y)I_D^U(y) \end{bmatrix}, \\ \begin{bmatrix} F_C^L(y) + F_D^L(y) - F_C^L(y)F_D^L(y), \\ F_C^U(y) + F_D^U(y) - F_C^U(y)T_D^U(y) \end{bmatrix} \end{pmatrix} \tag{3}$$

$$C - D = \begin{pmatrix} \begin{bmatrix} T_C^L(y) - T_D^U(y), \\ T_C^U(y) - T_D^L(y) \end{bmatrix}, \\ \begin{bmatrix} \max\left(I_C^L(y), I_D^L(y)\right), \\ \max\left(I_C^U(y), I_D^U(y)\right) \end{bmatrix}, \\ \begin{bmatrix} F_C^L(y) - F_D^U(y), \\ F_C^U(y) - F_D^L(y) \end{bmatrix} \end{pmatrix} \tag{4}$$

$$\frac{C}{\aleph} = \begin{pmatrix} \left[\min\left(\frac{T_C^L(y)}{\aleph}, 1\right), \min\left(\frac{T_C^U(y)}{\aleph}, 1\right)\right], \\ \left[\min\left(\frac{I_C^L(y)}{\aleph}, 1\right), \min\left(\frac{I_C^U(y)}{\aleph}, 1\right)\right] \\ \left[\min\left(\frac{F_C^L(y)}{\aleph}, 1\right), \min\left(\frac{F_C^U(y)}{\aleph}, 1\right)\right] \end{pmatrix} \tag{5}$$

$$\frac{C}{D}=\left(\begin{bmatrix}\min\left\{\frac{T_C^L(y)}{T_D^L(y)},\frac{T_C^L(y)}{T_D^U(y)},\\ \frac{T_C^U(y)}{T_D^L(y)},\frac{T_C^U(y)}{T_D^U(y)}\end{bmatrix},\\ \begin{bmatrix}\max\left\{\frac{T_C^L(y)}{T_D^L(y)},\frac{T_C^L(y)}{T_D^U(y)},\\ \frac{T_A^U(x)}{T_B^L(x)},\frac{T_A^U(x)}{T_B^U(x)}\end{bmatrix},\\ \begin{bmatrix}\min\left\{\frac{I_C^L(y)}{I_D^L(y)},\frac{I_C^L(y)}{I_D^U(y)}\right\},\\ \frac{I_C^U(y)}{I_D^L(y)},\frac{I_C^U(y)}{I_D^U(y)}\end{bmatrix},\\ \begin{bmatrix}\max\left\{\frac{I_C^L(y)}{I_D^L(y)},\frac{I_C^L(y)}{I_D^U(y)}\right\},\\ \frac{I_C^U(y)}{I_D^L(y)},\frac{I_C^U(y)}{I_D^U(y)}\end{bmatrix},\\ \begin{bmatrix}\min\left\{\frac{F_C^L(y)}{F_D^L(y)},\frac{F_C^L(y)}{F_D^U(y)}\right\},\\ \frac{F_C^U(y)}{F_D^L(y)},\frac{F_C^U(y)}{F_D^U(y)}\end{bmatrix},\\ \begin{bmatrix}\max\left\{\frac{F_C^L(y)}{F_D^L(y)},\frac{F_C^L(y)}{F_D^U(y)}\right\},\\ \frac{F_C^U(y)}{F_D^L(y)},\frac{F_C^U(y)}{F_D^U(y)}\end{bmatrix}\end{pmatrix} \tag{6}$$

$$C^{-1}=\begin{pmatrix}\left[\left(T_C^L(y)\right)^{-1},\left(T_C^U(y)\right)^{-1}\right],\\ \left[\left(I_C^L(y)\right)^{-1},\left(I_C^U(y)\right)^{-1}\right],\\ \left[\left(F_C^L(y)\right)^{-1},\left(F_C^U(y)\right)^{-1}\right]\end{pmatrix} \tag{7}$$

Definition 3.

We can compute the score function such as:

$$S(C)=\begin{pmatrix}\frac{T_C^L(y)+T_C^U(y)}{2}+\\ \left(1-\frac{\left(I_C^L(y)+I_C^U(y)\right)}{2}\right)\left(I_C^U(y)\right)-\\ \left(\frac{\left(F_C^L(y)+F_C^U(y)\right)}{2}\right)\left(1-F_C^U(y)\right)\end{pmatrix} \tag{8}$$

IVN-Entropy

This method is used to compute the criteria weights. The steps of the Entropy method are shown as follows[17], [18]:

Create the decision matrix. Experts are created the decision matrix using the IVNNs between the criteria and alternatives.

Normalize the decision matrix

$$x_{ij}=\frac{y_{ij}}{\sum_{i=1}^{m}y_{ij}} \tag{9}$$

Where y_{ij} refers to the value in the decision matrix, $i=1,\dots,m; j=1,\dots,n$

Determine the entropy

$$e_j = -t \sum_{i=1}^{m} x_{ij} \ln x_{ij} \tag{10}$$

$$t = \frac{1}{\ln m} \tag{11}$$

Compute the criteria weights

$$w_j = \frac{1-e_j}{\sum_{j=1}^{n} 1-e_j} \tag{12}$$

IVN-CoCoSo Method

This section shows the steps of the CoCoSo method to rank the alternatives[19], [20].
Normalize the decision matrix by the CoCoSo method for positive and negative criteria.

$$k_{ij} = \frac{y_{ij} - \min_i y_{ij}}{\max_i y_{ij} - \min_i y_{ij}} \tag{13}$$

$$k_{ij} = \frac{\max_i y_{ij} - y_{ij}}{\max_i y_{ij} - \min_i y_{ij}} \tag{14}$$

Determine the power weight of decision matrix and total weighted decision matrix

$$U_i = \sum_{j=1}^{n} w_j k_{ij} \tag{15}$$

$$Q_i = \sum_{j=1}^{n} (k_{ij})^{w_j} \tag{16}$$

Compute the relative weights

$$H_{ia} = \frac{U_i + Q_i}{\sum_{i=1}^{m} (Q_i + U_i)} \tag{17}$$

$$H_{ib} = \frac{U_i}{\min_i U_i} + \frac{Q_i}{\min_i Q_i} \tag{18}$$

$$H_{ic} = \frac{ב(U_i) + (1-ב)(Q_i)}{\left(ב \max_i U_i + (1-ב) \max_i Q_i\right)} \quad 0 \le ב \le 1 \tag{19}$$

Rank the alternatives such as

$$H_i = (H_{ia} H_{ib} H_{ic})^{\frac{1}{3}} + \frac{1}{3}(H_{ia} + H_{ib} + H_{ic}) \tag{20}$$

The aim of this study is to utilize a deep-federated approach for identifying malicious software in mobile environments. This section gives an overview of CNN-Fed model shown in Figure 1, and discusses data preparation, modelling, optimization, and federated training processes:

Data Preparation: This stage entails changing of data, generating numerical data and applying standardization and normalization methods.

General Overview: At the server level, the proposed CNN-Fed model has a global model, which is a convolutional neural network and an aggregator. The server shares a pre-trained global model with all clients who train their local models with their own data. Clients then send their updated parameters back to the server, which combines all local models' outputs. The input of the model contains logs on mobile network traffic while its output is classified into zero for benign traffic or one for malicious traffic.

A Suggested Federated Learning Procedure: To begin the training process for the CNN-Fed model, global CNN model parameters are initially set on a global server. These are altered using an optimization algorithm thus starting the training phase. The process of training occurs in rounds that are connected and represented by n_rounds. Each round involves a random selection of clients who participate to ensure heterogeneous client representation called n_client. During this selection, clients are notified about the current global model's parameters, which would help, make their training uniform based on the same basic conditions. Through this, they pass their data through local training for n_epocs epochs and share it with the global server after updating its parameters; this is what happens during local training phases. Clients undertake a local training phase to adjust the models according to their data properties — optimizing their performance. Several rounds of communication, local trainings, and model aggregations later resulted in finalizing them. They reflect the combined efforts of all contributing clients to develop a common pool of knowledge and expertise in solving different computing problems globally. Such parameters can be used for many other downstream tasks.

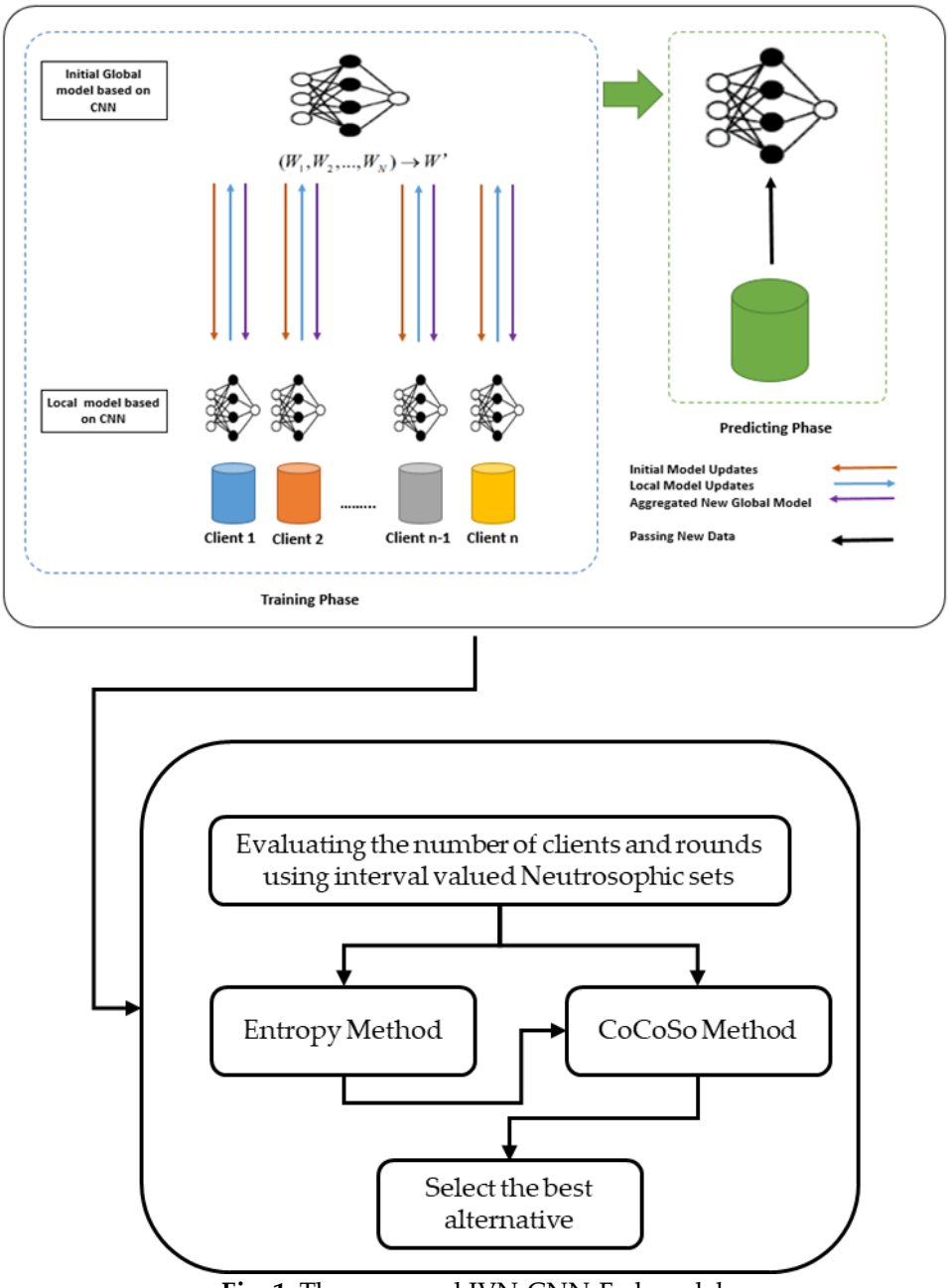

Fig 1. The proposed IVN-CNN-Fed model

. By enabling knowledge exchange among clients, the server eliminates the need for direct data access. As a result, there is no requirement to share data with other clients, and the server maintains data privacy.

3. Experiment

3.1 Dataset

Our work has used four publicly accessible datasets. Table 1 presents a summary of the datasets under consideration:

Table 1. Dataset description

Dataset Name	Ref	No. of Samples	No. of Features	Target Class	
				Benign	Malware
Drebin	[21]	15036	215	9476	5560
Malgenome	[22]	3799	215	2539	1260
Kronodrid	[23]	78137	463	36935	41382
Tunadromd	[24]	4465	241	903	3565

3.2 Performance Metric

The evaluation of the suggested CNN-Fed model for malware detection is assessed using different classification metrics. The primary metrics used in this study include:
True positive (γ): Occurrence in malware samples where the prediction was accurate.
False positive (μ): A test result indicates that a mobile device has malware while it does not contain malware.
True negative (σ): A true negative accurately predicts benign ware in samples.
False negative (ϱ): A test result that indicates that the mobile does not contain malware while the mobile does indeed contain malware.

• Precision:

Precision, also known as Positive Predictive Value, indicates the proportion of correct identifications. It is crucial to understand the reliability of the model in identifying malware.

$$\text{Precision} = \frac{\gamma}{(\gamma+\mu)} \quad (21)$$

Where γ represents true positives and μ represents false positives.

• Recall:

Recall, also known as Sensitivity or True Positive Rate, measures the ability of the model to correctly identify all relevant instances of the positive class (malware).

$$\text{Recall} = \frac{\gamma}{(\gamma+\rho)} \quad (22)$$

Where ϱ represents false negatives. A higher recall indicates that the model is effective at identifying most of the malware.

• Accuracy:

Accuracy measures the overall correctness of the model by calculating the ratio of correctly predicted instances (both true positives and true negatives) to the total number of instances.

$$\text{Accuracy}=(\gamma+\sigma)/(\gamma+\rho+\sigma+\mu) \quad (23)$$

• F1-Score:

The F1-Score is the harmonic mean of precision and recall, providing a single metric that balances both concerns. It is particularly useful when dealing with imbalanced datasets.

$$\text{F1-Score} = \frac{2*(\text{Precision}*\text{Recall})}{(\text{Precision}+\text{Recall})} \quad (24)$$

• False Positive Rate (FPR):

The False Positive Rate measures the proportion of benign instances that are incorrectly classified as malware. It is calculated as:

$$\text{FPR} = \frac{2*(\text{Precision}*\text{Recall})}{(\text{Precision}+\text{Recall})} \quad (25)$$

Where μ represents false positives and σ represents true negatives. A lower FPR indicates a better performance in distinguishing benign traffic from malware.

These metrics collectively offer a robust framework for evaluating the CNN-Fed model. By analysing these performance indicators, we can determine the efficacy of the proposed federated learning approach in accurately detecting malware in Mobile environments.

3.3 Results, Analysis, and Discussion

The experiments are conducted on four datasets for Android malware classification: Drebin, Malgenome, Kronodroid, and Tuandromd. Furthermore, a comparative analysis between the performance of our CNN-Fed model and other similar models was performed. The evaluation considered different numbers of clients and communication rounds to analyze the scalability and robustness of the models.

We examined three different scenarios based on client numbers, with 5, 10, and 15 clients. For each scenario, we conducted experiments with 10 and 20 rounds to assess the impact of the number of clients and rounds on model performance. Investigate how changes in client numbers and communication rounds influence the effectiveness of our proposed CNN-Fed model compared to Dw-FedAvg.

Figure 2 and Figure 3 show that CNN-Fed demonstrated competitive accuracy; on the Malgenome dataset (20 clients, five rounds), its highest accuracy was 0.9923. In general, DW-FedAvg exhibited larger loss values than CNN-Fed, suggesting that CNN-Fed performed better as a model in terms of minimising error. With twenty clients and five rounds in the Kronodroid dataset, CNN-Fed's lowest loss was 0.5151. With twenty clients and five rounds in the Tuandromd dataset, the lowest loss for DW-FedAvg was 0.3913. High F1 scores for both models indicate a favourable trade-off between precision and recall. With twenty clients and five rounds in the Tuandromd dataset, the greatest F1-score for DW-FedAvg was 0.9921. With twenty clients and five rounds in the Tuandromd dataset, the greatest F1-score for CNN-Fed was 0.9797. All scenarios exhibited great precision and recall, with both measures closely matched, suggesting consistent performance in minimising false negatives and recognizing real positives. The greatest accuracy and recall values for DW-FedAvg on the Malgenome dataset (10 clients, five rounds) were 0.9950 and 0.9950, respectively. CNN-Fed's best precision on the Malgenome dataset with ten clients and five rounds was 0.9934, while its greatest recall on the Tuandromd dataset (10 clients, five rounds) was 0.9950. With AUC values continuously over 0.99, the two models showed exceptional performance and a high degree of ability to distinguish between positive and negative classifications. With twenty clients and fifteen rounds in the Malgenome dataset showed that CNN-Fed's greatest AUC was 0.9997. FPR was low across all scenarios, with values mostly below 0.05, indicating that both models had a low rate of incorrectly classifying benign samples as malicious. The lowest FPR for CNN-Fed was 0.0054 on the Malgenome dataset (20 clients, five rounds).

The comparison in Figure 4 shows the Impact of the number of clients and number of rounds through the experiment.

Impact of the number of clients with every model CNN-Fed: Drebin Dataset: Accuracy slightly decreases as the number of clients increases (from 0.9845 with five clients to 0.9770 with fifteen clients in ten rounds). Malgenome Dataset: Accuracy decreases with more clients (0.9907 with five clients to 0.9892 with fifteen clients in ten rounds). Kronodroid Dataset: Accuracy decreases with more clients (0.9684 with five clients to 0.9628 with fifteen clients in ten rounds). Tuandromd Dataset: Accuracy decreases with more clients (0.9837 with five clients to 0.9778 with fifteen clients in ten rounds). DW-FedAvg: Drebin Dataset: Accuracy slightly decreases as the number of clients increases (from 0.9828 with five clients to 0.9734 with fifteen clients in ten rounds). Malgenome Dataset: Accuracy also shows a decrease with more clients (0.9943 with five clients to 0.9875 with fifteen clients in ten rounds). Kronodroid Dataset: Accuracy decreases with more clients (0.9597 with five clients to 0.9479 with fifteen clients in ten rounds). Tuandromd Dataset: Accuracy decreases with more clients (0.9861 with five clients to 0.9781 with fifteen clients in ten rounds).

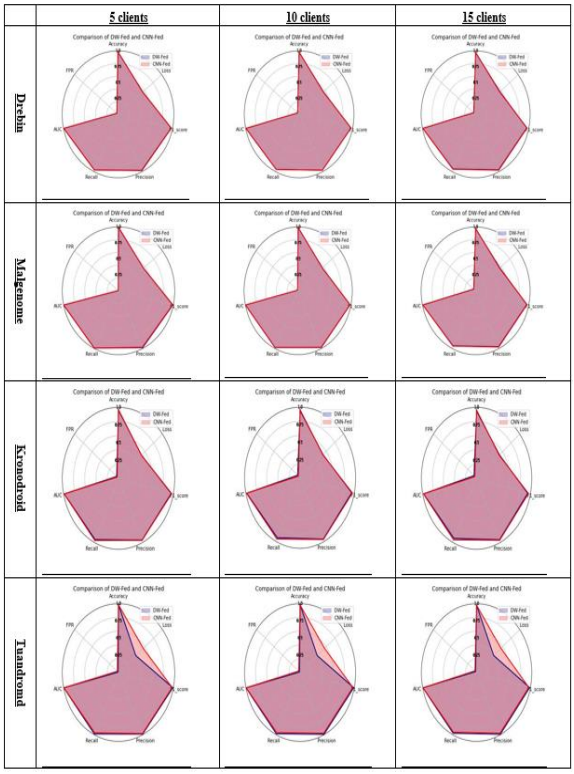

Figure 2: Performance comparison between CNN-Fed and DW-Fed for 10 Round

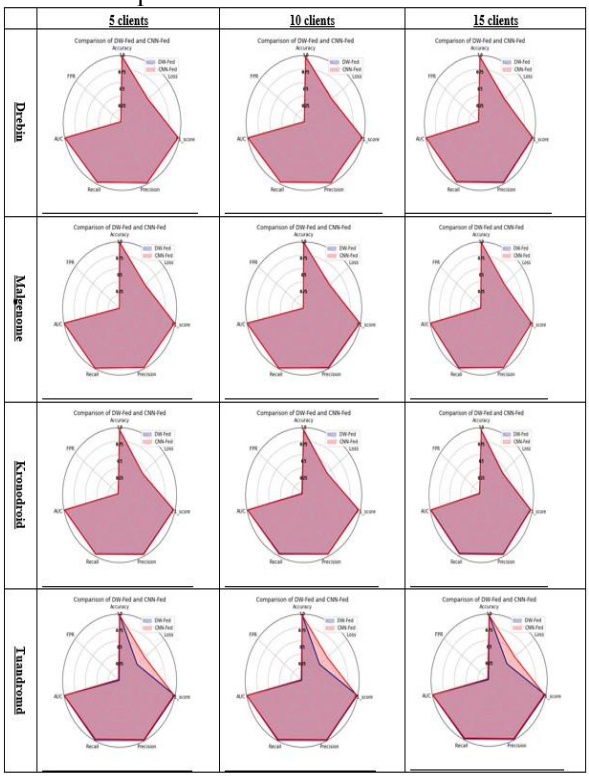

Figure 3: Performance comparison between CNN-Fed and DW-Fed for 20 Round

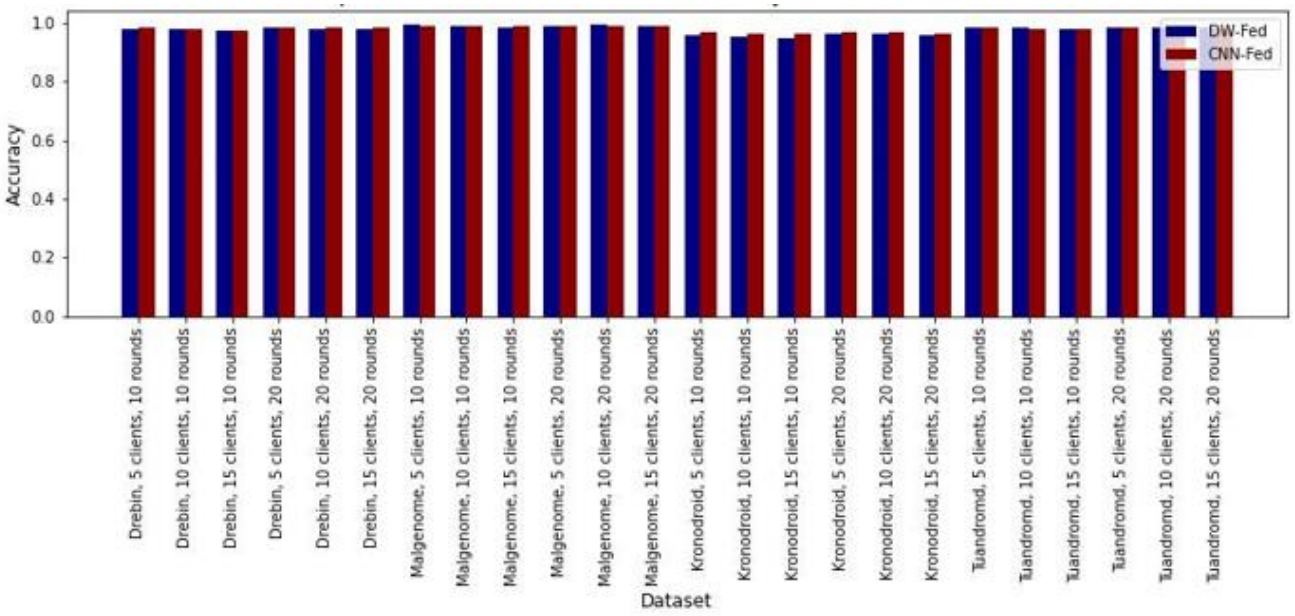

Figure 4: Accuracy comparison between CNN-Fed and DW-Fed

Impact of the number of rounds with every model CNN-Fed: Drebin Dataset: Accuracy increases with more rounds (from 0.9845 in 10 rounds to 0.9850 in 20 rounds with five clients). Malgenome Dataset: Accuracy increases with more rounds (from 0.9907 in 10 rounds to 0.9923 in 20 rounds with five clients). Kronodroid Dataset: Accuracy increases with more rounds (from 0.9684 in 10 rounds to 0.9713 in 20 rounds with five clients). DW-FedAvg: Drebin Dataset: Accuracy increases with more rounds (from 0.9828 in 10 rounds to 0.9841 in 20 rounds with five clients). Malgenome Dataset: Accuracy increases with more rounds (from 0.9943 in 10 rounds to 0.9923 in 20 rounds with five clients). Kronodroid Dataset: Accuracy increases with more rounds (from 0.9597 in 10 rounds to 0.9662 in 20 rounds with five clients). Tuandromd Dataset: Accuracy increases with more rounds (from 0.9861 in 10 rounds to 0.9871 in 20 rounds with five clients).

Then we evaluate the number of clients and rounds using the IVN-Entropy-CoCoSo method. The Entropy method is used to compute the criteria weights. Then we rank the alternatives using the CoCoSo method. The criteria of this study are accurate, precision, recall, f1-score, and false positive rate. The alternatives are shown the number of client and number of rounds.

In the first dataset

Results of IVN-Entropy

We show the results of the Entropy method. We applied the Entropy and CoCoSo methods into four datasets to show the criteria weights and ranking the alternatives.

Three experts are created the decision matrix using the IVNNs as shown in Table 2. Then we apply the score function to obtain the crisp values. Then we combine the decision matrix into a single matrix.

Eq. (9) is used to normalize the decision matrix as shown in Table 3.

Then we determine the entropy using Eqs. (10 and 11).

Then we compute the criteria weights using Eq. (12).

Table 2. Decision matrix.

	$IVNC_1$	$IVNC_2$	$IVNC_3$	$IVNC_4$	$IVNC_5$
$IVNA_1$	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
$IVNA_2$	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
$IVNA_3$	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])

IVNA4	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.1,0.2],[0.1,0.2],[0.8,0.9])
IVNA5	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
IVNA6	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.6,0.7],[0.4,0.5],[0.3,0.4])
	IVNC1	IVNC2	IVNC3	IVNC4	IVNC5
IVNA1	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
IVNA2	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
IVNA3	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])
IVNA4	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.1,0.2],[0.1,0.2],[0.8,0.9])
IVNA5	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.6],[0.5,0.6],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
IVNA6	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])
	IVNC1	IVNC2	IVNC3	IVNC4	IVNC5
IVNA1	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])
IVNA2	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])
IVNA3	([0.1,0.2],[0.1,0.2],[0.8,0.9])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])
IVNA4	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.2,0.3],[0.3,0.4],[0.7,0.8])	([0.2,0.3],[0.3,0.4],[0.7,0.8])
IVNA5	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.6,0.7],[0.4,0.5],[0.3,0.4])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.5,0.5],[0.6,0.7],[0.4,0.5])
IVNA6	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.5,0.5],[0.6,0.7],[0.4,0.5])	([0.3,0.4],[0.4,0.5],[0.6,0.7])	([0.4,0.5],[0.5,0.6],[0.5,0.6])	([0.4,0.5],[0.5,0.6],[0.5,0.6])

Table 3. Normalized decision matrix.

	IVNC1	IVNC2	IVNC3	IVNC4	IVNC5
IVNA1	0.150402	0.153207	0.183711	0.185383	0.192135
IVNA2	0.205511	0.160333	0.132272	0.153298	0.168539
IVNA3	0.144661	0.157957	0.183711	0.185383	0.182022
IVNA4	0.176808	0.13658	0.169014	0.170529	0.093258
IVNA5	0.174512	0.183492	0.18861	0.160428	0.179213
IVNA6	0.148106	0.208432	0.142682	0.144979	0.184831

Results of IVN-CoCoSo Method

We state with the combined decision matrix. Then we normalize the decision matrix using Eqs. (13 and 14) as shown in Table 4.

Then we determine the power weight of decision matrix and total weighted decision matrix suing Eq. (15 and 16) as shown in Tables 5 and 6.

Then we compute the relative weights using Eqs. (17-19)

Then we rank the alternatives based on Eq. (20).

Table 4. Normalized decision matrix by CoCoSo.

	IVNC1	IVNC2	IVNC3	IVNC4	IVNC5
IVNA1	0	0.326087	0.875	1	0.935065
IVNA2	1	0	0	0.4375	0.688312
IVNA3	0.796407	0	0.875	1	0.935065
IVNA4	0.700599	0.021739	0.4375	0.739583	0
IVNA5	0.676647	0	1	0.71875	0.939394
IVNA6	0.401198	1	0.322917	0	1

Table 5. weighed normalized decision matrix.

	IVNC1	IVNC2	IVNC3	IVNC4	IVNC5
IVNA1	0	0.057468	0.142019	0.07846	0.404042
IVNA2	0.150895	0	0	0.034326	0.297419
IVNA3	0.120174	0	0.142019	0.07846	0.404042

$IVNA_4$	0.105717	0.003831	0.07101	0.058028	0
$IVNA_5$	0.102103	0	0.162308	0.056393	0.405912
$IVNA_6$	0.060539	0.176237	0.052412	0	0.4321

Table 6. Power weighted normalized decision matrix.

	$IVNC_1$	$IVNC_2$	$IVNC_3$	$IVNC_4$	$IVNC_5$
$IVNA_1$	0.000	0.820789	0.97856	1	0.971406
$IVNA_2$	1	0	0	0.937197	0.850956
$IVNA_3$	0.966233	0	0.97856	1	0.971406
$IVNA_4$	0.947724	0.509286	0.874436	0.976609	0
$IVNA_5$	0.942763	0	1	0.974422	0.973347
$IVNA_6$	0.871262	1	0.83238	0	1

In the second dataset

Table 7 shows the normalization decision matrix.

Then we determine the entropy using Eqs. (10 and 11).

Then we compute the criteria weights using Eq. (12).

Table 7. Normalized decision matrix.

	$IVNC_1$	$IVNC_2$	$IVNC_3$	$IVNC_4$	$IVNC_5$
$IVNA_1$	0.175758	0.148276	0.185529	0.191882	0.192135
$IVNA_2$	0.202424	0.185632	0.150897	0.158672	0.168539
$IVNA_3$	0.117576	0.183333	0.197279	0.186962	0.182022
$IVNA_4$	0.198182	0.144828	0.166976	0.190652	0.093258
$IVNA_5$	0.149697	0.136207	0.155226	0.121771	0.179213
$IVNA_6$	0.156364	0.201724	0.144094	0.150062	0.184831

Then we normalize the decision matrix using Eqs. (13 and 14) as shown in Table 8 by the CoCoSo method.
Then we determine the power weight of decision matrix and total weighted decision matrix suing Eq. (15 and 16) as shown in Tables 9 and 10.

Then we compute the relative weights using Eqs. (17-19)

Then we rank the alternatives based on Eq. (20).

Table 8. Normalized decision matrix by CoCoSo.

	$IVNC_1$	$IVNC_2$	$IVNC_3$	$IVNC_4$	$IVNC_5$
$IVNA_1$	0.398496	0.557143	0.842975	0.791667	0.935065
$IVNA_2$	0.87218	0.847619	0.380165	0.541667	0.688312
$IVNA_3$	0	0.938095	1	0.736111	0.935065
$IVNA_4$	1	0.342857	0.719008	1	0
$IVNA_5$	0.398496	0	0	0	0.939394
$IVNA_6$	0.481203	1	0.404959	0.347222	1

Table 9. weighed normalized decision matrix.

	IVNC1	IVNC2	IVNC3	IVNC4	IVNC5
IVNA1	0.092222	0.087616	0.079111	0.142454	0.315609
IVNA2	0.201844	0.133296	0.035678	0.097468	0.232323
IVNA3	0	0.147525	0.093848	0.132457	0.315609
IVNA4	0.231425	0.053918	0.067477	0.179942	0
IVNA5	0.092222	0	0	0	0.31707
IVNA6	0.111362	0.15726	0.038004	0.06248	0.337526

Table 10. Power weighted normalized decision matrix.

	IVNC1	IVNC2	IVNC3	IVNC4	IVNC5
IVNA1	0.808	0.912117	0.984097	0.958834	0.977594
IVNA2	0.968846	0.974336	0.913233	0.895545	0.881553
IVNA3	0	0.990001	1	0.946363	0.977594
IVNA4	1	0.845069	0.969516	1	0
IVNA5	0.808217	0	0	0	0.979119
IVNA6	0.844273	1	0.918663	0.826677	1

In the third dataset

Table 11 shows the normalization decision matrix.

Then we determine the entropy using Eqs. (10 and 11).

Then we compute the criteria weights using Eq. (12).

Table 11. Normalized decision matrix.

	IVNC1	IVNC2	IVNC3	IVNC4	IVNC5
IVNA1	0.154572	0.14527	0.183711	0.187952	0.165637
IVNA2	0.211209	0.152027	0.132272	0.146988	0.152723
IVNA3	0.129204	0.163851	0.183711	0.166265	0.152723
IVNA4	0.171091	0.14527	0.169014	0.155422	0.147108
IVNA5	0.171091	0.195946	0.18861	0.177108	0.19708
IVNA6	0.162832	0.197635	0.142682	0.166265	0.184728

Then we normalize the decision matrix using Eqs. (13 and 14) as shown in Table 12 by the CoCoSo method. Then we determine the power weight of decision matrix and total weighted decision matrix suing Eq. (15 and 16) as shown in Tables 13 and 14.

Then we compute the relative weights using Eqs. (17-19)

Then we rank the alternatives based on Eq. (20).

Table 12. Normalized decision matrix by CoCoSo.

	IVNC1	IVNC2	IVNC3	IVNC4	IVNC5
IVNA1	0	0.176991	0.875	1	0.693333
IVNA2	1	0	0	0	0.186667
IVNA3	0.540323	0.469027	0.875	0.512195	0

$IVNA_4$	0.798387	0.176991	0.4375	0.512195	0.426667
$IVNA_5$	0.798387	0.628319	1	0.780488	1
$IVNA_6$	0.685484	1	0.322917	0.219512	1

Table 13. weighed normalized decision matrix.

	$IVNC_1$	$IVNC_2$	$IVNC_3$	$IVNC_4$	$IVNC_5$
$IVNA_1$	0	0.041333	0.204779	0.086499	0.110918
$IVNA_2$	0.285961	0	0	0	0.029862
$IVNA_3$	0.154511	0.109531	0.204779	0.044304	0
$IVNA_4$	0.228308	0.041333	0.10239	0.044304	0.068257
$IVNA_5$	0.228308	0.146731	0.234033	0.067511	0.159977
$IVNA_6$	0.196022	0.233529	0.075573	0.018988	0.159977

Table 14. Power weighted normalized decision matrix.

	$IVNC_1$	$IVNC_2$	$IVNC_3$	$IVNC_4$	$IVNC_5$
$IVNA_1$	0.000	0.667382	0.969232	1	0.943093
$IVNA_2$	1	0	0	0	0.764517
$IVNA_3$	0.838589	0.837944	0.969232	0.943771	0
$IVNA_4$	0.937642	0.667382	0.824094	0.943771	0.872615
$IVNA_5$	0.937642	0.897158	1	0.978791	1
$IVNA_6$	0.897639	1	0.767557	0.877076	1

In the fourth dataset

Table 15 shows the normalization decision matrix.

Then we determine the entropy using Eqs. (10 and 11).

Then we compute the criteria weights using Eq. (12).

Table 15. Normalized decision matrix.

	$IVNC_1$	$IVNC_2$	$IVNC_3$	$IVNC_4$	$IVNC_5$
$IVNA_1$	0.180193	0.175749	0.225225	0.219873	0.239161
$IVNA_2$	0.229023	0.134877	0.162162	0.154334	0.20979
$IVNA_3$	0.114168	0.132153	0.145646	0.179704	0.125874
$IVNA_4$	0.136176	0.156676	0.148649	0.134602	0.116084
$IVNA_5$	0.189821	0.161444	0.183183	0.139535	0.173427
$IVNA_6$	0.150619	0.239101	0.135135	0.171952	0.135664

Then we normalize the decision matrix using Eqs. (13 and 14) as shown in Table 16 by the CoCoSo method.
Then we determine the power weight of decision matrix and total weighted decision matrix suing Eq. (15 and 16) as shown in Tables 17 and 18.

Then we compute the relative weights using Eqs. (17-19)

Then we rank the alternatives based on Eq. (20).

Table 16. Normalized decision matrix by CoCoSo.

	$IVNC_1$	$IVNC_2$	$IVNC_3$	$IVNC_4$	$IVNC_5$

$IVNA_1$	0.260417	0.557143	1	1	1
$IVNA_2$	1	0	0.314465	0.363014	0.736111
$IVNA_3$	0.130208	0	0	0.219178	0
$IVNA_4$	0	0.357143	0.157233	0	0
$IVNA_5$	0.557292	0	0.314465	0	0.347222
$IVNA_6$	0.203125	1	0.333333	0.171233	0

Table 17. weighed normalized decision matrix.

	$IVNC_1$	$IVNC_2$	$IVNC_3$	$IVNC_4$	$IVNC_5$
$IVNA_1$	0.058674	0.105871	0.138234	0.126807	0.319626
$IVNA_2$	0.225308	0	0.04347	0.046033	0.23528
$IVNA_3$	0.029337	0	0	0.027793	0
$IVNA_4$	0	0.067866	0.021735	0	0
$IVNA_5$	0.125562	0	0.04347	0	0.110981
$IVNA_6$	0.045766	0.190025	0.046078	0.021714	0

Table 18. Power weighted normalized decision matrix.

	$IVNC_1$	$IVNC_2$	$IVNC_3$	$IVNC_4$	$IVNC_5$
$IVNA_1$	0.738	0.894803	1	1	1
$IVNA_2$	1	0	0.852212	0.879418	0.906717
$IVNA_3$	0.631715	0	0	0.824914	0
$IVNA_4$	0	0.822297	0.774346	0	0
$IVNA_5$	0.876578	0	0.852212	0	0.713126
$IVNA_6$	0.698286	1	0.859104	0.799491	0

Then we compute the criteria in each dataset as shown in Fig 5. Then we ranked the alternatives as shown in Fig 6. We show the large number of clients and large number of rounds are the best alternative in each dataset.

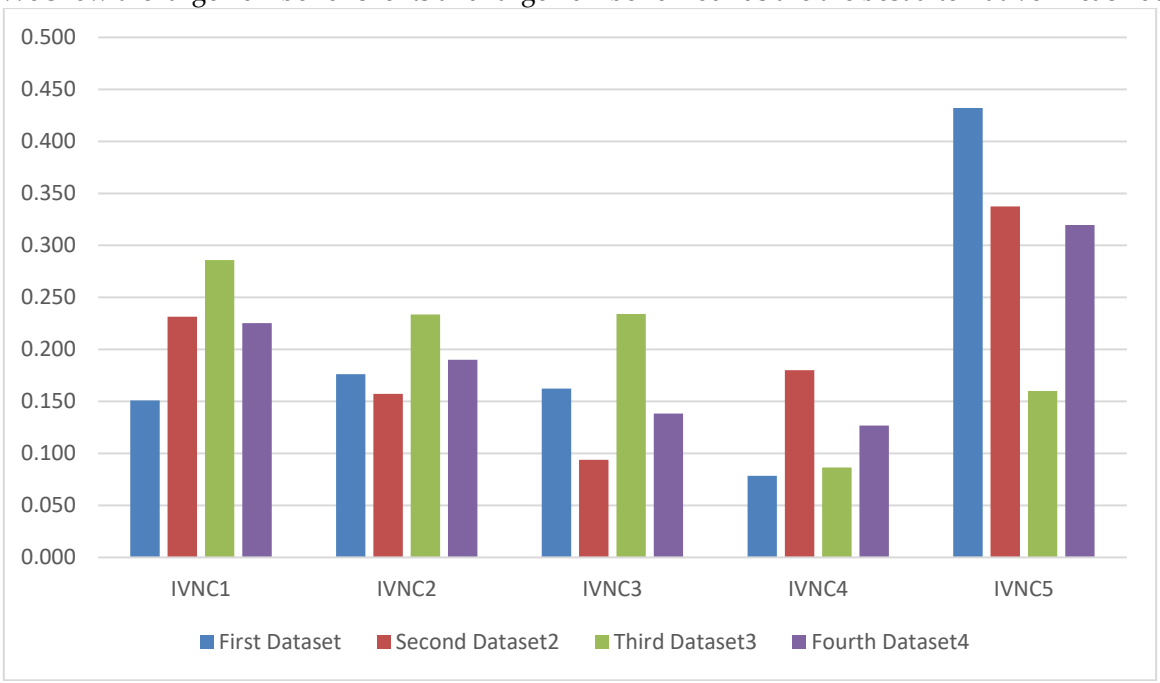

Fig 5. Criteria weights of four datasets.

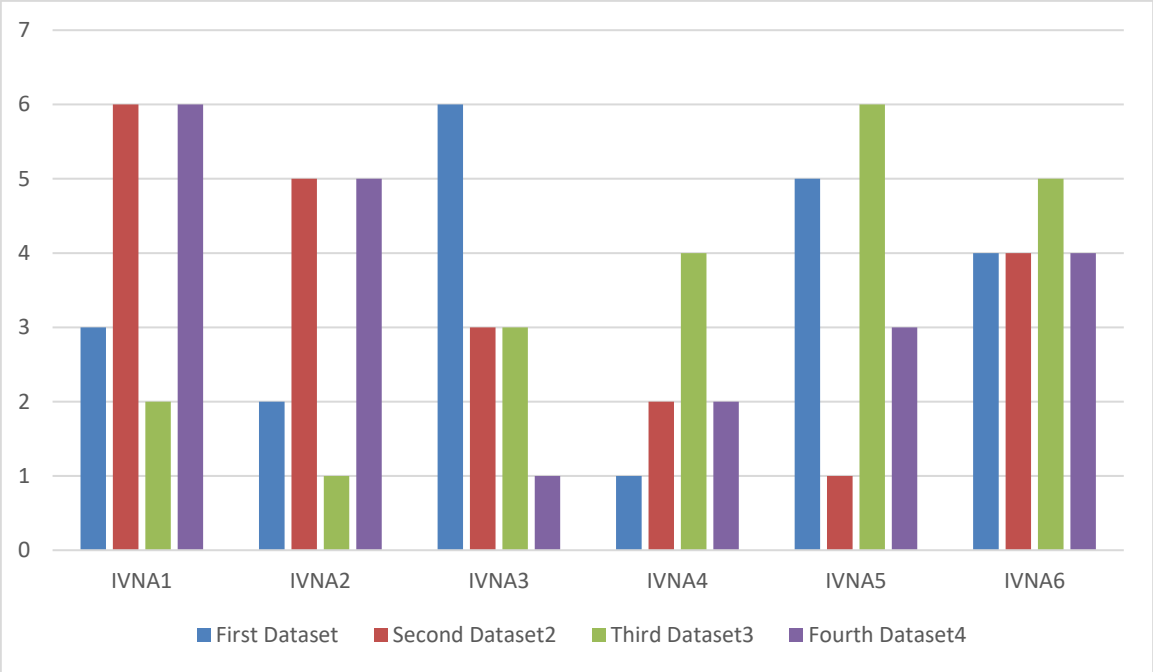

Fig 6. Rank the alternatives of four datasets.

4. Conclusions

In this study, we introduced CNN-Fed, a novel federated learning framework based on CNN, for mobile malware detection. The primary objective was to improve the accuracy of the global classifier while preserving data privacy through decentralized model training. By leveraging a dynamic weighted aggregation mechanism, CNN-Fed improves the contribution of high-performing local models to the global model, resulting in superior classification performance. Using four datasets for Android malware classification: Malgenome, Drebin, Kronodroid, and Tuandromd We evaluated CNN-Fed and compared the results with the Dw-FedAvg technique across all datasets. These findings prove that CNN-Fed is a promising approach for enhancing the accuracy of federated learning models in malware detection applications, making it a valuable tool for cybersecurity. We used the interval valued neutrosophic set (IVNSs) methodology to evaluate the number of clients and number of rounds. We used five criteria such as accuracy, precision, recall, f1-score, and false positive rate. Then we used two MCDM methods such as Entropy to compute the criteria weights and the CoCoSo method to rank the alternatives. IVNSs are uses to deal with indeterminacy. We applied these methods into four datasets. The results show the larger clients and rounds are the best alternatives.

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A Neutrosophic Approach for Opponent Analysis and Game Strategy Formulation in College Volleyball

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Abstract: In competitive college volleyball, effective opponent analysis is crucial for strategic decision-making, particularly when facing teams with varying performance dynamics. This study introduces a Q-Neutrosophic Soft Set (QNSS) approach to opponent analysis, enabling multi-context decision evaluation based on a granular assessment of opponent characteristics across different game conditions. Our QNSS proposes a novel Context-wise Aggregation Operator (CWA) that captures the nuanced relationships between contexts and parameters in QNSS, providing a flexible aggregation mechanism that not only accounts for the varying importance of different contexts but also emphasizes the multiplicative relationships between membership values. Our framework presents neutrosophic best-worst method to compute criteria weights by ensuring more diverse and well-distributed criteria values and introduces a regularization-based spread maximization function to prevent weight convergence and ensure proper consistency handling through a nonlinear optimization model. Then, a customized fuzzy ARAS is computed to assess and rank the different playing strategies. Further, we introduce a real case study on European volleyball championship for senior men's national teams in Europe, in which decision-makers. Proof of concept analysis is conducted on real case studies validated to the advantages of the proposed approach, and compared the methods against state-of-the-art studies, implying insightful boosting to the opponent analysis and game planning.

Keywords: Neutrosophic Logic (NL), Neutrosophic Sets (NS), Uncertainty Modeling, Multi-valued Logic, Tactical Decision-Making, Volleyball Match Analysis.

1. Introduction

In the competitive landscape of college volleyball, the ability to analyzes opponents and formulate effective gaming strategies is crucial for team success [1]. Coaches and analysts traditionally rely on statistical models, game footage analyses, and expert intuition to devise strategies. However, these conventional methods often struggle to account for the inherent uncertainties and dynamics nature of the game [2]. Factors such as player fatigue, unpredictable injuries, real-time tactical adjustments, and psychological factors introduce a degree of indeterminacy that traditional analysis technique fail to capture effectively [3].

Traditional opponent analysis methods depend on crisp values, predefined probabilistic models, and deterministic approachs that assume complete knowledge about game situations [4].

However, real-world volleyball matches are dynamic, where performances of players varied, tactical formations change, and unexpected circumstances arise [5]. For example, the player's past performance did not guarantee the same efficiency in the subsequent matches because off outside variables e.g., stress, fatigue, or variety within game environments. Coaches often face ambiguous situations where multiple strategies appear viable, but the best decision remained unclear. Existing systems depend on pre-analyzed data, making it difficult to regulate to unforeseen in-game developments [6].

Neutrosophic theory introduces a three-dimensional representation of knowledge—Truth (T), Indeterminacy (I), and Falsity (F)—which can better model the complexities and uncertainties in volleyball strategy formulation [7], [8]. Apart from out-of-date binary or probabilistic models, neutrosophic logic enabled partial truths and indeterminate knowledge, which make it it well-suited for decision-making (DM) in sports [9].

In this article, we aimed to develop a neutrosophic patterns of player performance variability and game uncertainties. Then, we aim to integrate a decision-support system to dynamically adjusted strategies with respects to historical data. This collectively aims to build a unified framework that can be applied to coaching strategies thereby improving the DM in competitive volleyball. To achieve this objective, this study contributes to the body of knowledge through introducing a novel application of neutrosophic logic in opponent analysis as well as strategy formulation for volleyball colleges. The key contributions are summarized as follows. First, a mathematical model for aggregating uncertainty in multi factor assessment of volleyball strategies. Second, a customized weighting mechanism to highlight the importance of different components in analyzing opponent teams. To validate decision-making ability of our approach, we introduce a practical case study where a college volleyball coach analyzes different game strategies under uncertain conditions. A reasonable set of conclusive analysis established the compensations of our methods over previously developed approaches.

The left part of this articles contains five main sections. Background and related literature are described in Section 2. Section 3 explain the methodology of proposed framework in details. The experimental design is debated in Section 4. The results and analysis are discussed in Section 5. The conclusion of article is derived in Section 6.

2. Background and Literature

This section presents a background on the main concepts neutrosophic methods along with the related studies, including essential definitions.

Definition 1 ([7], [8]**)**. With an assumption of $\mathcal{U}$ as a universe of discourse, neutrosophic set $\aleph$ can be defined as:

$$\aleph = \{\langle u,(T_\aleph(u),I_\aleph(u),F_\aleph(u))\rangle : u \in \mathcal{U}\},$$
$$\text{where} \tag{1}$$
$$T_\aleph(u), I_\aleph(u), F_\aleph(u): \mathcal{U} \to]^-0, 1^+[\text{ and } {}^-0 \le T_\aleph(u) + I_\aleph(u) + F_\aleph(u) \le 3^+$$

Definition 2 ([10]**)**. With assumption of $\mathcal{U}$ as a universe of discourse and Q as a nonempty set, then the Q-neutrosophic set, $\aleph_Q$, is articulated follows:

$$\aleph_Q = \{\langle(u,q),T_{\aleph_Q}(u,q),I_{\aleph_Q}(u,q),F_{\aleph_Q}(u,q)\rangle : u \in \mathcal{U}, q \in Q\},$$
$$\text{where} \quad (2)$$
$$T_{\aleph_Q}, I_{\aleph_Q}, F_{\aleph_Q}: \mathcal{U} \times Q \rightarrow]^{-}0, 1^{+}[\text{ and } {}^{-}0 \leq T_{\aleph_Q} + I_{\aleph_Q} + F_{\aleph_Q} \leq 3^{+}$$

Definition 3 ([10]**)**. With assumption of $\mathcal{U}$ as a universe of discourse and Q as a nonempty set, l be any positive integer and I be a unit interval [0,1], then multi Q-neutrosophic set $\widetilde{\aleph}_Q$ in $\mathcal{U}$ and Q is a set of ordered sequences

$$\widetilde{\aleph}_Q = \{\langle(u,q),T_{\widetilde{\aleph}_{Q_i}}(u,q),I_{\widetilde{\aleph}_{Q_i}}(u,q),F_{\widetilde{\aleph}_{Q_i}}(u,q)\rangle: u \in \mathcal{U}, q \in Q$$
$$\text{where} \quad (3)$$
$$T_{\widetilde{\aleph}_{Q_i}}, I_{\widetilde{\aleph}_{Q_i}}, F_{\widetilde{\aleph}_{Q_i}}: \mathcal{U} \times Q \rightarrow I^l \; for \; all \; i = 1,2,\dots,l$$
$$0 \leq T_{\widetilde{\aleph}_{Q_i}} + I_{\widetilde{\aleph}_{Q_i}} + F_{\widetilde{\aleph}_{Q_i}} \leq 3 \; for \; all \; i = 1,2,\dots,l$$

Definition 4 ([11]**)**. Given $\mathcal{U}$ as a universe of discourse, Q as a nonempty set, and E as set of parameters, and Q-neutrosophic sets, $\mu^l QNS(\mathcal{U})$, with dimension $l = 1$, then, a Q-neutrosophic soft set (QNSS) can be define as follows:

$$\aleph_Q: X \rightarrow \mu^l QNS(\mathcal{U}) \text{ such that } X \subseteq E \text{ and } \aleph_Q(e) = \phi \text{ if } e \notin X \quad (4)$$

QNSS can be represented by the set of ordered pairs

$$(\aleph_Q, X) = \{(e, \aleph_Q(e)): e \in X, \aleph_Q \in \mu^l QNS(\mathcal{U})\}. \quad (5)$$

Example 1: In college volleyball, coaches analyze opponent teams, $\mathcal{U} = \{T1, T2, T3\}$, to formulate the best game strategy. The location at which match is played can affect team performance, leading two conditions $Q = \{played - home\ (q_1),\ or\ away\ (q_2)\}$.The analysis is based on key performance indicators, $E = \{\mathrm{e}_1 = \text{Offensive Strength}, \mathrm{e}_2 = \text{Defensive Weakness}\}$ as a set of decision parameters. Then the QNSS $(\aleph_Q, \mathrm{A})$ can be derived as follows:

$$(\aleph_Q, X) = \left\{ \begin{array}{l} \langle e_1, [(u_1,q_1), 0.8,0.2,0.3], [(u_1,q_2), 0.6,0.3,0.4], [(u_2,q_1), 0.9,0.1,0.2], [(u_2,q_2), 0.7,0.2,0.3], \\ \quad [(u_3,q_1), 0.4,0.5,0.6], [(u_3,q_2), 0.5,0.3,0.4] \rangle \\ \langle e_2, [(u_1,p), 0.3,0.4,0.8], [(u_1,q_2), 0.8,0.4,0.1], [(u_2,q_1), 0.5,0.5,0.8], [(u_2,q_2), 0.2,0.4,0.5], \\ \quad [(u_3,q_1), 0.7,0.3,0.2], [(u_3,q_2), 0.6,0.4,0.3] \rangle \end{array} \right\}$$

Definition 5 ([10],[11]**)**. Given two subsets $(\aleph_Q, X) = \left\{\left(T_{\aleph_Q(e_j)}(u,q)_i, I_{\aleph_Q(e_j)}(u,q)_i, F_{\aleph_Q(e_j)}(u,q)_i\right) : \forall e_j \in X, (u,q)_i \in \mathcal{U} \times Q\right\} \in QNSS(\mathcal{U})$ and $(\Psi_Q, Y) = \left\{\left(T_{\Psi_Q(e_j)}(u,q)_i, I_{\Psi_Q(e_j)}(u,q)_i, F_{\Psi_Q(e_j)}(u,q)_i\right) : \forall e_j \in Y, (u,q)_i \in \mathcal{U} \times Q\right\} \in Q -$

$NSS(\mathcal{U})$, then the (Ψ_Q, Y) can be declared as subset of $(\aleph_Q, X)$ if $Y \subseteq X$ and $\Psi_Q(u) \subseteq \aleph_Q(u)$ for all $u \in \mathcal{U}$.

$$X \subseteq Y \;\&\; \aleph_Q(e) \subseteq \Psi_Q(e) \forall e \in X$$

where

$$T_{\aleph_Q(e)}(u,q) \leq T_{\Psi_Q(e)}(u,q),$$
$$I_{\aleph_Q(e)}(u,q) \geq I_{\Psi_Q(e)}(u,q), F_{\aleph_Q(e)}(u,q) \geq F_{\Psi_Q(e)}(u,q), \forall (u,q) \in \mathcal{U} \times Q \quad (6)$$

Definition 6 ([10],[11]). Given two QNSSs $(\aleph_Q, X) = \left\{\left(T_{\aleph_Q(e)}(u,q), I_{\aleph_Q(e)}(u,q), F_{\aleph_Q(e)}(u,q)\right) : \forall e \in X, (u,q) \in \mathcal{U} \times Q\right\}$ and $(\Psi_Q, Y) = \left\{\left(T_{\Psi_Q(e)}(u,q), I_{\Psi_Q(e)}(u,q), F_{\Psi_Q(e)}(u,q)\right) : \forall e \in Y, (u,q) \in \mathcal{U} \times Q\right\}$, the union of these two sets can be symbolized as $(\cup_Q, C) = (\aleph_Q, A) \cup (\Psi_Q, B)$, in which the membership functions are computed as follows:

$$T_{\cup_Q(c)}(u,q) = \begin{cases} T_{\aleph_Q(c)}(u,q) & if c \in X - Y, \\ T_{\Psi_Q(c)}(u,q) & if c \in Y - X, \\ max\{T_{\Lambda_Q(c)}(u,q), T_{\Psi_Q(c)}(u,q)\} & if c \in X \cap Y, \end{cases}$$
$$I_{\cup_Q(c)}(u,q) = \begin{cases} I_{\aleph_Q(c)}(u,q) & if c \in X - Y, \\ I_{\Psi_Q(c)}(u,q) & if c \in Y - X, \\ \min\left\{I_{\aleph_Q(c)}(u,q), I_{\Psi_Q(c)}(u,q)\right\} & if c \in X \cap Y, \end{cases} \quad (7)$$
$$F_{\cup_Q(c)}(u,q) = \begin{cases} F_{\aleph_Q(c)}(u,q) & if c \in X - Y, \\ F_{\Psi_Q(c)}(u,q) & if c \in Y - X, \\ min\{F_{\aleph_Q(c)}(u,q), F_{\Psi_Q(c)}(u,q)\} & if c \in X \cap Y. \end{cases}$$

where $C = A \cup B$ and $c \in C, (u,q) \in U \times Q$.

Definition 7 ([10],[11]). Given two QNSSs $(\aleph_Q, X) = \left\{\left(T_{\aleph_Q(e)}(u,q), I_{\aleph_Q(e)}(u,q), F_{\aleph_Q(e)}(u,q)\right) : \forall e \in X, (u,q) \in \mathcal{U} \times Q\right\}$ and $(\Psi_Q, Y) = \left\{\left(T_{\Psi_Q(e)}(u,q), I_{\Psi_Q(e)}(u,q), F_{\Psi_Q(e)}(u,q)\right) : \forall e \in Y, (u,q) \in \mathcal{U} \times Q\right\}$, the intersection of these two sets can be symbolized as $(\cap_Q, C) = (\aleph_Q, X) \cap (\Psi_Q, Y)$, in which the membership functions are computed as follows:

$$T_{\cap_Q(c)}(u,q) = \min\left\{T_{\aleph_Q(c)}(u,q), T_{\Psi_Q(c)}(u,q)\right\},$$
$$I_{\cap_Q(c)}(u,q) = \max\left\{I_{\aleph_Q(c)}(u,q), I_{\Psi_Q(c)}(u,q)\right\}, \quad (8)$$
$$F_{\cap_Q(c)}(u,q) = \max\left\{F_{\aleph_Q(c)}(u,q), F_{\Psi_Q(c)}(u,q)\right\}.$$

Definition 8 ([10]**,**[11]**)**. Given an *QNSSs* $(\aleph_Q, X) = \left\{\left(T_{\aleph_Q(e)}(u,q), I_{\aleph_Q(e)}(u,q), F_{\aleph_Q(e)}(u,q)\right): \forall e \in X, (u,q) \in \mathcal{U} \times Q\right\}$, then, its complement is defined as $(\aleph_Q, X)^c = (\aleph_Q^c, X)$,

$$(\aleph_Q, X)^c = \left\{\langle e, T^c_{\aleph_Q(e)}(u,q), I^c_{\aleph_Q(e)}(u,q), F^c_{\aleph_Q(e)}(u,q)\rangle : e \in X, (u,q) \in U \times Q\right\}, \quad (9)$$

such that $\forall e \in X, (u,q) \in \mathcal{U} \times Q$

$$\begin{aligned} T^c_{\aleph_Q(e)}(u,q) &= 1 - T_{\aleph_Q(e)}(u,q), \\ I^c_{\aleph_Q(e)}(u,q) &= 1 - I_{\aleph_Q(e)}(u,q), \\ F^c_{\aleph_Q(e)}(u,q) &= 1 - F_{\aleph_Q(e)}(u,q). \end{aligned}$$

Example 2. Based on example 1, the complement of $(\aleph_Q, X)$ is expressed as follows:

$$(\aleph_Q, X) = \left\{ \begin{array}{c} \langle e_1, [(u_1,q_1), 0.2,0.8,0.7], [(u_1,q_2), 0.4,0.7,0.6], [(u_2,q_1), 0.1,0.9,0.8], [(u_2,q_2), 0.3,0.8,0.7], \\ [(u_3,q_1),0.6,0.5,0.4], [(u_3,q_2),0.5,0.7,0.6] \rangle \\ \langle e_2, [(u_1,p), 0.7,0.6,0.2], [(u_1,q_2), 0.8,0.4,0.1], [(u_2,q_1), 0.5,0.5,0.8], [(u_2,q_2), 0.8,0.6,0.5], \\ [(u_3,q_1),0.3,0.7,0.8], [(u_3,q_2),0.4,0.6,0.7] \rangle \end{array} \right\}$$

Definition 9 ([10]**,**[11]**)**. Given $(\aleph_Q, X) \in QNSS(\mathcal{U})$, $\aleph_Q(e) = \phi$ for all $e \in X$, then $(\aleph_Q, X)$ can be declared as a null $QNSS(\mathcal{U})$, and is referred to as (ϕ, X).

Definition 10 ([10]**,**[11]). Given an QNSSs $(\aleph_Q, X) = \left\{\left(T_{\aleph_Q(e)}(u,q), I_{\aleph_Q(e)}(u,q), F_{\aleph_Q(e)}(u,q)\right): \forall e \in X, (u,q) \in \mathcal{U} \times Q\right\}$, then, the necessity operation can be defined as:

$$\oplus(\aleph_Q, X) = \left\{\langle e, [(u,q), T_{\aleph_Q}(u,q), I_{\aleph_Q}(u,q), 1 - T_{\aleph_Q}(u,q)]\rangle : (u,q) \in U \times Q\right\}. \quad (10)$$

Example 3. Based on example 1, the necessity operation of $(\aleph_Q, X)$ is computed as follows:

$$\oplus(\aleph_Q, X) = \left\{ \begin{array}{c} \langle e_1, [(u_1,q_1), 0.8,0.2,0.2], [(u_1,q_2), 0.6,0.3,0.4], [(u_2,q_1), 0.9,0.1,0.1], [(u_2,q_2), 0.7,0.2,0.3], \\ [(u_3,q_1),0.4,0.5,0.6], [(u_3,q_2),0.5,0.3,0.5] \rangle \\ \langle e_2, [(u_1,p), 0.3,0.4,0.7], [(u_1,q_2), 0.8,0.4,0.2], [(u_2,q_1), 0.5,0.5,0.5], [(u_2,q_2), 0.2,0.4,0.8], \\ [(u_3,q_1),0.7,0.3,0.3], [(u_3,q_2),0.6,0.4,0.4] \rangle \end{array} \right\}$$

Definition 11 ([10]**,**[11]**)**.Given an QNSSs $(\aleph_Q, X) = \left\{\left(T_{\aleph_Q(e)}(u,q), I_{\aleph_Q(e)}(u,q), F_{\aleph_Q(e)}(u,q)\right): \forall e \in X, (u,q) \in \mathcal{U} \times Q\right\}$, then, the possibility operation can be defined as:

$$\otimes(\aleph_Q, X) = \left\{\langle e, [(u,q), 1 - F_{\aleph_Q}(u,q), I_{\aleph_Q}(u,q), F_{\aleph_Q}(u,q)]\rangle : (u,q) \in U \times Q\right\}. \quad (11)$$

Definition 12 ([12]**)**. Given $X = \{u_1, u_2, \dots, u_m\}$, $Q = \{q_1, q_2, \dots, q_l\}$, and $X = \{e_1, e_2, \dots, e_n\}$, then, the hamming distance between two QNSSs $(\aleph_Q, X)$, and $(\aleph_Q, Y)$ is defined as follows:

$$d_{QNSS}^{Hamm}\left((\aleph_Q, X), (\Psi_Q, Y)\right) = \sum_{j=1}^{n} \sum_{i=1}^{lm} \frac{\left|T_{\aleph_Q}(e_j^X)(u,q)_i - T_{\Psi_Q}(e_j^Y)(u,q)_i\right| + \left|I_{\aleph_Q}(e_j^X)(u,q)_i - I_{\Psi_Q}(e_j^Y)(u,q)_i\right| + \left|F_{\aleph_Q}(e_j^X)(u,q)_i - F_{\Psi_Q}(e_j^Y)(u,q)_i\right|}{3}, \quad (12)$$

Definition 13 ([12]**)**. Given $X = \{u_1, u_2, \dots, u_m\}$, $Q = \{q_1, q_2, \dots, q_l\}$, and $X = \{e_1, e_2, \dots, e_n\}$, then, the excluding distance between two QNSSs $(\aleph_Q, X)$, and $(\aleph_Q, Y)$ is defined as follows:

$$d_{QNSS}^{Eclud}\left((\aleph_Q, X), (\Psi_Q, Y)\right) = \sum_{j=1}^{n} \sum_{i=1}^{lm} \sqrt{\frac{\left(T_{\aleph_Q}(e_j^X)(u,q)_i - T_{\Psi_Q}(e_j^Y)(u,q)_i\right)^2 + \left(I_{\aleph_Q}(e_j^X)(u,q)_i - I_{\Psi_Q}(e_j^Y)(u,q)_i\right)^2 + \left(F_{\aleph_Q}(e_j^X)(u,q)_i - F_{\Psi_Q}(e_j^Y)(u,q)_i\right)^2}{3}}, \quad (13)$$

3. Theoretical Framework & Methodology

In this section, we present a neutrosophic-based methodology for opponent analysis in college volleyball. Our proposed approach integrates a sequence of systematic steps for modeling opponent performance and evaluating strategic choices.

3.1. Context-aware Aggregator

Definition 14: Given a QNSS $(\aleph_Q, A)$ over U, a QNS aggregator for $(\aleph_Q, A)$, symbolized by $\aleph_Q^{agg}$ and is $\aleph_Q^{agg} = \{\langle (u,t), T_Q^{agg}(u,t), I_Q^{agg}(u,t), F_Q^{agg}(u,t)\rangle : (u,t) \in U \times Q\}$, such that

$$T_Q^{agg} = \frac{1}{|A|} \sum_{(u,t)\in U\times Q} T_{\aleph_Q}(u,t), I_Q^{agg} = \frac{1}{|A|} \sum_{(u,t)\in U\times Q} I_{\aleph_Q}(u,t), \quad F_Q^{agg} = \frac{1}{|A|} \sum_{(u,t)\in U\times Q} F_{\aleph_Q}(u,t). \quad (14)$$

where $T_Q^{agg}, I_Q^{agg}, F_Q^{agg} : U \times Q \to [0,1]$.

Example 4: Consider a college volleyball team is preparing for an important playoff match and must decide on the best defensive strategy against a strong attacking opponent. The coaching staff evaluates three opponent teams, $U = \{u_1, u_2, u_3\}$, based on their attacking styles under

different match scenarios, $Q = \{q_1: Serve\ Reception\ Phase\ , q_2: Transition\ Phase\}$, using key defensive evaluation criteria, $E = \{e_1: Blocking\ Efficiency\ , e_2: Digging\ Success\ , e_3: Opponent\ Disruption\}$. Then, expert drive the following QNSS:

$$(\aleph_Q, A) = \left\{\begin{matrix}\langle e_1, [(u_1, q_1), 0.3,0.4,0.3], [(u_1, q_2), 0.7,0.3,0.1], [(u_2, q_1), 0.2,0.5,0.3], [(u_3, q_2), 0.4,0.2,0.4]\rangle \\ \langle e_2, [(u_1, q_1), 0.5,0.2,0.3], [(u_2, q_1), 0.6,0.3,0.1], [(u_3, q_1), 0.4,0.4,0.2], [(u_3, q_2), 0.3,0.3,0.4]\rangle \\ \langle e_3, [(u_1, q_2), 0.4,0.3,0.3], [(u_2, q_2), 0.8,0.2,0.1]\rangle\end{matrix}\right\}$$

By applying Q-NS Aggregation Operator, the following QNSS is obtained:

$$\aleph_Q^{agg} = \left\{\begin{matrix}[(u_1, q_1), 0.267,0.20,0.20], \\ [(u_1, q_2), 0.366,0.20,0.133], \\ [(u_2, q_1), 0.267,0.267,0.133], \\ [(u_2, q_2), 0.267,0.067,0.033], \\ [(u_3, q_1), 0.133,0.133,0.066], \\ [(u_3, q_2), 0.233,0.166,0.267]\end{matrix}\right\}$$

While the Q-NS aggregation operator was developed as a robust tool for DM within uncertainty problems, it has some notable confines that can be prudently considered in real-world application. One major limitation is that the aggregation operator chiefly emphases on pairing elements from the universe of alternatives (i.e., volleyball teams) with the set of contexts (i.e., match scenarios), while neglecting direct relationships between the criteria themselves and the alternatives. This results in a loss of inter-criteria dependencies. In many DM scenarios, criteria are not independent but rather interrelated. For example, in the hiring scenario, experience might have a strong correlation with language fluency, meaning that candidates with more experience are also likely to have better fluency. However, the aggregation method collapses these relationships by averaging them, rather than maintaining their interactions. Moreover, the method does not preserve the original decision space structure, making it difficult to justify why a particular alternative was ranked higher, as the detailed assessment per criterion is lost in the overall summary values.

To address these limitations, we propose a context-wise aggregation operator (CWAO) multi-context behavior in QNSS effectually by carrying out aggregation distinctly from all contexts $q \in Q$ and then combine the resulting component in a meaningful way for each parameter. This way, our CWAO provide a more nuanced and flexible approach to capture the diverse features of each context while addressing issues such as unequal importance, and uncertainty in membership values.

Definition 15. Given a QNSS $(\aleph_Q, A)$ over U, a QNS aggregator for $(\aleph_Q, A)$, symbolized by $\aleph_Q^{agg}$ and is $\aleph_Q^{agg} = \{\langle (u,t), T_Q^{agg}(u,t), I_Q^{agg}(u,t), F_Q^{agg}(u,t)\rangle : (u,t) \in U \times Q\}$, such that

(15)

$$T^{agg}(u,e) = \frac{1}{|Q|}\sum_{(u,e)\in U\times E}\left(T_{\aleph_Q}(u,e)\right) - \left(\prod_{(u,e)\in U\times E}\left(T_{\aleph_Q}(u,e)\right)^{w_q}\right),$$
$$I^{agg}(u,e) = \frac{1}{|Q|}\sum_{(u,e)\in U\times E}\left(I_{\aleph_Q}(u,e)\right) - \left(\prod_{(u,e)\in U\times E}\left(I_{\aleph_Q}(u,e)\right)^{w_q}\right),$$
$$F^{agg}(u,e) = \frac{1}{|Q|}\sum_{(u,e)\in U\times E}\left(F_{\aleph_Q}(u,e)\right) - \left(\prod_{(u,e)\in U\times E}\left(F_{\aleph_Q}(u,e)\right)^{w_q}\right).$$

where w_q is the weight for context q, such that $\sum_{q\in Q} w_q = 1$.

Example 5: Given the weights $w_{q_1} = 0.6$, and $w_{q_2} = 0.4$, the result of applying CWAO to Example result in the following QNSS:

$$\aleph_Q^{agg} = \left\{\begin{matrix}[(e_1,u_1),0.554,0.346,0.195],[(e_1,u_1),0.2,0.5,0.3],[(e_1,u_3),0.4,0.2,0.4]\\ [(e_2,u_1),0.5,0.2,0.3],[(e_2,u_2),0.6,0.3,0.1],[(e_2,u_3),0.346,0.346,0.202]\\ [(e_3,u_1),0.4,0.3,0.3],[(e_3,u_2),0.8,0.2,0.1]\end{matrix}\right\}$$

To voids excessive reduction of values caused by the multiplications of fractional membership values in, we decide to modify the CWAO formula by raising the membership values to inverse of weight value.

Definition 16. Given a QNSS $(\aleph_Q, A)$ over U, a QNS aggregator for $(\aleph_Q, A)$, symbolized by $\aleph_Q^{agg}$ and is $\aleph_Q^{agg} = \{\langle (u,t), T_Q^{agg}(u,t), I_Q^{agg}(u,t), F_Q^{agg}(u,t)\rangle : (u,t) \in U \times Q\}$, such that

$$T^{agg}(u,e) = \frac{1}{|Q|}\sum_{(u,e)\in U\times E}\left(T_{\aleph_Q}(u,e)\right) - \left(\prod_{(u,e)\in U\times E}\left(T_{\aleph_Q}(u,e)\right)^{\frac{1}{w_q}}\right),$$
$$I^{agg}(u,e) = \frac{1}{|Q|}\sum_{(u,e)\in U\times E}\left(I_{\aleph_Q}(u,e)\right) - \left(\prod_{(u,e)\in U\times E}\left(I_{\aleph_Q}(u,e)\right)^{\frac{1}{w_q}}\right), \quad (16)$$
$$F^{agg}(u,e) = \frac{1}{|Q|}\sum_{(u,e)\in U\times E}\left(F_{\aleph_Q}(u,e)\right) - \left(\prod_{(u,e)\in U\times E}\left(F_{\aleph_Q}(u,e)\right)^{\frac{1}{w_q}}\right).$$

where w_q is the weight for context q, such that $\sum_{q\in Q} w_q = 1$.

3.1. Decision-making application

The Fuzzy Best-Worst Method (Fuzzy BWM) [13] was developed as DM tool that integrate expert preferences using fuzzy logic to handle uncertainty. However, in our case, uncertainty arises in multiple contexts, which motivate proposing new extension called QNSS-BWM. QNSS-BWM is

developed to compute criteria weights in a the aggregated QNSS decision matrix. given a set of decision criteria, the QNSS-BWM follows these steps to compute the weights.

In step 1, we define the decision criteria $C = \{C_1, C_2, \dots, C_n\}$ with best criterion denoted as C_B, and worst criterion noted as C_W. Both of them are determined by expert knowledge.

In step 2, we drive preference vector for the best criterion as:

$$QBO_B = \{QNN_{B1}, QNN_{B2}, \dots, QNN_{Bn}\} \tag{17}$$

where $QNN_{Bi} = (T_{Bi}, I_{Bi}, F_{Bi})$

Each entry represents how much better the best criterion C_B is compared to C_i. For the best criterion itself:$QNN_{BB} = (1,0,0)$.

In step 3, we calculate Q-Neutrosophic Best-to-Others (QBO) Matrix

$$QB_M = \begin{bmatrix} (1,0,0) & QNN_{B1} & QNN_{B2} & \cdots & QNN_{Bn} \\ \frac{1}{QNN_{B1}} & (1,0,0) & \frac{QNN_{B2}}{QNN_{B1}} & \cdots & \frac{QNN_{Bn}}{QNN_{B1}} \\ \frac{1}{QNN_{B2}} & \frac{QNN_{B1}}{QNN_{B2}} & (1,0,0) & \cdots & \frac{QNN_{Bn}}{QNN_{B2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{QNN_{Bn}} & \frac{QNN_{B1}}{QNN_{Bn}} & \frac{QNN_{B2}}{QNN_{Bn}} & \cdots & (1,0,0) \end{bmatrix} \tag{18}$$

and similarly calculate Neutrosophic Others-to-Worst (QOW) Matrix

$$QW_M = \begin{bmatrix} (0,1,1) & \frac{1}{QNN_{1W}} & \frac{1}{QNN_{2W}} & \cdots & \frac{1}{QNN_{nW}} \\ QNN_{1W} & (0,1,1) & \frac{QNN_{2W}}{QNN_{1W}} & \cdots & \frac{QNN_{nW}}{QNN_{1W}} \\ QNN_{2W} & \frac{QNN_{1W}}{QNN_{2W}} & (0,1,1) & \dots & \frac{QNN_{nW}}{QNN_{2W}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ QNN_{nW} & \frac{QNN_{1W}}{QNN_{nW}} & \frac{QNN_{2W}}{QNN_{nW}} & \cdots & (0,1,1) \end{bmatrix} \tag{19}$$

In step 4, we calculate QNSS-BWM weights $W = (w_1, w_2, \dots, w_n)$ by solving the following:

$$\min \xi \text{ subject to:} \tag{20}$$

$$\left|\frac{w_B}{w_i} - T_{Bi}\right| \le \xi,\ \forall i$$
$$\left|\frac{w_i}{w_W} - T_{iW}\right| \le \xi,\ \forall i$$
$$\sum_{i=1}^{n} w_i = 1,\ w_i \ge 0,\ \forall i$$

By computing the weights, we can get back to the aggregated decision matrix is formulated as:

$$D = \begin{bmatrix} (T_{11}, I_{11}, F_{11}) & (T_{12}, I_{12}, F_{12}) & \dots & (T_{1n}, I_{1n}, F_{1n}) \\ (T_{21}, I_{21}, F_{21}) & (T_{22}, I_{22}, F_{22}) & \dots & (T_{2n}, I_{2n}, F_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{m1}, I_{m1}, F_{m1}) & (T_{m2}, I_{m2}, F_{m2}) & \dots & (T_{mn}, I_{mn}, F_{mn}) \end{bmatrix} \tag{21}$$

In step 5, we normalize the matrix to ensure all criteria are comparable. For benefit criteria and cost criteria, the normalization formulas are:

$$\begin{aligned} T_{ij}^* = \frac{T_{ij}}{\max(T_j)},\ I_{ij}^* = \frac{I_{ij}}{\max(I_j)},\ F_{ij}^* = \frac{F_{ij}}{\max(F_j)}, \\ T_{ij}^* = \frac{\min(T_j)}{T_{ij}},\ I_{ij}^* = \frac{\min(I_j)}{I_{ij}},\ F_{ij}^* = \frac{\min(F_j)}{F_{ij}} \end{aligned} \tag{22}$$

The normalized matrix is formulated as:

$$D^* = \begin{bmatrix} (T_{11}^*, I_{11}^*, F_{11}^*) & (T_{12}^*, I_{12}^*, F_{12}^*) & \dots & (T_{1n}^*, I_{1n}^*, F_{1n}^*) \\ (T_{21}^*, I_{21}^*, F_{21}^*) & (T_{22}^*, I_{22}^*, F_{22}^*) & \dots & (T_{2n}^*, I_{2n}^*, F_{2n}^*) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{m1}^*, I_{m1}^*, F_{m1}^*) & (T_{m2}^*, I_{m2}^*, F_{m2}^*) & \dots & (T_{mn}^*, I_{mn}^*, F_{mn}^*) \end{bmatrix}. \tag{23}$$

In step 6, we assign weights w_j to each criterion e_j, such that $\sum_{j=1}^{n} w_j = 1$. The weighted normalized matrix is:

$$D^w = \begin{bmatrix} (w_1 T_{11}^*, w_1 I_{11}^*, w_1 F_{11}^*) & (w_2 T_{12}^*, w_2 I_{12}^*, w_2 F_{12}^*) & \dots & (w_n T_{1n}^*, w_n I_{1n}^*, w_n F_{1n}^*) \\ (w_1 T_{21}^*, w_1 I_{21}^*, w_1 F_{21}^*) & (w_2 T_{22}^*, w_2 I_{22}^*, w_2 F_{22}^*) & \dots & (w_n T_{2n}^*, w_n I_{2n}^*, w_n F_{2n}^*) \\ \vdots & \vdots & \ddots & \vdots \\ (w_1 T_{m1}^*, w_1 I_{m1}^*, w_1 F_{m1}^*) & (w_2 T_{m2}^*, w_2 I_{m2}^*, w_2 F_{m2}^*) & \dots & (w_n T_{mn}^*, w_n I_{mn}^*, w_n F_{mn}^*) \end{bmatrix} \tag{24}$$

In step 7, we compute the neutrosophic score

$$S_i = \left(\sum_{j=1}^{n} w_j T_{ij}^*, \sum_{j=1}^{n} w_j I_{ij}^*, \sum_{j=1}^{n} w_j F_{ij}^* \right) \tag{25}$$

Finally, we convert the optimality function into a crisp value using a defuzzification method. Then, rank alternatives based on utility degree:

$$S_i = \frac{\left(S_i^T + S_i^I + S_i^F\right)}{3}$$
$$D_i = \frac{S_i}{S} \quad (26)$$

We rank the alternatives according to D_i. The alternative with the highest D_i is the best choice.

4. Data Collection & Opponent Profiling

To validate the effectiveness of the proposed neutrosophic-based framework in opponent analysis and game strategy formulation, we apply it to a real-world college volleyball scenario. This case study aims to demonstrate how a multi-criteria decision model can improve strategy selection under indeterminate conditions. In this case study, we analyze potential starting lineups for an upcoming championship match, evaluating them based on key performance criteria. A college volleyball team is preparing for a crucial championship match and must decide on the best starting lineup. The selection process is complex due to contradicting factors such as player performance and fatigue/injury risks. The coach must evaluate potential starting lineups while considering different critical criteria. These criteria include Offensive Efficiency (Benefit), Defensive Stability (Benefit), Opponent Disruption (Benefit), Energy Consumption (Cost), Tactical Complexity (Cost), and Injury Risk (Cost). Using these set of criteria, the coach carefully evaluates multiple strategic alternatives to optimize the team's performance against a strong opponent. As tabulated in Table 1, the coach considers eight alternative strategies, each with distinct advantages and trade-offs. These strategies are assessed with respect to different criteria, which include both benefit-oriented factors and cost-related factors.

Table 1: Overview of Alternative Game Strategies and Their Tactical Characteristics in College Volleyball.

Alternative Tactics	Description	**ID**
Aggressive Attack Focus	Prioritizing the power of spiking, fast tempo offense, as well as fast attacks.	A1
Defensive Block-Oriented Play	Emphasizing hard obstructive creations and robust back-row defense.	A2
Serve-Pressure Strategy	Usage of high-risk, high-reward serves for disruption of opponent's transitory.	A3
Balanced Offense-Defense Play	Mechanism for moderating intensity of attack and organized defense.	A4
Quick Rotation & Substitution Strategy	Recurrent rotations of player to uphold energy and deed matchups.	A5
Targeted Weakness Exploitation	Emphasis attacking the weakest opponent defenders.	A6
Adaptive Tactical Shifts	Instantons adjustments in response to opponent's strategy shifts.	A7
Set-Variation Strategy	Vicissitudes offensive play style unpredictably to confuse the opponent.	A8

To ensure a robust and objective assessment of each game strategy, we obtained team evaluations from three coaching experts. These experts, with extensive experience in college volleyball coaching, assessed the strategies based on key performance criteria. Each expert provided an independent evaluation of eight strategic alternatives under three different match conditions namely Home Matches (H), Away Matches (A), and Neutral Venue Matches (N). Tables 2-4

resents the neutrosophic evaluations of the eight strategic alternatives according to assessments from three coaching experts.

Table 2. Evaluation of volleyball team based on expert 1 assessments.

Strategy	Q	C1	C2	C3	C4	C5	C6
A1	H	(0.52, 0.48, 0.0)	(0.74, 0.34, 0.16)	(0.39, 0.12, 0.49)	(0.82, 0.34, 0.38)	(0.31, 0.49, 0.2)	(0.8, 0.18, 0.02)
	A	(0.41, 0.17, 0.42)	(0.48, 0.31, 0.21)	(0.56, 0.22, 0.22)	(0.67, 0.16, 0.17)	(0.48, 0.25, 0.27)	(0.57, 0.41, 0.02)
	N	(0.42, 0.31, 0.27)	(0.66, 0.12, 0.22)	(0.66, 0.17, 0.17)	(0.34, 0.48, 0.18)	(0.88, 0.42, 0.22)	(0.36, 0.37, 0.27)
A2	H	(0.56, 0.15, 0.29)	(0.6, 0.11, 0.29)	(0.85, 0.2, 0.37)	(0.49, 0.31, 0.2)	(0.63, 0.17, 0.2)	(0.88, 0.41, 0.48)
	A	(0.84, 0.34, 0.47)	(0.35, 0.18, 0.47)	(0.33, 0.23, 0.44)	(0.53, 0.21, 0.26)	(0.8, 0.24, 0.21)	(0.63, 0.16, 0.21)
	N	(0.78, 0.13, 0.09)	(0.89, 0.41, 0.18)	(0.3, 0.43, 0.27)	(0.72, 0.39, 0.41)	(0.34, 0.24, 0.42)	(0.37, 0.45, 0.18)
A3	H	(0.67, 0.23, 0.1)	(0.34, 0.22, 0.44)	(0.5, 0.39, 0.11)	(0.68, 0.45, 0.29)	(0.37, 0.39, 0.24)	(0.76, 0.32, 0.41)
	A	(0.6, 0.31, 0.09)	(0.56, 0.11, 0.33)	(0.36, 0.11, 0.53)	(0.68, 0.23, 0.09)	(0.61, 0.46, 0.2)	(0.55, 0.4, 0.05)
	N	(0.44, 0.13, 0.43)	(0.47, 0.16, 0.37)	(0.86, 0.42, 0.35)	(0.82, 0.42, 0.17)	(0.84, 0.32, 0.42)	(0.84, 0.23, 0.14)
A4	H	(0.44, 0.27, 0.29)	(0.79, 0.44, 0.1)	(0.61, 0.27, 0.12)	(0.43, 0.15, 0.42)	(0.5, 0.48, 0.02)	(0.49, 0.31, 0.2)
	A	(0.72, 0.25, 0.03)	(0.88, 0.48, 0.2)	(0.6, 0.22, 0.18)	(0.47, 0.11, 0.42)	(0.67, 0.3, 0.03)	(0.33, 0.21, 0.46)
	N	(0.84, 0.2, 0.16)	(0.59, 0.49, 0.2)	(0.7, 0.41, 0.23)	(0.74, 0.25, 0.01)	(0.68, 0.35, 0.31)	(0.35, 0.43, 0.22)
A5	H	(0.49, 0.17, 0.34)	(0.32, 0.34, 0.34)	(0.71, 0.11, 0.18)	(0.61, 0.19, 0.2)	(0.69, 0.17, 0.14)	(0.71, 0.25, 0.04)
	A	(0.86, 0.16, 0.24)	(0.37, 0.47, 0.16)	(0.83, 0.2, 0.36)	(0.79, 0.32, 0.31)	(0.45, 0.14, 0.41)	(0.84, 0.46, 0.35)
	N	(0.5, 0.24, 0.26)	(0.74, 0.46, 0.45)	(0.77, 0.36, 0.13)	(0.4, 0.46, 0.14)	(0.66, 0.1, 0.24)	(0.36, 0.37, 0.27)
A6	H	(0.3, 0.16, 0.54)	(0.63, 0.38, 0.36)	(0.43, 0.38, 0.19)	(0.44, 0.23, 0.33)	(0.75, 0.36, 0.44)	(0.69, 0.33, 0.14)
	A	(0.52, 0.21, 0.27)	(0.45, 0.49, 0.06)	(0.54, 0.46, 0.0)	(0.68, 0.42, 0.3)	(0.65, 0.3, 0.05)	(0.42, 0.39, 0.19)
	N	(0.47, 0.11, 0.42)	(0.69, 0.17, 0.14)	(0.86, 0.48, 0.47)	(0.52, 0.11, 0.37)	(0.86, 0.27, 0.49)	(0.88, 0.44, 0.22)
A7	H	(0.53, 0.44, 0.03)	(0.49, 0.17, 0.34)	(0.63, 0.47, 0.38)	(0.64, 0.14, 0.22)	(0.67, 0.5, 0.16)	(0.61, 0.45, 0.4)
	A	(0.72, 0.38, 0.24)	(0.48, 0.42, 0.1)	(0.79, 0.45, 0.47)	(0.61, 0.3, 0.09)	(0.78, 0.36, 0.38)	(0.78, 0.46, 0.24)
	N	(0.53, 0.14, 0.33)	(0.65, 0.11, 0.24)	(0.58, 0.32, 0.1)	(0.47, 0.34, 0.19)	(0.32, 0.11, 0.57)	(0.79, 0.24, 0.15)
A8	H	(0.61, 0.41, 0.19)	(0.67, 0.13, 0.2)	(0.33, 0.31, 0.36)	(0.62, 0.35, 0.03)	(0.74, 0.49, 0.31)	(0.49, 0.42, 0.09)
	A	(0.46, 0.28, 0.26)	(0.35, 0.11, 0.54)	(0.88, 0.43, 0.38)	(0.55, 0.17, 0.28)	(0.39, 0.2, 0.41)	(0.63, 0.39, 0.36)
	N	(0.47, 0.48, 0.05)	(0.74, 0.32, 0.34)	(0.55, 0.2, 0.25)	(0.51, 0.4, 0.09)	(0.31, 0.15, 0.54)	(0.33, 0.12, 0.55)

Table 3. Evaluation of volleyball team based on expert 2 assessments.

Strategy	Q	C1	C2	C3	C4	C5	C6
A1	H	(0.81, 0.38, 0.29)	(0.36, 0.3, 0.34)	(0.58, 0.17, 0.25)	(0.56, 0.26, 0.18)	(0.67, 0.35, 0.12)	(0.52, 0.35, 0.13)
	A	(0.6, 0.44, 0.36)	(0.4, 0.13, 0.47)	(0.69, 0.11, 0.2)	(0.65, 0.48, 0.33)	(0.53, 0.36, 0.11)	(0.57, 0.32, 0.11)
	N	(0.86, 0.25, 0.48)	(0.84, 0.18, 0.13)	(0.36, 0.11, 0.53)	(0.36, 0.37, 0.27)	(0.34, 0.23, 0.43)	(0.81, 0.11, 0.08)
A2	H	(0.79, 0.21, 0.0)	(0.37, 0.38, 0.25)	(0.68, 0.45, 0.39)	(0.78, 0.21, 0.01)	(0.41, 0.4, 0.19)	(0.78, 0.5, 0.27)
	A	(0.52, 0.41, 0.07)	(0.5, 0.47, 0.03)	(0.82, 0.27, 0.4)	(0.75, 0.14, 0.11)	(0.84, 0.3, 0.43)	(0.49, 0.46, 0.05)
	N	(0.53, 0.1, 0.37)	(0.84, 0.14, 0.02)	(0.49, 0.48, 0.03)	(0.87, 0.33, 0.35)	(0.57, 0.22, 0.21)	(0.5, 0.37, 0.13)
A3	H	(0.75, 0.42, 0.42)	(0.35, 0.3, 0.35)	(0.33, 0.32, 0.35)	(0.56, 0.46, 0.24)	(0.37, 0.16, 0.47)	(0.76, 0.35, 0.14)
	A	(0.35, 0.38, 0.27)	(0.34, 0.43, 0.23)	(0.72, 0.13, 0.15)	(0.35, 0.49, 0.16)	(0.52, 0.25, 0.23)	(0.79, 0.48, 0.49)
	N	(0.75, 0.25, 0.0)	(0.35, 0.41, 0.24)	(0.64, 0.27, 0.09)	(0.84, 0.14, 0.02)	(0.6, 0.1, 0.3)	(0.58, 0.12, 0.3)

A4	H	(0.37, 0.15, 0.48)	(0.69, 0.4, 0.33)	(0.88, 0.25, 0.21)	(0.82, 0.19, 0.49)	(0.31, 0.49, 0.2)	(0.33, 0.46, 0.21)
	A	(0.62, 0.5, 0.13)	(0.63, 0.49, 0.31)	(0.68, 0.38, 0.28)	(0.68, 0.33, 0.46)	(0.33, 0.21, 0.46)	(0.87, 0.46, 0.28)
	N	(0.67, 0.21, 0.12)	(0.41, 0.29, 0.3)	(0.51, 0.33, 0.16)	(0.35, 0.49, 0.16)	(0.89, 0.38, 0.31)	(0.49, 0.43, 0.08)
A5	H	(0.71, 0.17, 0.12)	(0.85, 0.43, 0.48)	(0.74, 0.35, 0.27)	(0.86, 0.45, 0.12)	(0.32, 0.25, 0.43)	(0.79, 0.49, 0.16)
	A	(0.66, 0.25, 0.09)	(0.88, 0.44, 0.44)	(0.58, 0.27, 0.15)	(0.46, 0.12, 0.42)	(0.82, 0.43, 0.5)	(0.9, 0.32, 0.41)
	N	(0.87, 0.44, 0.2)	(0.57, 0.15, 0.28)	(0.87, 0.34, 0.19)	(0.7, 0.35, 0.24)	(0.37, 0.37, 0.26)	(0.61, 0.41, 0.31)
A6	H	(0.81, 0.32, 0.32)	(0.83, 0.26, 0.15)	(0.32, 0.4, 0.28)	(0.67, 0.38, 0.19)	(0.38, 0.11, 0.51)	(0.51, 0.34, 0.15)
	A	(0.54, 0.27, 0.19)	(0.84, 0.24, 0.31)	(0.77, 0.26, 0.35)	(0.82, 0.48, 0.16)	(0.86, 0.3, 0.2)	(0.58, 0.49, 0.3)
	N	(0.5, 0.35, 0.15)	(0.44, 0.13, 0.43)	(0.38, 0.15, 0.47)	(0.39, 0.16, 0.45)	(0.68, 0.17, 0.15)	(0.51, 0.46, 0.03)
A7	H	(0.58, 0.37, 0.05)	(0.4, 0.18, 0.42)	(0.32, 0.17, 0.51)	(0.47, 0.17, 0.36)	(0.35, 0.15, 0.5)	(0.58, 0.18, 0.24)
	A	(0.52, 0.3, 0.18)	(0.71, 0.12, 0.17)	(0.78, 0.35, 0.13)	(0.82, 0.47, 0.12)	(0.47, 0.42, 0.11)	(0.75, 0.17, 0.08)
	N	(0.43, 0.25, 0.32)	(0.59, 0.35, 0.06)	(0.52, 0.29, 0.19)	(0.75, 0.11, 0.14)	(0.45, 0.39, 0.16)	(0.84, 0.3, 0.31)
A8	H	(0.36, 0.28, 0.36)	(0.62, 0.2, 0.18)	(0.46, 0.25, 0.29)	(0.31, 0.23, 0.46)	(0.43, 0.23, 0.34)	(0.37, 0.46, 0.17)
	A	(0.66, 0.37, 0.42)	(0.6, 0.13, 0.27)	(0.62, 0.33, 0.05)	(0.75, 0.27, 0.15)	(0.47, 0.25, 0.28)	(0.69, 0.33, 0.24)
	N	(0.89, 0.34, 0.19)	(0.36, 0.16, 0.48)	(0.45, 0.16, 0.39)	(0.41, 0.21, 0.38)	(0.4, 0.46, 0.14)	(0.35, 0.31, 0.34)

Table 4. Evaluation of volleyball team based on expert 3 assessments.

Strategy	Q	C1	C2	C3	C4	C5	C6
A1	H	(0.55, 0.49, 0.14)	(0.54, 0.49, 0.45)	(0.79, 0.2, 0.01)	(0.4, 0.37, 0.23)	(0.86, 0.32, 0.33)	(0.47, 0.41, 0.12)
	A	(0.41, 0.23, 0.36)	(0.56, 0.3, 0.14)	(0.45, 0.15, 0.4)	(0.67, 0.22, 0.11)	(0.65, 0.16, 0.19)	(0.59, 0.31, 0.1)
	N	(0.33, 0.23, 0.44)	(0.38, 0.13, 0.49)	(0.89, 0.23, 0.42)	(0.45, 0.37, 0.18)	(0.76, 0.34, 0.29)	(0.55, 0.24, 0.21)
A2	H	(0.86, 0.43, 0.49)	(0.37, 0.39, 0.24)	(0.86, 0.17, 0.13)	(0.74, 0.33, 0.44)	(0.38, 0.42, 0.2)	(0.42, 0.17, 0.41)
	A	(0.4, 0.43, 0.17)	(0.7, 0.31, 0.24)	(0.83, 0.26, 0.43)	(0.56, 0.25, 0.19)	(0.58, 0.22, 0.2)	(0.75, 0.3, 0.19)
	N	(0.84, 0.25, 0.32)	(0.84, 0.35, 0.15)	(0.86, 0.35, 0.23)	(0.38, 0.42, 0.2)	(0.67, 0.31, 0.02)	(0.84, 0.42, 0.16)
A3	H	(0.49, 0.2, 0.31)	(0.75, 0.11, 0.14)	(0.64, 0.4, 0.45)	(0.51, 0.43, 0.06)	(0.37, 0.44, 0.19)	(0.38, 0.26, 0.36)
	A	(0.78, 0.16, 0.06)	(0.44, 0.39, 0.17)	(0.73, 0.36, 0.38)	(0.63, 0.2, 0.17)	(0.51, 0.17, 0.32)	(0.85, 0.33, 0.26)
	N	(0.58, 0.48, 0.16)	(0.65, 0.3, 0.05)	(0.67, 0.11, 0.22)	(0.82, 0.47, 0.33)	(0.72, 0.47, 0.38)	(0.39, 0.33, 0.28)
A4	H	(0.66, 0.27, 0.07)	(0.74, 0.47, 0.47)	(0.57, 0.15, 0.28)	(0.89, 0.44, 0.15)	(0.85, 0.45, 0.31)	(0.65, 0.26, 0.09)
	A	(0.33, 0.23, 0.44)	(0.78, 0.1, 0.12)	(0.5, 0.26, 0.24)	(0.62, 0.47, 0.24)	(0.51, 0.4, 0.09)	(0.57, 0.19, 0.24)
	N	(0.57, 0.16, 0.27)	(0.41, 0.3, 0.29)	(0.55, 0.47, 0.24)	(0.65, 0.35, 0.0)	(0.31, 0.37, 0.32)	(0.41, 0.48, 0.11)
A5	H	(0.39, 0.27, 0.34)	(0.35, 0.5, 0.15)	(0.6, 0.34, 0.06)	(0.34, 0.4, 0.26)	(0.43, 0.46, 0.11)	(0.42, 0.18, 0.4)
	A	(0.32, 0.29, 0.39)	(0.64, 0.13, 0.23)	(0.77, 0.28, 0.31)	(0.56, 0.26, 0.18)	(0.64, 0.16, 0.2)	(0.41, 0.44, 0.15)
	N	(0.87, 0.25, 0.21)	(0.69, 0.26, 0.05)	(0.32, 0.16, 0.52)	(0.73, 0.36, 0.11)	(0.43, 0.19, 0.38)	(0.7, 0.11, 0.19)
A6	H	(0.36, 0.42, 0.22)	(0.41, 0.36, 0.23)	(0.44, 0.14, 0.42)	(0.45, 0.39, 0.16)	(0.81, 0.43, 0.26)	(0.7, 0.18, 0.12)
	A	(0.48, 0.46, 0.06)	(0.31, 0.13, 0.56)	(0.42, 0.11, 0.47)	(0.41, 0.33, 0.26)	(0.55, 0.46, 0.43)	(0.51, 0.2, 0.29)
	N	(0.53, 0.34, 0.13)	(0.46, 0.35, 0.19)	(0.55, 0.32, 0.13)	(0.56, 0.22, 0.22)	(0.87, 0.41, 0.16)	(0.82, 0.29, 0.46)
A7	H	(0.78, 0.27, 0.11)	(0.46, 0.32, 0.22)	(0.68, 0.2, 0.12)	(0.38, 0.43, 0.19)	(0.89, 0.31, 0.17)	(0.46, 0.11, 0.43)
	A	(0.85, 0.15, 0.0)	(0.65, 0.21, 0.14)	(0.63, 0.36, 0.01)	(0.8, 0.18, 0.02)	(0.31, 0.15, 0.54)	(0.84, 0.45, 0.34)
	N	(0.66, 0.37, 0.17)	(0.85, 0.27, 0.25)	(0.61, 0.12, 0.27)	(0.4, 0.4, 0.2)	(0.35, 0.34, 0.31)	(0.45, 0.26, 0.29)

A8	H	(0.47, 0.24, 0.29)	(0.73, 0.22, 0.05)	(0.64, 0.29, 0.07)	(0.7, 0.47, 0.39)	(0.43, 0.11, 0.46)	(0.46, 0.34, 0.2)
	A	(0.33, 0.3, 0.37)	(0.66, 0.23, 0.11)	(0.76, 0.14, 0.1)	(0.35, 0.39, 0.26)	(0.6, 0.38, 0.02)	(0.56, 0.2, 0.24)
	N	(0.79, 0.42, 0.38)	(0.46, 0.34, 0.2)	(0.52, 0.14, 0.34)	(0.85, 0.15, 0.0)	(0.87, 0.28, 0.17)	(0.63, 0.45, 0.39)

5. Results & Discussion

This section presents the findings of the proposed approach for opponent analysis and strategy selection framework applied to college volleyball. The results are analyzed based on the case study described in the previous section. The discussion focuses on interpreting the results, comparing the selected strategy with traditional DM approaches, and examining the practical implications of our approach in real-world volleyball coaching.

The results of applying the proposed CWAO scheme to various QNSS decision matrices provided by different volleyball experts are shown in Tables 5-7. These tables showcase the aggregated QNSS components for each strategy A_i across the criteria C_1 *to* C_6, considering the varying contexts. The aggregation process employs the CWAO, which syndicated hybrid weighted geometric averaging to guarantee a balanced and robust capturing of the relative importance of each context. The results highlighted the suitablity of each strategy under various criteria, which provide a inclusive and nuanced DM for optimizing the defensive strategy against a strong attacking opponent.

Table 5: Context-Aware Aggregation Results for QNSS Decision Matrix from Expert 1.

Strategy	C1	C2	C3	C4	C5	C6
A1	(0.449, 0.320, 0.230)	(0.614, 0.257, 0.197)	(0.534, 0.170, 0.293)	(0.603, 0.327, 0.243)	(0.554, 0.387, 0.230)	(0.572, 0.320, 0.103)
A2	(0.677, 0.207, 0.283)	(0.607, 0.233, 0.313)	(0.493, 0.287, 0.360)	(0.573, 0.303, 0.290)	(0.585, 0.217, 0.277)	(0.618, 0.340, 0.290)
A3	(0.564, 0.223, 0.207)	(0.456, 0.163, 0.380)	(0.570, 0.307, 0.330)	(0.672, 0.367, 0.183)	(0.600, 0.390, 0.287)	(0.673, 0.317, 0.200)
A4	(0.648, 0.240, 0.160)	(0.684, 0.469, 0.167)	(0.620, 0.300, 0.177)	(0.543, 0.170, 0.283)	(0.605, 0.377, 0.120)	(0.390, 0.317, 0.293)
A5	(0.607, 0.190, 0.280)	(0.476, 0.423, 0.317)	(0.677, 0.223, 0.223)	(0.593, 0.323, 0.217)	(0.591, 0.137, 0.263)	(0.627, 0.360, 0.220)
A6	(0.430, 0.160, 0.410)	(0.583, 0.347, 0.187)	(0.602, 0.439, 0.220)	(0.543, 0.253, 0.333)	(0.680, 0.310, 0.327)	(0.647, 0.386, 0.183)
A7	(0.585, 0.320, 0.200)	(0.536, 0.233, 0.227)	(0.643, 0.413, 0.317)	(0.567, 0.260, 0.167)	(0.585, 0.323, 0.370)	(0.674, 0.383, 0.263)
A8	(0.511, 0.390, 0.167)	(0.581, 0.187, 0.360)	(0.583, 0.313, 0.330)	(0.555, 0.307, 0.133)	(0.479, 0.280, 0.420)	(0.482, 0.310, 0.333)

Table 6: Context-Aware Aggregation Results for QNSS Decision Matrix from Expert 2.

Strategy	C1	C2	C3	C4	C5	C6
A1	(0.684, 0.357, 0.377)	(0.532, 0.203, 0.313)	(0.540, 0.130, 0.327)	(0.521, 0.370, 0.260)	(0.512, 0.313, 0.220)	(0.619, 0.260, 0.107)
A2	(0.603, 0.240, 0.147)	(0.566, 0.330, 0.100)	(0.643, 0.400, 0.273)	(0.668, 0.227, 0.157)	(0.599, 0.307, 0.277)	(0.583, 0.443, 0.150)
A3	(0.609, 0.350, 0.230)	(0.347, 0.380, 0.273)	(0.560, 0.240, 0.197)	(0.579, 0.363, 0.140)	(0.495, 0.170, 0.333)	(0.668, 0.317, 0.310)
A4	(0.550, 0.287, 0.243)	(0.571, 0.393, 0.313)	(0.662, 0.320, 0.217)	(0.609, 0.337, 0.370)	(0.509, 0.360, 0.323)	(0.561, 0.449, 0.190)
A5	(0.679, 0.287, 0.137)	(0.689, 0.340, 0.400)	(0.678, 0.320, 0.203)	(0.652, 0.307, 0.260)	(0.502, 0.350, 0.396)	(0.685, 0.406, 0.293)
A6	(0.606, 0.313, 0.220)	(0.674, 0.210, 0.297)	(0.489, 0.270, 0.367)	(0.617, 0.340, 0.267)	(0.629, 0.193, 0.287)	(0.530, 0.430, 0.160)
A7	(0.508, 0.307, 0.183)	(0.562, 0.217, 0.217)	(0.538, 0.270, 0.277)	(0.656, 0.250, 0.207)	(0.423, 0.320, 0.257)	(0.675, 0.217, 0.210)

A8	(0.627, 0.330, 0.323)	(0.524, 0.163, 0.310)	(0.508, 0.247, 0.243)	(0.489, 0.237, 0.330)	(0.433, 0.313, 0.253)	(0.469, 0.367, 0.250)

Table 7: Context-Aware Aggregation Results for QNSS Decision Matrix from Expert 3.

Strategy	C1	C2	C3	C4	C5	C6
A1	(0.430, 0.317, 0.313)	(0.492, 0.307, 0.360)	(0.678, 0.193, 0.277)	(0.505, 0.320, 0.173)	(0.680, 0.273, 0.270)	(0.533, 0.320, 0.143)
A2	(0.676, 0.370, 0.327)	(0.626, 0.350, 0.210)	(0.619, 0.260, 0.263)	(0.556, 0.333, 0.277)	(0.540, 0.317, 0.140)	(0.651, 0.297, 0.253)
A3	(0.606, 0.280, 0.177)	(0.603, 0.267, 0.120)	(0.649, 0.290, 0.350)	(0.635, 0.367, 0.187)	(0.531, 0.360, 0.297)	(0.538, 0.307, 0.300)
A4	(0.518, 0.220, 0.260)	(0.630, 0.290, 0.293)	(0.536, 0.293, 0.253)	(0.674, 0.420, 0.130)	(0.554, 0.406, 0.240)	(0.540, 0.310, 0.147)
A5	(0.525, 0.270, 0.313)	(0.556, 0.297, 0.143)	(0.560, 0.260, 0.297)	(0.541, 0.340, 0.183)	(0.498, 0.270, 0.230)	(0.508, 0.243, 0.247)
A6	(0.456, 0.406, 0.137)	(0.393, 0.280, 0.327)	(0.469, 0.190, 0.340)	(0.472, 0.313, 0.213)	(0.685, 0.433, 0.283)	(0.652, 0.223, 0.290)
A7	(0.680, 0.263, 0.093)	(0.637, 0.267, 0.203)	(0.622, 0.227, 0.133)	(0.525, 0.337, 0.137)	(0.516, 0.267, 0.340)	(0.578, 0.273, 0.353)
A8	(0.528, 0.320, 0.347)	(0.606, 0.263, 0.120)	(0.624, 0.190, 0.170)	(0.624, 0.337, 0.217)	(0.622, 0.257, 0.217)	(0.546, 0.330, 0.277)

Following the derivation of the final aggregation matrix based on the CWAO, we compute the weights of different criteria in Figure 1.

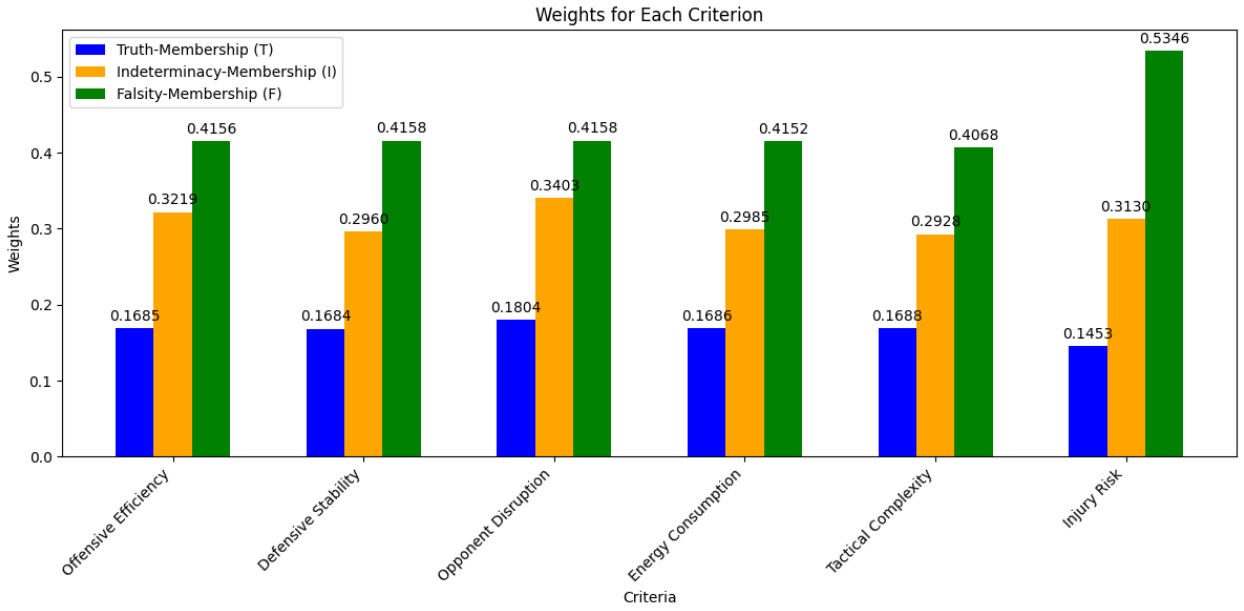

Figure 1. Visualization of computed weights for different criteria in terms of components of the aggregated QNSS.

To assess the competitiveness of the proposed approach against conventional DM approaches (fuzzy COPRA [14], Fuzzy OCRA [15], Fuzzy Edas [16]) , we provide a summary of results of comparative analysis in Table 8. It can be noted that our model outperforms previous methods in handling highly uncertain, vague, as well as incomplete information inherent in volleyball analysis. The fuzzy COPRA, while effective in dealing with uncertainty to some extent, lacks the indeterminate component needed to capture real-world ambiguity fully. In similar way, Fuzzy

OCRA, though mathematically robust, require well-defined probability distributionss. The QNSS approach, by contrast, show a great evidence on more flexible and comprehensive framework by integration multi-context components, which lead to more robust and adaptable DM.

Table 8. Comparative analysis of our approach against different methods.

Method	
Our	A1> A2> A5> A4> A6> A8> A3> A7
Fuzzy COPRA [14]	A1> A2> A5> A4> A8> A6> A7> A3
Fuzzy Edas [16]	A1> A2> A4> A5 > A8> A7> A6> A3
Fuzzy OCRA [15]	A1> A2> A6> A4 > A5 > A8> A3> A7

To further validate the reliability of our approach model, we performed a sensitivity analysis to assess how variations in uncertainty levels impact opponent analysis (Table 9.). Sensitivity analysis is crucial in evaluating whether small perturbations in weights lead to significant changes in output, which ensured that the model remains stable and practical under dynamic startup environments.

Table 9. Sensitivity Analysis of our approach

Weight Adjustment	**Rank**
-40%	A1> A2> A5> A4> A6> A8> A3> A7
-30%	A1> A2> A5> A4> A6> A8> A3> A7
-10%	A1> A2> A5> A4> A6> A8> A3> A7
+40%	A1> A2> A5> A4> A6> A8> A3> A7
+30%	A1> A2> A5> A4> A6> A8> A3> A7
+10%	A1> A2> A5> A4> A6> A8> A3> A7

6. Conclusion

This study research investigates a Q-Neutrosophic Soft Set (QNSS) approach for managing uncertainty in opponent analysis and game strategy DM in college volleyball. Our framework proposes a multi-context aggregation operator for generating insightful and multi-dimension decision matrix. Then, the aggregated QNSS is then used to develop a customized Best-Worst Method to rate the contribution or importance of different parameters for different DM analysis. Our approach was applied to realistic case study for opponent analysis in college volleyball, which demonstrated its practicality in sports strategy formulation under multi-context, and multi-factor scenarios. Future research will extend this framework to integrate dynamic optimization methods to solve complex sports DM problems.

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An Introduction to Advanced Soft Set Variants: SuperHyperSoft Sets, IndetermSuperHyperSoft Sets, IndetermTreeSoft Sets, BiHyperSoft sets, GraphicSoft sets, and Beyond

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Abstract. This paper focuses on the study of Soft Sets, a concept that has led to the development of various extensions, including Double-Framed Soft Sets [5, 6, 10, 21], Hypersoft Sets [1], SuperHyperSoft Sets [36], ForestSoft Sets, TreeSoft Sets [3], IndetermSoft Sets [35], and IndetermHyperSoft Sets [35]. These extensions have been actively explored in recent research. Smarandache [https://fs.unm.edu/TSS/] introduced six new types of Soft Sets, such as: the HyperSoft Set (2018), IndetermSoft Set (2022), IndetermHyperSoft Set (2022), SuperHyperSoft Set, TreeSoft Set (2022) and ForestSoft Set (2024).

In this paper, we review several advanced soft set concepts, including IndetermSuperHyperSoft Sets, IndetermForestSoft Sets, IndetermTreeSoft Sets, and other related structures. We hope that this work will inspire more researchers to explore Soft Sets and uncertainty modeling, further advancing studies in this field.

Keywords: Soft Set, Hypersoft set, Superhypersoft set, IndetermSoft Set, Treesoft set

1. Soft Set, Hypersoft Set, and SuperHypersoft Set

A Soft Set offers a straightforward approach to parameterized decision modeling by associating attributes (or parameters) with subsets of a universal set, effectively addressing uncertainty in a structured manner [25, 26]. Several related mathematical frameworks, such as Fuzzy Sets [44], Neutrosophic Sets [14, 32, 41], and Rough Sets [30], have been introduced to handle different aspects of uncertainty. Building upon this foundation, the Hypersoft Set extends Soft Sets by incorporating multi-attribute decision modeling. Instead of assigning a single parameter to a subset of the universal set, a Hypersoft Set maps combinations of multiple attributes to subsets of the universal set, thereby improving its capability for handling complex decision-making scenarios [20, 33]. Further generalizing the concept, SuperHypersoft Sets expand the functionality of Hypersoft Sets by mapping power set combinations of multiple attribute values to subsets of a universal set. This higher-order approach allows for multidimensional decision-making and captures intricate interrelationships among attributes, making it a powerful tool for modeling complex systems [15, 16, 36, 42].

Definition 1.1 (Soft Set)**.** [25, 26] Let U be a universal set and A be a set of attributes. A soft set over U is a pair $(\mathcal{F}, S)$, where $S \subseteq A$ and $\mathcal{F} : S \to \mathcal{P}(U)$. Here, $\mathcal{P}(U)$ denotes the power set of U. Mathematically, a soft set is represented as:

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \in \mathcal{P}(U)\}.$$

Each $\alpha \in S$ is called a parameter, and $\mathcal{F}(\alpha)$ is the set of elements in U associated with α.

Definition 1.2 (Hypersoft Set)**.** [33] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be attribute domains. Define $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m$, the Cartesian product of these domains. A hypersoft set over U is a pair $(G, \mathcal{C})$, where $G : \mathcal{C} \to \mathcal{P}(U)$. The hypersoft set is expressed as:

$$(G, \mathcal{C}) = \{(\gamma, G(\gamma)) \mid \gamma \in \mathcal{C}, G(\gamma) \in \mathcal{P}(U)\}.$$

For an m-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathcal{C}$, where $\gamma_i \in \mathcal{A}_i$ for $i = 1, 2, \ldots, m$, $G(\gamma)$ represents the subset of U corresponding to the combination of attribute values $\gamma_1, \gamma_2, \ldots, \gamma_m$.

Definition 1.3 (SuperHyperSoft Set)**.** [36] Let U be a universal set, and let $\mathcal{P}(U)$ denote the power set of U. Consider n distinct attributes $a_1, a_2, \ldots, a_n$, where $n \geq 1$. Each attribute a_i is associated with a set of attribute values A_i, satisfying the property $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Define $\mathcal{P}(A_i)$ as the power set of A_i for each $i = 1, 2, \ldots, n$. Then, the Cartesian product of the power sets of attribute values is given by:

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n).$$

A SuperHyperSoft Set over U is a pair $(F, \mathcal{C})$, where:

$$F : \mathcal{C} \to \mathcal{P}(U),$$

and F maps each element $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$ (with $\alpha_i \in \mathcal{P}(A_i)$) to a subset $F(\alpha_1, \alpha_2, \ldots, \alpha_n) \subseteq U$. Mathematically, the SuperHyperSoft Set is represented as:

$$(F, \mathcal{C}) = \{(\gamma, F(\gamma)) \mid \gamma \in \mathcal{C}, F(\gamma) \in \mathcal{P}(U)\}.$$

Here, $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$, where $\alpha_i \in \mathcal{P}(A_i)$ for $i = 1, 2, \ldots, n$, and $F(\gamma)$ corresponds to the subset of U defined by the combined attribute values $\alpha_1, \alpha_2, \ldots, \alpha_n$.

2. Double-Framed Hypersoft Set

The Double-Framed Soft Set [5,6,21] and Double-Framed Hypersoft Set [10] are extended concepts of the Soft Set and Hypersoft Set, incorporating two frames for enhanced representation. Their definitions are provided below.

Definition 2.1 (Double Framed Soft Set)**.** [5, 6, 21] Let U be the universal set, and let A be a set of parameters. A *Double-Framed Soft Set* is a triple $\langle(\alpha, \beta); A\rangle$, where:

(1) $\alpha : A \to P(U)$ and $\beta : A \to P(U)$ are mappings from the parameter set A to the power set of U.

(2) $\alpha(x)$ represents the *positive frame* and $\beta(x)$ represents the *negative frame* for each parameter $x \in A$.

A Double-Framed Soft Set satisfies the condition:

$$\forall x, y \in A, \quad \alpha(x * y) \supseteq \alpha(x) \cap \alpha(y), \quad \beta(x * y) \subseteq \beta(x) \cup \beta(y),$$

where $*$ is a binary operation defined on A.

Definition 2.2 (Double-Framed Hypersoft Set (DFHSS))**.** [10] Let U be the universal set and $P(U)$ denote the power set of U. Let $\{a_1, a_2, \ldots, a_n\}$ represent n distinct attributes, where each attribute a_i is associated with a set of attribute values ϕ_i, satisfying the conditions:

$$\phi_i \cap \phi_j = \emptyset \quad \text{for } i \neq j, \quad i, j \in \{1, 2, \ldots, n\}.$$

A *Double-Framed Hypersoft Set (DFHSS)* is defined as a tuple:

$$(\pi_1, \pi_2; \phi_1 \times \phi_2 \times \cdots \times \phi_n),$$

where:

- $\phi_1 \times \phi_2 \times \cdots \times \phi_n$ is the Cartesian product of the attribute value sets.
- $\pi_1, \pi_2 : \phi_1 \times \phi_2 \times \cdots \times \phi_n \to P(U)$ are mappings that associate each tuple of attribute values with subsets of the universal set U.

Definition 2.3 (Double-Framed SuperHypersoft Set)**.** Let U be a universal set. Suppose there are n distinct attributes $a_1, a_2, \ldots, a_n$, each associated with a set of possible values A_i such that $A_i \cap A_j = \varnothing$ for all $i \neq j$. For each $i \in \{1, 2, \ldots, n\}$, let

$$\mathcal{P}(A_i)$$

denote the power set of A_i. Define

$$\mathcal{C} \;=\; \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n),$$

which is the Cartesian product of these power sets.

A *Double-Framed SuperHypersoft Set (DFSHSS)* over U is then a triple

$$\big(\Theta_1, \Theta_2; \mathcal{C}\big),$$

where

$$\Theta_1 : \mathcal{C} \;\to\; \mathcal{P}(U), \quad \Theta_2 : \mathcal{C} \;\to\; \mathcal{P}(U).$$

That is, both Θ_1 and Θ_2 map each element of $\mathcal{C}$ (i.e., each n-tuple $\gamma = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathcal{P}(A_i)$) to a subset of U.

Informally, $\Theta_1(\gamma)$ and $\Theta_2(\gamma)$ can be viewed as two distinct but related "frames" (e.g., a *positive* vs. *negative*, or *lower* vs. *upper* approximation) for the combined attribute values in γ.

3. Treesoft set and ForestSoft Set

A *TreeSoft Set* maps the power set of a hierarchical tree-like structure of attributes, Tree(A), to subsets of a universal set [40]. It is known that concepts such as TreeSoft Sets can generalize MultiSoft Sets and related frameworks.

Definition 3.1. [37] Let U be a universe of discourse, and let H be a non-empty subset of U, with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes (parameters, factors, etc.), for some integer $n \geq 1$, where each attribute A_i (for $1 \leq i \leq n$) is considered a first-level attribute.

Each first-level attribute A_i consists of sub-attributes, defined as:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

where the elements $A_{i,j}$ (for $j = 1, 2, \ldots$) are second-level sub-attributes of A_i. Each second-level sub-attribute $A_{i,j}$ may further contain sub-sub-attributes, defined as:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\},$$

and so on, allowing for as many levels of refinement as needed. Thus, we can define sub-attributes of an m-th level with indices $A_{i_1,i_2,\ldots,i_m}$, where each i_k (for $k = 1, \ldots, m$) denotes the position at each level.

This hierarchical structure forms a tree-like graph, which we denote as Tree(A), with root A (level 0) and successive levels from 1 up to m, where m is the depth of the tree. The terminal nodes (nodes without descendants) are called *leaves* of the graph-tree.

A *TreeSoft Set* F is defined as a function:

$$F : P(\text{Tree}(A)) \to P(H),$$

where Tree(A) represents the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and $P(\text{Tree}(A))$ denotes its power set.

A *ForestSoft Set* is formed by taking a collection of TreeSoft Sets and "gluing" (uniting) them together so as to obtain a single function whose domain is the union of all tree-nodes' power sets and whose values in $P(H)$ combine the images given by the individual TreeSoft Sets [31, 38].

Definition 3.2 (ForestSoft Set)**.** [38] Let U be a universe of discourse, $H \subseteq U$ be a non-empty subset, and $P(H)$ be the power set of H. Suppose we have a finite (or countable) collection of TreeSoft Sets

$$\left\{ F_t : P(\text{Tree}(A^{(t)})) \to P(H) \right\}_{t \in T},$$

where each F_t is a TreeSoft Set corresponding to a tree Tree($A^{(t)}$) of attributes $A^{(t)}$.

We construct a *forest* by taking the (disjoint) union of all these trees:

$$\text{Forest}\big(\{A^{(t)}\}_{t \in T}\big) \;=\; \bigsqcup_{t \in T} \text{Tree}\big(A^{(t)}\big).$$

A *ForestSoft Set*, denoted by

$$\mathbf{F} : P\big(\text{Forest}(\{A^{(t)}\})\big) \longrightarrow P(H),$$

is defined as the *union* of all TreeSoft Set mappings F_t. Concretely, for any element $X \in P\big(\text{Forest}(\{A^{(t)}\})\big)$, we set

$$\mathbf{F}(X) = \bigcup_{\substack{t\in T \\ X\cap \text{Tree}(A^{(t)})\neq \varnothing}} F_t\big(X \cap \text{Tree}(A^{(t)})\big),$$

where we only apply F_t to that portion of X belonging to the tree $\text{Tree}(A^{(t)})$.

4. Double-Framed Treesoft Set

We now define the Double-Framed Treesoft Set, extending the idea of a Treesoft Set (which maps subsets of a hierarchical attribute tree to subsets of the universe) by introducing two frames [17].

Definition 4.1 (Double-Framed Treesoft Set)**.** [17] Let:

- U be a universal set.
- $\text{Tree}(A)$ be a hierarchical attribute tree constructed from an attribute set $A = \{A_1, A_2, \ldots, A_n\}$ (with possibly multiple levels of sub-attributes, sub-sub-attributes, etc.).
- $P(\text{Tree}(A))$ be the power set of all nodes (including leaves) in the tree $\text{Tree}(A)$.

A *Double-Framed Treesoft Set (DFTS)* is a triple

$$\big(\Phi_1, \Phi_2; \text{Tree}(A)\big),$$

where

$$\Phi_1 : P\big(\text{Tree}(A)\big) \to \mathcal{P}(U), \quad \Phi_2 : P\big(\text{Tree}(A)\big) \to \mathcal{P}(U).$$

For each subset of nodes $X \subseteq \text{Tree}(A)$, $\Phi_1(X)$ and $\Phi_2(X)$ represent two distinct frames (e.g., *positive* vs. *negative* or *lower* vs. *upper*) for the elements of U relevant to the portion of the tree in X.

5. New Concepts: Double-Framed Forestsoft Set

In this paper, we introduce a new concept, the Double-Framed Forestsoft Set, which is defined as follows. We anticipate that future research will further explore its applications and mathematical properties.

Definition 5.1 (Double-Framed Forestsoft Set)**.** Let U be a universal set, $H \subseteq U$ be a non-empty subset, and let

$$\text{Forest}\big(\{A^{(t)}\}_{t\in T}\big)$$

be a forest formed by the disjoint union of attribute trees $\text{Tree}(A^{(t)})$ for $t \in T$. Let

$$P\big(\text{Forest}(\{A^{(t)}\}_{t\in T})\big)$$

denote the power set of the forest. A *Double-Framed Forestsoft Set* (DFFS) is defined as a pair of mappings

$$\big(\Psi_1, \Psi_2\big) : P\big(\text{Forest}(\{A^{(t)}\}_{t\in T})\big) \to P(H),$$

such that for each $X \subseteq \text{Forest}(\{A^{(t)}\}_{t\in T})$,

$$\Psi_1(X) \quad \text{and} \quad \Psi_2(X)$$

represent two distinct frames (for example, a positive frame and a negative frame) associated with the subset X of the forest. These frames provide dual evaluations of the objects in H with respect to the hierarchical attributes in X.

Example 5.2 (Evaluating Environmental Impact)**.** Suppose

$$U = \{\text{Site1, Site2, Site3, Site4}\}$$

represents a set of environmental sites, and let $H = U$. Assume that the attributes are organized into trees reflecting factors such as *Pollution Level* and *Biodiversity*. For simplicity, consider a forest formed by two trees:

- Tree($A^{(1)}$) representing *Pollution* with nodes {Low, Medium, High}.
- Tree($A^{(2)}$) representing *Biodiversity* with nodes {Rich, Moderate, Poor}.

A traditional Forestsoft Set $\mathbf{F}$ might assign:

$$\mathbf{F}(X) = \{\text{Site2, Site3}\} \quad \text{for } X = \{\text{High Pollution}, \text{Poor Biodiversity}\}.$$

Now, a Double-Framed Forestsoft Set introduces two mappings:

$$\Psi_1, \Psi_2 : P\big(\text{Forest}(\{A^{(1)}, A^{(2)}\})\big) \to P(U).$$

For the same X, suppose:

$$\Psi_1(X) = \{\text{Site2}\} \quad \text{(a positive evaluation, e.g., based on recovery potential)},$$

$$\Psi_2(X) = \{\text{Site3}\} \quad \text{(a negative evaluation, e.g., based on risk assessment)}.$$

Thus, the Double-Framed Forestsoft Set provides a dual perspective for the environmental assessment of sites under the attributes in X.

Example 5.3 (Evaluating Agricultural Land)**.** Let

$$U = \{\text{Field1, Field2, Field3}\}$$

be a set of agricultural fields, and suppose $H = U$. Assume the forest is built from two trees representing:

- Tree($A^{(1)}$): *Soil Quality* with nodes {Poor, Average, Excellent}.
- Tree($A^{(2)}$): *Irrigation Availability* with nodes {Scarce, Moderate, Abundant}.

A Double-Framed Forestsoft Set assigns two evaluations to each subset $X \subseteq \text{Forest}(\{A^{(1)}, A^{(2)}\})$. For instance, for

$$X = \{\text{Excellent Soil, Abundant Irrigation}\},$$

one might have:

$$\Psi_1(X) = \{\text{Field1, Field3}\} \quad \text{(indicating fields highly favorable for production)},$$

$$\Psi_2(X) = \{\text{Field2}\} \quad \text{(indicating fields with potential risks, such as waterlogging)}.$$

This dual evaluation allows decision-makers to consider both the positive and negative aspects of the agricultural land.

6. Bipolar Hypersoft Set and Bipolar SuperHypersoft Set

A Bipolar Soft Set represents positive and negative memberships, ensuring consistency by mapping parameters to subsets of a universal set [2,8,9,19,22,43]. A Bipolar Hypersoft Set extends Bipolar Soft Sets by incorporating multi-attribute combinations for positive and negative memberships in decision-making frameworks [4, 27–29].

Definition 6.1 (Bipolar Soft Set)**.** [8, 9, 43] A *Bipolar Soft Set* over a universal set U is a triple (F, G, A), where:

- $F : A \to P(U)$ is the *positive membership mapping*,
- $G : \neg A \to P(U)$ is the *negative membership mapping*,
- $A \subseteq E$, $\neg A = E \setminus A$, where E is a set of parameters.

The mappings satisfy the *consistency constraint*:

$$F(e) \cap G(\neg e) = \emptyset, \quad \forall e \in A.$$

A Bipolar Soft Set is represented as:

$$(F, G, A) = \{(e, F(e), G(\neg e)) \mid e \in A, F(e) \cap G(\neg e) = \emptyset\}.$$

Definition 6.2 (Bipolar Hypersoft Set)**.** [27–29] A *Bipolar Hypersoft Set (BHS-Set)* is a triple (F, G, A) over a universe of discourse U, where:

- $F : A \to \mathcal{P}(U)$ and $G : \neg A \to \mathcal{P}(U)$, with $\mathcal{P}(U)$ denoting the power set of U.
- The mappings satisfy the *consistency constraint*:

$$F(\alpha) \cap G(\neg\alpha) = \emptyset, \quad \forall \alpha \in A.$$

- $A = A_1 \times A_2 \times \cdots \times A_n$, where $A_i \subseteq E_i$ and $E = E_1 \times E_2 \times \cdots \times E_n$.
- $\neg A = \neg A_1 \times \neg A_2 \times \cdots \times \neg A_n$, where $\neg A_i = E_i \setminus A_i$.

The BHS-Set (F, G, A) is represented as:

$$(F, G, A) = \{(\alpha, F(\alpha), G(\neg\alpha)) \mid \alpha \in A \text{ and } F(\alpha) \cap G(\neg\alpha) = \emptyset\}.$$

The Bipolar SuperHypersoft Set, a generalization of the Bipolar Hypersoft Set, is defined as follows.

Definition 6.3 (Bipolar SuperHypersoft Set)**.** [13] Let U be a universe, and let $E_1, \ldots, E_n$ be pairwise disjoint sets of parameter values. Define

$$\mathcal{A} \;=\; \mathcal{P}(E_1) \;\times\; \cdots \times\; \mathcal{P}(E_n).$$

Let $A \subseteq \mathcal{A}$, and denote by $\neg A$ its complement in $\mathcal{A}$. A *Bipolar SuperHypersoft Set (BSHS-Set)* is a triple (F, G, A) where

$$F : A \to \mathcal{P}(U), \quad G : \neg A \to \mathcal{P}(U),$$

and for every $\alpha \in A$, the following *consistency constraint* holds:

$$F(\alpha) \;\cap\; G(\neg\alpha) \;=\; \varnothing.$$

Equivalently, we may write

$$(F, G, A) \;=\; \big\{\, (\alpha, F(\alpha), G(\neg\alpha)) \;\big|\; \alpha \in A,\; F(\alpha) \cap G(\neg\alpha) = \varnothing \big\}.$$

7. Forest SuperHypersoft Set

A Forest Hypersoft Set models multi-level attributes as trees, mapping their combinations to subsets of a universal set for decision-making.

Definition 7.1 (Forest Hypersoft Set)**.** [31] Let U be a universal set, and let $H \subseteq U$ be a non-empty subset relevant to the decision or classification context. Suppose we have a finite set of root attributes $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, with $n \geq 1$. Each attribute A_i can be expanded into a *tree* of sub-attributes:

$$A_i \;\to\; \begin{cases} A_{i1}, A_{i2}, \ldots & \text{(level 1)} \\ A_{i1k}, A_{i2\ell}, \ldots & \text{(level 2)} \\ \ldots \end{cases}$$

Collecting all such trees for $A_1, A_2, \ldots, A_n$ produces a *forest* of attributes, denoted

$$\text{Forest}(\mathcal{A}) \;=\; \{\text{Tree}(A_1),\, \text{Tree}(A_2), \ldots, \text{Tree}(A_n)\}.$$

Leaf-Level Sub-Attributes. For each tree $\text{Tree}(A_i)$, let $\Gamma\big(\text{Tree}(A_i)\big)$ be the set of all possible *leaf-level sub-attributes* stemming from A_i. A single leaf-level sub-attribute might represent a path in the tree from A_i down to one of its final sub-sub-attributes. Then define

$$\Gamma\big(\text{Forest}(\mathcal{A})\big) \;=\; \bigcup_{i=1}^{n} \Gamma\big(\text{Tree}(A_i)\big),$$

so that any element $\alpha \in \Gamma(\text{Forest}(\mathcal{A}))$ is a final-level attribute value from one of the trees in the forest. Forest Combinations. Consider the family of all possible subsets of $\Gamma(\text{Forest}(\mathcal{A}))$:

$$\mathcal{C}_{\text{forest}} \;\subseteq\; \mathcal{P}\big(\Gamma(\text{Forest}(\mathcal{A}))\big).$$

Each $\gamma \in \mathcal{C}_{\text{forest}}$ is understood as a *forest-based combination* of leaf-level sub-attributes (potentially from different root attributes).

Mapping to the Universe. A *Forest Hypersoft Set* over (U, H) is given by a pair

$$\big(G,\ \mathcal{C}_{\text{forest}}\big),$$

where $G : \mathcal{C}_{\text{forest}} \to \mathcal{P}(H)$ satisfies the condition that for each $\gamma \in \mathcal{C}_{\text{forest}}$, $G(\gamma) \subseteq H$. Concretely,

$$(G,\ \mathcal{C}_{\text{forest}})\ =\ \Big\{\big(\gamma,\ G(\gamma)\big)\ \Big|\ \gamma \in \mathcal{C}_{\text{forest}},\ G(\gamma) \subseteq H\Big\}.$$

In words, $G(\gamma)$ is the subset of H that corresponds to the collective influence or membership of all final-level sub-attributes in γ.

Definition 7.2 (Forest SuperHypersoft Set (FSHS))**.** [12] Let:

- U be a universal set, and let $H \subseteq U$ be a non-empty subset.
- $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be $n \geq 1$ root attributes, each expanding into a tree of sub-attributes, collectively forming a forest Forest($\mathcal{A}$).
- $\Gamma(\text{Forest}(\mathcal{A}))$ denote the set of all final-level sub-attributes (the leaves) across all attribute trees.
- Define the *forest super-domain* as the power set of leaf-level sub-attributes:

$$\widetilde{\mathcal{C}}_{\text{forest}}\ =\ \mathcal{P}\big(\Gamma(\text{Forest}(\mathcal{A}))\big).$$

  An element $\beta \in \widetilde{\mathcal{C}}_{\text{forest}}$ is thus a subset of final-level sub-attributes (possibly from different root attributes).

A **Forest SuperHypersoft Set** (FSHS) over (U, H) is a pair

$$\big(G,\ \widetilde{\mathcal{C}}_{\text{forest}}\big),$$

where

$$G : \widetilde{\mathcal{C}}_{\text{forest}}\ \longrightarrow\ \mathcal{P}(H)$$

maps each set of leaf-level sub-attributes $\beta \subseteq \Gamma(\text{Forest}(\mathcal{A}))$ to a subset $G(\beta) \subseteq H$. In other words,

$$\big(G,\ \widetilde{\mathcal{C}}_{\text{forest}}\big)\ =\ \Big\{\big(\beta,\ G(\beta)\big)\ \Big|\ \beta \in \widetilde{\mathcal{C}}_{\text{forest}},\ G(\beta) \subseteq H\Big\}.$$

8. IndetermSoft set and IndetermHyperSoft Set

An *IndetermSoft Set* models uncertainty by associating attribute values with subsets of a universal set where indeterminacy exists in attributes or images. And an *IndetermHyperSoft Set* generalizes IndetermSoft Sets to multiple distinct attributes, mapping combinations of indeterminate attribute values to subsets of a universal set.

$$F : A_1 \times A_2 \times \cdots \times A_n \to P(H),$$

where indeterminacy can occur in any A_i, $P(H)$, or $F(a_1, a_2, \ldots, a_n)$ [24, 34, 37, 39].

Definition 8.1 (IndetermSoft set)**.** [24, 34, 37, 39] Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the powerset of H. Let A be the set of attribute values for an attribute a. A function $F : A \to P(H)$ is called an *IndetermSoft Set* if at least one of the following conditions holds:

(1) A has some indeterminacy.

(2) $P(H)$ has some indeterminacy.

(3) There exists at least one $v \in A$ such that $F(v)$ is indeterminate (unclear, uncertain, or not unique).

(4) Any two or all three of the above conditions.

An IndetermSoft Set is represented mathematically as:

$$F : A \to H(\cap, \cup, \oplus, \neg),$$

where $H(\cap, \cup, \oplus, \neg)$ represents a structure closed under the IndetermSoft operators.

Definition 8.2 (IndetermHyperSoft Set)**.** [24, 34, 37, 39] Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the powerset of H. Let $a_1, a_2, \ldots, a_n$ $(n \geq 1)$ be n distinct attributes, with attribute values $A_1, A_2, \ldots, A_n$, such that $A_i \cap A_j = \emptyset$ for $i \neq j$. The pair $(F, A_1 \times A_2 \times \cdots \times A_n)$, where

$$F : A_1 \times A_2 \times \cdots \times A_n \to P(H),$$

is called an *IndetermHyperSoft Set* if:

(1) Any A_i or $P(H)$ exhibits indeterminacy.

(2) For $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n$, $F(a_1, a_2, \ldots, a_n)$ is indeterminate.

9. IndetermTreeSoft Set and IndetermForestSoft Set

The IndetermTreeSoft Set integrates the concepts of TreeSoft Sets and IndetermSoft Sets into a unified framework [18]. Its definition and related details are provided below.

Definition 9.1 (IndetermTreeSoft Set)**.** [18] Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the power set of H. Let Tree(A) be the hierarchical tree of attributes (as defined in the TreeSoft Set, with levels of sub-attributes) and $P(\text{Tree}(A))$ its power set.

A mapping

$$F \ : \ P(\text{Tree}(A)) \ \longrightarrow \ P(H)$$

is called an *IndetermTreeSoft Set* if one or more of the following conditions of indeterminacy hold:

(1) The tree of attributes, Tree(A), includes some indeterminate or uncertain nodes (e.g., an attribute node has unspecified or overlapping sub-attributes).

(2) The codomain $P(H)$ contains uncertain or partially unknown subsets (e.g., membership of some elements of H is not uniquely determined).

(3) For at least one $X \in P(\text{Tree}(A))$, the image $F(X) \subseteq H$ itself is indeterminate or ambiguous.

Mathematically, one may allow each value $F(X)$ to be an element of a more general structure $H(\cap, \cup, \oplus, \neg)$ under partial or fuzzy membership (if desired), provided it accommodates indeterminacy.

The IndetermForestSoft Set combines the concepts of IndetermSoft Sets and ForestSoft Sets. Its definitions and related details are provided below [18].

Definition 9.2 (IndetermForestSoft Set)**.** Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and let

$$\big\{F_t : P(\mathrm{Tree}(A^{(t)})) \to P(H)\big\}_{t\in T}$$

be a finite or countable collection of TreeSoft Sets, each possibly containing some indeterminacy as described in Definition 9.1. Let

$$\mathrm{Forest}\big(\{A^{(t)}\}\big) \;=\; \bigsqcup_{t\in T} \mathrm{Tree}\big(A^{(t)}\big)$$

be the disjoint union of all attribute trees.

A mapping

$$\mathbf{F} : P\big(\mathrm{Forest}(\{A^{(t)}\})\big) \;\longrightarrow\; P(H)$$

is called an *IndetermForestSoft Set* if for each $X \in P\big(\mathrm{Forest}(\{A^{(t)}\})\big)$,

$$\mathbf{F}(X) \;=\; \bigcup_{\substack{t\in T \\ X\cap \mathrm{Tree}(A^{(t)})\neq \varnothing}} F_t\big(X \cap \mathrm{Tree}(A^{(t)})\big),$$

and at least one of the following holds:

(1) There is indeterminacy in some $\mathrm{Tree}(A^{(t)})$ or in the way these trees are combined.

(2) The codomain $P(H)$ has elements that are uncertain or ambiguous.

(3) For some input X, the image $\mathbf{F}(X)$ is not uniquely defined or is subject to partial information.

In other words, an IndetermForestSoft Set unites multiple IndetermTreeSoft Sets (or TreeSoft Sets, some or all of which may have indeterminacy) into a single framework while preserving the possibility of ambiguity.

10. **IndetermSuperHyperSoft Set**

The IndetermSuperHyperSoft Set combines the concepts of SuperHyperSoft Sets and IndetermHyperSoft Sets. Its definition is provided below [18].

Definition 10.1 (IndetermSuperHyperSoft Set)**.** Let U be a universal set, and let $\{a_1, a_2, \ldots, a_n\}$ be n distinct attributes with corresponding sets of values A_i, each of which may exhibit partial or total indeterminacy. Define

$$\mathcal{C} \;=\; \mathcal{P}(A_1) \;\times\; \mathcal{P}(A_2) \;\times\; \cdots \;\times\; \mathcal{P}(A_n),$$

which forms the domain of a *SuperHyperSoft Set* (Definition [36]).

A mapping

$$F : \mathcal{C} \;\longrightarrow\; \mathcal{P}(U)$$

is called an *IndetermSuperHyperSoft Set* if one or more of the following indeterminacy conditions hold:

(1) Some $\mathcal{P}(A_i)$ is uncertain or incompletely specified (i.e., the attribute values or their subsets are not fully determined).

(2) The codomain $\mathcal{P}(U)$ contains partially unknown subsets or ambiguous memberships.

(3) For at least one $\gamma \in \mathcal{C}$, the set $F(\gamma) \subseteq U$ is not uniquely defined or is subject to ambiguity.

Additionally, higher-level nesting (if each A_i itself is replaced by $\tilde{\mathcal{P}}(A_i)$ for extended hyperstructures) may introduce further levels of indeterminacy, consistent with the definition of IndetermHyperSoft Sets.

11. **Incomplete HyperSoft Set and Incomplete SuperHyperSoft Set**

The incomplete soft set is known as a special case of the IndetermSoft set, and the following definition of the incomplete soft set has been introduced [23].

Definition 11.1. [23] An *incomplete soft set* is defined as a tuple:

$$S = (U, E, f),$$

where:

- U is a non-empty set of objects, referred to as the universe.
- E is a non-empty set of parameters.
- $f : E \to \mathcal{P}(U) \cup \{*\}$, a mapping from the parameter set E to the power set of U ($\mathcal{P}(U)$) or the special symbol $*$.

Here, $*$ indicates missing or undefined information for a given parameter $e \in E$. If $f(e) = *$, the information associated with the parameter e is incomplete. Otherwise, $f(e) \subseteq U$ as in a standard soft set.

The Incomplete HyperSoft Set is a generalized concept derived from the Incomplete Soft Set. Its formal definition is provided below.

Definition 11.2 (Incomplete HyperSoft Set)**.** [18] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be m (possibly indeterminate) attribute domains. Define the Cartesian product

$$\mathcal{C} \;=\; \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m.$$

An *Incomplete HyperSoft Set* over U is defined as a tuple

$$\Big(U,\ \mathcal{A}_1,\ \mathcal{A}_2,\ \ldots,\ \mathcal{A}_m,\ G\Big),$$

where $G : \mathcal{C} \to \big(\mathcal{P}(U) \cup \{*\}\big)$. Concretely,

$$G(\gamma) \;=\; \begin{cases} S_\gamma \subseteq U, & \text{if sufficient information is available,} \\ *, & \text{if the information is missing or undefined,} \end{cases}$$

for each $\gamma \in \mathcal{C}$. Here, $*$ denotes incomplete or missing data in analogy with the incomplete soft set.

In this definition:

(1) U is a non-empty universe of discourse.

(2) $\mathcal{A}_i$ are attribute domains, each of which may be partially incomplete or fully defined.

(3) $\mathcal{C} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$ represents all possible combinations of attribute values.

(4) $G(\gamma)$ lies in $\mathcal{P}(U) \cup \{*\}$. If $G(\gamma) = *$, the mapping is incomplete for the given γ.

The Incomplete SuperHyperSoft Set is a generalized concept derived from the Incomplete HyperSoft Set. Its formal definition is provided below.

Definition 11.3 (Incomplete SuperHyperSoft Set)**.** [18] Let U be a universal set, and let $\{a_1, a_2, \ldots, a_n\}$ be $n \geq 1$ distinct attributes, each with a set of possible values A_i. Define

$$\mathcal{C} \;=\; \mathcal{P}(A_1) \;\times\; \mathcal{P}(A_2) \;\times\; \cdots \;\times\; \mathcal{P}(A_n),$$

as in the definition of a *SuperHyperSoft Set*. An *Incomplete SuperHyperSoft Set* is a tuple

$$\Big(U,\; \{A_i\}_{i=1}^{n},\; F\Big),$$

where

$$F : \mathcal{C} \;\longrightarrow\; \big(\mathcal{P}(U) \cup \{*\}\big).$$

Specifically, for each $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \subseteq A_i$,

$$F(\gamma) \;=\; \begin{cases} S_\gamma \subseteq U, & \text{if the information is complete,} \\ *, & \text{if the information is incomplete or undefined.} \end{cases}$$

Here, $*$ indicates a missing or indeterminate value at the superhyper level (e.g., incomplete data about chosen subsets α_i or how they map into U).

12. New Concept: GraphicSoft Set

In this section, we introduce a new type of Soft Set. A GraphicSoft Set extends the concept of Soft Sets by incorporating a graph structure to represent relationships between attributes, mapping subsets of the graph to subsets of a universal set.

Definition 12.1 (Graph and Its Power Set)**.** (cf. [11]) Let $G = (V, E)$ be a finite graph, where V is the set of vertices (each representing an attribute) and $E \subseteq V \times V$ is the set of edges (representing relationships among these attributes). Define a *subgraph* H of G as a graph $H = (V_H, E_H)$ with $V_H \subseteq V$ and $E_H \subseteq E \cap (V_H \times V_H)$. The *power set of* G, denoted $\mathcal{P}(G)$, is defined as the collection of all subgraphs of G:

$$\mathcal{P}(G) = \{\, H \mid H \text{ is a subgraph of } G \,\}.$$

Definition 12.2 (GraphicSoft Set)**.** Let U be a universe of discourse, and let $G = (V, E)$ be a graph representing a set of attributes and their relationships. A *GraphicSoft Set* is defined as a mapping

$$F : \mathcal{P}(G) \to \mathcal{P}(U),$$

which assigns to each subgraph $H \in \mathcal{P}(G)$ a subset $F(H) \subseteq U$. Intuitively, $F(H)$ represents the set of objects in U that possess the combined attributes described by the subgraph H.

Example 12.3 (Attributes of Individuals)**.** Let

$$U = \{\text{Alice}, \text{Bob}, \text{Charlie}, \text{Diana}\}$$

be a set of individuals. Define a graph

$$G = (V, E)$$

with

$$V = \{\text{Smart}, \text{Friendly}, \text{Athletic}\}$$

and let the edges in E indicate potential relationships (e.g., "Smart" might be related to "Friendly"). Define a GraphicSoft Set $F : \mathcal{P}(G) \to \mathcal{P}(U)$ as follows:

- For the subgraph H_1 containing only the vertex $\{\text{Smart}\}$, set

$$F(H_1) = \{\text{Alice}, \text{Charlie}\}.$$

- For the subgraph H_2 with only $\{\text{Friendly}\}$, set

$$F(H_2) = \{\text{Bob}, \text{Diana}\}.$$

- For the subgraph H_3 with vertices $\{\text{Smart}, \text{Friendly}\}$ (and the edge connecting them), define

$$F(H_3) = F(\{\text{Smart}\}) \cap F(\{\text{Friendly}\}) = \{\text{Alice}\}.$$

- For the entire graph G itself, let

$$F(G) = \{\text{Alice}, \text{Bob}\},$$

  indicating that only Alice and Bob exhibit all the attributes in G.

Example 12.4 (Houses and Their Features)**.** Let

$$U = \{\text{House1}, \text{House2}, \text{House3}, \text{House4}\}$$

be a set of houses. Define a graph

$$G = (V, E)$$

with

$$V = \{\text{Red}, \text{Big}, \text{Modern}\}.$$

Assume the edges in E express compatibility (e.g., "Red" might be frequently seen with "Modern"). Define a GraphicSoft Set F as follows:

- For the subgraph H_1 with vertex $\{\text{Red}\}$, let

$$F(H_1) = \{\text{House1}, \text{House3}\}.$$

- For the subgraph H_2 with vertex $\{\text{Big}\}$, let

$$F(H_2) = \{\text{House2}, \text{House3}\}.$$

- For the subgraph H_3 with vertices $\{\text{Modern}\}$, let

$$F(H_3) = \{\text{House3}, \text{House4}\}.$$

- For the subgraph H_4 with vertices $\{\text{Red}, \text{Big}\}$, define

$$F(H_4) = F(\{\text{Red}\}) \cap F(\{\text{Big}\}) = \{\text{House3}\}.$$

- For the full graph G, let

$$F(G) = \{\text{House3}\},$$

meaning that only House3 exhibits all three features.

Theorem 12.5 (Monotonicity of GraphicSoft Sets)**.** *Assume $F : \mathcal{P}(G) \to \mathcal{P}(U)$ is defined such that for any subgraph $H \in \mathcal{P}(G)$,*

$$F(H) = \bigcap_{v \in V(H)} F(\{v\}),$$

where $F(\{v\})$ is the set of objects in U possessing attribute v. Then, if H_1 and H_2 are subgraphs of G with $H_1 \subseteq H_2$ (i.e., $V(H_1) \subseteq V(H_2)$), we have

$$F(H_2) \subseteq F(H_1).$$

Proof. Since $H_1 \subseteq H_2$, we have $V(H_1) \subseteq V(H_2)$. Then,

$$F(H_2) = \bigcap_{v \in V(H_2)} F(\{v\}) \quad \text{and} \quad F(H_1) = \bigcap_{v \in V(H_1)} F(\{v\}).$$

Because intersecting over a larger set of indices yields a smaller (or equal) set, it follows that

$$\bigcap_{v \in V(H_2)} F(\{v\}) \subseteq \bigcap_{v \in V(H_1)} F(\{v\}),$$

which implies $F(H_2) \subseteq F(H_1)$. □

13. New Concept: ClusterSoft Set and Cluster-HyperSoft Set

In this section, we introduce a new type of Soft Set. A ClusterSoft Set groups multiple Soft Sets, capturing relationships among clustered attributes and mapping them to subsets of a universal set for decision modeling.

Definition 13.1 (ClusterSoft Set)**.** Let $\{F_i\}_{i \in I}$ be a finite family of soft sets over a universe U, where each soft set F_i is a mapping

$$F_i : A_i \to \mathcal{P}(U)$$

for some set of attributes A_i. Suppose the index set I is partitioned into clusters $\{C_j\}_{j \in J}$ with each $C_j \subseteq I$ and $C_j \cap C_k = \emptyset$ for $j \neq k$. A *ClusterSoft Set* is defined as a mapping

$$G : \{C_j : j \in J\} \to \mathcal{P}(U)$$

given by

$$G(C_j) = \bigcup_{i \in C_j} F_i^*(A_i),$$

where $F_i^*(A_i)$ denotes the set of objects in U associated with soft set F_i (possibly after appropriate aggregation or normalization). The union is taken in the usual set-theoretic sense.

Example 13.2 (Houses Grouped by Features)**.** Let

$$U = \{\text{House1}, \text{House2}, \text{House3}, \text{House4}, \text{House5}\}$$

be a set of houses. Suppose we have two soft sets:

- $F_1 : \{\text{Color}\} \to \mathcal{P}(U)$ with $F_1(\text{Red}) = \{\text{House1}, \text{House2}\}$.
- $F_2 : \{\text{Size}\} \to \mathcal{P}(U)$ with $F_2(\text{Large}) = \{\text{House2}, \text{House3}, \text{House4}\}$.

Let the index set be $I = \{1, 2\}$ and define a single cluster $C_1 = \{1, 2\}$. Then the ClusterSoft Set G is given by

$$G(C_1) = F_1(\text{Red}) \cup F_2(\text{Large}) = \{\text{House1}, \text{House2}\} \cup \{\text{House2}, \text{House3}, \text{House4}\} = \{\text{House1}, \text{House2}, \text{House3}, \text{House4}\}.$$

Example 13.3 (Students Grouped by Skill Sets)**.** Let

$$U = \{\text{Student1}, \text{Student2}, \text{Student3}, \text{Student4}\}$$

be a set of students. Define two soft sets:

- $F_1 : \{\text{Mathematics}\} \to \mathcal{P}(U)$ with $F_1(\text{Strong}) = \{\text{Student1}, \text{Student3}\}$.
- $F_2 : \{\text{Language}\} \to \mathcal{P}(U)$ with $F_2(\text{Fluent}) = \{\text{Student2}, \text{Student3}, \text{Student4}\}$.

Let the index set be $I = \{1, 2\}$ and form a single cluster $C_1 = \{1, 2\}$. Then, the ClusterSoft Set G is defined as

$$G(C_1) = F_1(\text{Strong}) \cup F_2(\text{Fluent}) = \{\text{Student1}, \text{Student3}\}$$

$$\cup \{\text{Student2}, \text{Student3}, \text{Student4}\} = \{\text{Student1}, \text{Student2}, \text{Student3}, \text{Student4}\}.$$

Theorem 13.4 (Disjoint Union Property)**.** *Let $\{F_i\}_{i \in I}$ be a family of soft sets over U and let $C_1, C_2 \subseteq I$ be two disjoint clusters (i.e., $C_1 \cap C_2 = \emptyset$). Then, the ClusterSoft Set satisfies*

$$G(C_1 \cup C_2) = G(C_1) \cup G(C_2).$$

Proof. By definition,

$$G(C_1 \cup C_2) = \bigcup_{i \in C_1 \cup C_2} F_i^*(A_i).$$

Since C_1 and C_2 are disjoint, the union can be split as

$$\bigcup_{i \in C_1 \cup C_2} F_i^*(A_i) = \left(\bigcup_{i \in C_1} F_i^*(A_i) \right) \cup \left(\bigcup_{i \in C_2} F_i^*(A_i) \right).$$

This equals $G(C_1) \cup G(C_2)$, proving the assertion. □

Theorem 13.5 (Non-emptiness)**.** *Assume that for every $i \in I$, the soft set F_i is non-empty (i.e., $F_i(A_i) \neq \emptyset$). Then for any non-empty cluster $C \subseteq I$, the ClusterSoft Set $G(C)$ is non-empty.*

Proof. Since each $F_i(A_i)$ is non-empty for $i \in C$ and $C \neq \emptyset$, the union

$$G(C) = \bigcup_{i \in C} F_i^*(A_i)$$

is a union of non-empty sets. Therefore, $G(C) \neq \emptyset$. □

Definition 13.6 (Cluster-HyperSoft Set)**.** Let U be a universal set. Suppose that for each $i \in I$, there is a HyperSoft Set

$$G_i : \mathcal{C}_i \to \mathcal{P}(U),$$

where $\mathcal{C}_i \subseteq A_{i1} \times A_{i2} \times \cdots \times A_{im_i}$ is the Cartesian product of attribute domains corresponding to the i-th evaluation. Assume that the index set I is partitioned into disjoint clusters $\{C_k\}_{k \in K}$ (i.e., $C_k \subseteq I$ and $C_k \cap C_{k'} = \emptyset$ for $k \neq k'$).

Then, the *Cluster-HyperSoft Set* is defined as a mapping

$$H : \{C_k \mid k \in K\} \to \mathcal{P}(U)$$

given by

$$H(C_k) = \bigcup_{i \in C_k} G_i(\gamma_i),$$

where for each $i \in C_k$, a specific element $\gamma_i \in \mathcal{C}_i$ is chosen (or determined by an aggregation operator). In other words, the Cluster-HyperSoft Set aggregates the hypersoft evaluations of the soft sets within each cluster.

Example 13.7 (Cluster-HyperSoft Set: Product Quality Evaluation)**.** Let

$$U = \{\text{Product1, Product2, Product3, Product4}\}$$

be a set of products. Assume we have two hyper soft sets:

- $G_1 : \mathcal{C}_1 \to \mathcal{P}(U)$ where $\mathcal{C}_1 = \{(\text{High}, \text{Low}), (\text{High}, \text{Medium})\}$ represents quality and cost parameters. Let
$$G_1(\text{High}, \text{Low}) = \{\text{Product1}, \text{Product2}\}.$$
- $G_2 : \mathcal{C}_2 \to \mathcal{P}(U)$ where $\mathcal{C}_2 = \{(\text{Medium}, \text{High}), (\text{Medium}, \text{Medium})\}$. Let
$$G_2(\text{Medium}, \text{High}) = \{\text{Product3}\}.$$

Let the index set be $I = \{1, 2\}$ and form a single cluster $C_1 = \{1, 2\}$. Then, by choosing $\gamma_1 = (\text{High}, \text{Low})$ for G_1 and $\gamma_2 = (\text{Medium}, \text{High})$ for G_2, we define the Cluster-HyperSoft Set as:

$$H(C_1) = G_1(\text{High}, \text{Low}) \cup G_2(\text{Medium}, \text{High}) = \{\text{Product1}, \text{Product2}\} \cup \{\text{Product3}\} = \{\text{Product1}, \text{Product2}, \text{Product3}\}.$$

Theorem 13.8 (Monotonicity of Cluster-HyperSoft Set Aggregation)**.** *Let C_1 and C_2 be clusters with $C_1 \subseteq C_2 \subseteq I$. Then,*

$$H(C_1) \subseteq H(C_2).$$

Proof. Since $C_1 \subseteq C_2$, every index $i \in C_1$ is also contained in C_2. Therefore, the union over C_1 is a subset of the union over C_2:

$$H(C_1) = \bigcup_{i \in C_1} G_i(\gamma_i) \subseteq \bigcup_{i \in C_2} G_i(\gamma_i) = H(C_2).$$

Thus, the monotonicity property holds. □□

14. New Concept: CycleSoft Set and Cycle-Hypersoft set

In this section, we introduce a new type of Soft Set. A CycleSoft Set extends Soft Sets by organizing parameters in a cycle graph, mapping cycle subgraphs to subsets of a universal set for structured decision-making.

Definition 14.1 (CycleSoft Set)**.** Let U be a universal set and let $C = (A, E_C)$ be a cycle graph, where A is a set of parameters arranged in a cycle and

$$E_C = \{(a_i, a_{i+1}) \mid a_i, a_{i+1} \in A\} \cup \{(a_n, a_1)\}$$

describes the cyclic adjacency among the parameters. Define the power set of C as

$$\mathcal{P}(C) = \{H \mid H \text{ is a subgraph of } C\}.$$

A *CycleSoft Set* is a mapping

$$F : \mathcal{P}(C) \to \mathcal{P}(U),$$

where for each subgraph $H \in \mathcal{P}(C)$, $F(H) \subseteq U$ represents the set of objects associated with the combination of parameters corresponding to H. A common aggregation is to define, for each H,

$$F(H)(x) = \bigcap_{a \in V(H)} f(a)(x),$$

with $f(a) : U \to [0, 1]$ (or characteristic functions in the crisp case).

Example 14.2 (CycleSoft Set: Social Event Preferences)**.** Let

$$U = \{\text{Event1, Event2, Event3, Event4}\}$$

be a set of social events. Consider parameters arranged in a cycle:

$$A = \{\text{Music, Food, Dancing, Art}\},$$

with the cycle graph

$$C = \big(A,\ \{(\text{Music}, \text{Food}),\ (\text{Food}, \text{Dancing}),\ (\text{Dancing}, \text{Art}),\ (\text{Art}, \text{Music})\}\big).$$

Define the basic assignments:

$$F(\{\text{Music}\}) = \{\text{Event1}, \text{Event2}\}, \quad F(\{\text{Food}\}) = \{\text{Event2}, \text{Event3}\},$$

$$F(\{\text{Dancing}\}) = \{\text{Event3}, \text{Event4}\}, \quad F(\{\text{Art}\}) = \{\text{Event1}, \text{Event4}\}.$$

For the subgraph $H = \{\text{Music}, \text{Food}\}$ (i.e., two adjacent parameters), we define

$$F(H) = F(\{\text{Music}\}) \cap F(\{\text{Food}\}) = \{\text{Event2}\}.$$

Thus, the CycleSoft Set captures the joint effect of consecutive attributes arranged in a cyclic order.

Theorem 14.3 (Monotonicity of CycleSoft Set)**.** *Let $H_1, H_2 \in \mathcal{P}(C)$ be subgraphs with $H_1 \subseteq H_2$ (i.e., $V(H_1) \subseteq V(H_2)$). If F is defined via the intersection operator as*

$$F(H)(x) = \bigcap_{a \in V(H)} f(a)(x),$$

then for all $x \in U$,

$$F(H_2)(x) \subseteq F(H_1)(x).$$

Proof. Since $H_1 \subseteq H_2$, every parameter in H_1 is contained in H_2. Hence,

$$F(H_1)(x) = \bigcap_{a \in V(H_1)} f(a)(x) \quad \text{and} \quad F(H_2)(x) = \bigcap_{a \in V(H_2)} f(a)(x).$$

Since the intersection over a larger set of indices yields a subset (or equal set) of the intersection over a smaller set, we have

$$F(H_2)(x) \subseteq F(H_1)(x),$$

for all $x \in U$. □

Definition 14.4 (Cycle-Hypersoft Set)**.** Let U be a universal set, and let $C = (A, E_C)$ be a cycle graph where $A = \{a_1, a_2, \ldots, a_n\}$ is a set of parameters arranged in a cyclic order and

$$E_C = \{(a_i, a_{i+1}) \mid 1 \leq i < n\} \cup \{(a_n, a_1)\}$$

describes the cyclic adjacency among the parameters. Define the hypersoft structure on C by considering the Cartesian product of the power sets of attribute values for each a_i; that is, let

$$\mathcal{H}(C) = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n),$$

where for each a_i, A_i is the set of possible attribute values. A *Cycle-Hypersoft Set* is a mapping

$$F : \mathcal{H}(C) \to \mathcal{P}(U),$$

which assigns to each n-tuple $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ (with $\alpha_i \in \mathcal{P}(A_i)$) a subset $F(\gamma) \subseteq U$. This mapping aggregates the information from all cyclically arranged attributes in a higher-dimensional manner.

Example 14.5 (Cycle-Hypersoft Set: Social Media Preferences)**.** Let

$$U = \{\text{User1}, \text{User2}, \text{User3}, \text{User4}\}$$

represent a set of social media users. Suppose the parameters are arranged in a cycle:

$$A = \{\text{Music}, \text{Movies}, \text{Sports}, \text{Art}\},$$

with corresponding attribute value sets:

$$A_{\text{Music}} = \{\text{Rock}, \text{Pop}\}, \quad A_{\text{Movies}} = \{\text{Action}, \text{Drama}\},$$

$$A_{\text{Sports}} = \{\text{Football}, \text{Basketball}\}, \quad A_{\text{Art}} = \{\text{Modern}, \text{Classic}\}.$$

Then, the domain of the Cycle-Hypersoft Set is

$$\mathcal{H}(C) = \mathcal{P}(A_{\text{Music}}) \times \mathcal{P}(A_{\text{Movies}}) \times \mathcal{P}(A_{\text{Sports}}) \times \mathcal{P}(A_{\text{Art}}).$$

For instance, one may define:

$$F\Big(\{\text{Rock}\}, \{\text{Action}\}, \{\text{Football}\}, \{\text{Modern}\}\Big) = \{\text{User1}, \text{User3}\},$$

and

$$F\Big(\{\text{Pop}\}, \{\text{Drama}\}, \{\text{Basketball}\}, \{\text{Classic}\}\Big) = \{\text{User2}, \text{User4}\}.$$

Thus, the Cycle-Hypersoft Set models the joint effect of cyclically arranged attributes on user preferences.

Theorem 14.6 (Monotonicity in Cycle-Hypersoft Set)**.** *Let* $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *and* $\gamma' = (\alpha_1', \alpha_2', \ldots, \alpha_n')$ *be two elements of* $\mathcal{H}(C)$ *such that for each* i, $\alpha_i \subseteq \alpha_i'$*. Then,*

$$F(\gamma) \subseteq F(\gamma'),$$

provided that the aggregation operator used in F *is monotonic with respect to set inclusion.*

Proof. Since for each i we have $\alpha_i \subseteq \alpha_i'$, the combination of attribute values in γ is a subset of that in γ'. Assuming that the aggregation operator in F (for example, a union or intersection operator) is monotonic with respect to inclusion, it follows that

$$F(\gamma) \subseteq F(\gamma'),$$

for all $x \in U$. □□

15. **New Concept: Bipartite-Soft Set (Bisoft Set)**

In this section, we introduce a new type of Soft Set. A Bipartite-Soft Set (Bisoft Set) extends Soft Sets by structuring attributes in a bipartite graph, mapping subsets of partitions to subsets of a universal set.

Definition 15.1 (Bipartite-Soft Set (Bisoft Set))**.** Let U be a universal set and let X and Y be two disjoint sets of parameters (i.e., $X \cap Y = \emptyset$). Consider the bipartite graph $B = (X \cup Y, E_B)$ where $E_B \subseteq X \times Y$ represents the interrelationships between parameters in X and those in Y. Define the Cartesian product of the power sets as

$$\mathcal{C} = \mathcal{P}(X) \times \mathcal{P}(Y).$$

A *Bipartite-Soft Set* (or *Bisoft Set*) is defined as a mapping

$$F : \mathcal{C} \to \mathcal{P}(U),$$

such that for each pair $(A, B) \in \mathcal{P}(X) \times \mathcal{P}(Y)$, $F(A, B) \subseteq U$ represents the set of objects associated with the combination of attributes in A (from the first partition) and B (from the second partition).

Example 15.2 (Bipartite-Soft Set: Employee Evaluation)**.** Let

$$U = \{\text{Employee1}, \text{Employee2}, \text{Employee3}, \text{Employee4}\}$$

be a set of employees. Suppose the evaluation criteria are divided into two disjoint groups:

$$X = \{\text{Technical Skill}, \text{Experience}\} \quad \text{and} \quad Y = \{\text{Communication}, \text{Teamwork}\}.$$

Define a Bipartite-Soft Set $F : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(U)$ by:

- $F(\{\text{Technical Skill}\}, \{\text{Communication}\}) = \{\text{Employee1}, \text{Employee3}\}$.
- $F(\{\text{Experience}\}, \{\text{Teamwork}\}) = \{\text{Employee2}, \text{Employee4}\}$.
- $F(\{\text{Technical Skill}, \text{Experience}\}, \{\text{Communication}, \text{Teamwork}\}) = \{\text{Employee3}\}$.

Thus, the Bipartite-Soft Set aggregates information from the two groups of attributes to provide an evaluation of employees.

Example 15.3 (Bipartite-Soft Set: Product Analysis)**.** Let

$$U = \{\text{Product1}, \text{Product2}, \text{Product3}\}$$

be a set of products. Suppose the attributes are divided into two categories:

$$X = \{\text{Design}, \text{Durability}\} \quad \text{and} \quad Y = \{\text{Price}, \text{Market Appeal}\}.$$

Define the Bipartite-Soft Set $F : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(U)$ as follows:

- $F(\{\text{Design}\}, \{\text{Price}\}) = \{\text{Product1}, \text{Product2}\}$.
- $F(\{\text{Durability}\}, \{\text{Market Appeal}\}) = \{\text{Product2}, \text{Product3}\}$.
- $F(\{\text{Design}, \text{Durability}\}, \{\text{Price}, \text{Market Appeal}\}) = \{\text{Product2}\}$.

This example demonstrates how the Bipartite-Soft Set can be used to integrate and analyze product attributes from two different domains.

Theorem 15.4 (Union Property for Bipartite-Soft Sets)**.** *Let (A_1, B_1) and (A_2, B_2) be two pairs of subsets where $A_1, A_2 \subseteq X$ and $B_1, B_2 \subseteq Y$. Suppose that the Bipartite-Soft Set F is defined such that for any $(A, B) \in \mathcal{P}(X) \times \mathcal{P}(Y)$,*

$$F(A, B) = F(A_1, B_1) \cup F(A_2, B_2),$$

whenever $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Then,

$$F(A, B) = F(A_1, B_1) \cup F(A_2, B_2).$$

Proof. By the definition of the Bipartite-Soft Set, if $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, then

$$F(A, B) = \{x \in U \mid x \text{ is associated with either } (A_1, B_1) \text{ or } (A_2, B_2)\}.$$

Thus,

$$F(A, B) = F(A_1, B_1) \cup F(A_2, B_2),$$

which completes the proof. □

Theorem 15.5 (Monotonicity of Bipartite-Soft Set)**.** *Let $A_1 \subseteq A_2 \subseteq X$ and $B_1 \subseteq B_2 \subseteq Y$. Then, for every $x \in U$,*

$$F(A_1, B_1)(x) \subseteq F(A_2, B_2)(x),$$

assuming that the mapping F is monotonic with respect to set inclusion.

Proof. Since $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$, the set of parameters considered in (A_1, B_1) is a subset of those in (A_2, B_2). Thus, the set of objects associated with (A_1, B_1) is contained in the set of objects associated with (A_2, B_2), i.e.,

$$F(A_1, B_1) \subseteq F(A_2, B_2).$$

This proves the monotonicity property. □

16. **New Concept: Bipartite-HyperSoft Set and Bipartite-SuperhyperSoft Set**

In this section, we introduce a new type of Soft Set. The Bipartite-HyperSoft Set and Bipartite-SuperhyperSoft Set are extensions of the Bipartite-Soft Set. Their definitions and related properties are provided below.

Definition 16.1 (Bipartite-HyperSoft Set)**.** Let U be a universal set. Let X and Y be two disjoint sets of parameters, and assume that each parameter in X is associated with a set of attribute values A and each parameter in Y is associated with a set of attribute values B. Denote by $\mathcal{P}(A)$ and $\mathcal{P}(B)$ the power sets of A and B, respectively. A *Bipartite-HyperSoft Set* is a mapping

$$F : \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(U),$$

which assigns to each pair (S, T) with $S \subseteq A$ and $T \subseteq B$ a subset $F(S, T) \subseteq U$. This mapping captures the combined influence of attribute values from both disjoint groups.

Example 16.2 (Bipartite-HyperSoft Set: Product Evaluation)**.** Let

$$U = \{\text{Product1, Product2, Product3, Product4}\}.$$

Assume we evaluate products based on two criteria:

- Design attributes: Let $A = \{\text{Modern, Classic}\}$.
- Performance attributes: Let $B = \{\text{Efficient, Robust}\}$.

Thus, the domain of the mapping is $\mathcal{P}(A) \times \mathcal{P}(B)$. Define $F : \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(U)$ as follows:

- $F(\{\text{Modern}\}, \{\text{Efficient}\}) = \{\text{Product1}, \text{Product2}\}$.
- $F(\{\text{Classic}\}, \{\text{Robust}\}) = \{\text{Product3}\}$.
- $F(\{\text{Modern}, \text{Classic}\}, \{\text{Efficient}, \text{Robust}\}) = U$.

This Bipartite-HyperSoft Set captures the joint effects of design and performance attributes on the set of products.

Theorem 16.3 (Monotonicity of Bipartite-HyperSoft Set)**.** *Let (S_1, T_1) and (S_2, T_2) be two pairs in $\mathcal{P}(A) \times \mathcal{P}(B)$ with $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$. Then, for all $x \in U$,*

$$F(S_1, T_1)(x) \subseteq F(S_2, T_2)(x).$$

Proof. Since $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$, the combination of attribute values in (S_1, T_1) is a subset of that in (S_2, T_2). Under standard aggregation, the set $F(S_2, T_2)(x)$ includes all objects corresponding to the larger collection of attributes, hence

$$F(S_1, T_1)(x) \subseteq F(S_2, T_2)(x),$$

for all $x \in U$. □

Definition 16.4 (Bipartite-SuperhyperSoft Set)**.** Let U be a universal set. Let A and B be two disjoint sets of attribute values corresponding to two disjoint parameter groups. Consider the power sets $\mathcal{P}(A)$ and $\mathcal{P}(B)$. Then, define the double power sets $\mathcal{P}(\mathcal{P}(A))$ and $\mathcal{P}(\mathcal{P}(B))$. A *Bipartite-SuperhyperSoft Set* is a mapping

$$F : \mathcal{P}(\mathcal{P}(A)) \times \mathcal{P}(\mathcal{P}(B)) \to \mathcal{P}(U),$$

which assigns to each pair (S,T) with $S \subseteq \mathcal{P}(A)$ and $T \subseteq \mathcal{P}(B)$ a subset $F(S,T) \subseteq U$. This higher-order aggregation allows for modeling more complex interrelationships by considering sets of attribute value subsets.

Example 16.5 (Bipartite-SuperhyperSoft Set: Re-Conscientization in Education)**.** (cf. [7]) Let

$$U = \{\text{Program1, Program2, Program3, Program4}\},$$

be a set of educational programs offered by a university. In order to evaluate these programs in a manner that promotes re-conscientization—that is, a process of re-evaluating established educational paradigms and fostering both critical inquiry and ethical reflection—we introduce two disjoint sets of attribute values:

- $A = \{\text{Critical Inquiry, Philosophical Rigor}\}$, representing the intellectual and reflective dimensions.
- $B = \{\text{Ethical Reflection, Social Engagement}\}$, representing the ethical and societal dimensions.

To capture complex, higher-order relationships among these attributes, we consider the double power sets $\mathcal{P}(\mathcal{P}(A))$ and $\mathcal{P}(\mathcal{P}(B))$. We then define the mapping

$$F : \mathcal{P}(\mathcal{P}(A)) \times \mathcal{P}(\mathcal{P}(B)) \to \mathcal{P}(U)$$

as follows:

- $F\Big(\{\{\text{Critical Inquiry}\}\}, \{\{\text{Ethical Reflection}\}\}\Big) = \{\text{Program1}\}$.
  This indicates that Program1 is evaluated highly for its ability to promote both critical inquiry and ethical reflection, essential components of re-conscientization.
- $F\Big(\{\{\text{Philosophical Rigor}\}\}, \{\{\text{Social Engagement}\}\}\Big) = \{\text{Program2}\}$.
  Here, Program2 is recognized for its strong theoretical foundations paired with active social engagement, thereby challenging conventional educational norms.
- $F\Big(\{\{\text{Critical Inquiry, Philosophical Rigor}\}\}, \{\{\text{Ethical Reflection, Social Engagement}\}\}\Big) = \{\text{Program3, Program4}\}$.
  This captures programs that integrate both layers of intellectual depth and comprehensive ethical-social considerations, embodying a holistic approach to re-conscientization.

Thus, the Bipartite-SuperhyperSoft Set framework here models a multi-layered aggregation of attributes that reflect not only academic performance but also the transformative potential of education through re-conscientization.

Theorem 16.6 (Consistency Under Aggregation for Bipartite-SuperhyperSoft Set)**.** *Let* (S_1,T_1) *and* (S_2,T_2) *be elements of* $\mathcal{P}(\mathcal{P}(A)) \times \mathcal{P}(\mathcal{P}(B))$ *with* $S_1 \subseteq S_2$ *and* $T_1 \subseteq T_2$*. Then,*

$$F(S_1,T_1) \subseteq F(S_2,T_2).$$

Proof. Since $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$, the set of attribute value subsets considered in (S_1, T_1) is a subset of that in (S_2, T_2). Assuming the aggregation function F is monotonic with respect to set inclusion, it follows that

$$F(S_1, T_1) \subseteq F(S_2, T_2).$$

Thus, the aggregation is consistent under the inclusion of additional attribute value subsets. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Complex Neutrosophic Approach for Uncertainty Management in Employment Competence for Higher Vocational College Students

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Abstract: In an era of rapid technological revolutions as well as ever-changing job market dynamics, the analysis of employment competence for higher vocational college students becomes a complicated challenge. Legacy methods become unable to manage the intrinsic uncertainty related to workforce readiness, skill relevancies, and career decision-making. In this work, we propose a multi-criteria complex neutrosophic approach integrating for optimization and uncertainty management of vocational student employment. Our approach provides a novel application of the Interval Complex Neutrosophic Set (ICNS) to handle indefinite, hesitant, and inconsistent information in vocational student employment. It also presents a novel ICNS de-neutrosophication method to provide representative and expressive scoring of ICNS decisions. Then, algorithmic analytical weighting is applied to infer the significance of ICNS-valued criteria. Following, the ICNS-VIKOR method is integrated to compute evaluation results and rank vocational employment strategies according to either compromise Measure, Regret Measure, or Q-VIKOR Index scores for each alternative. We introduce a real-world case study to optimize the evaluation of different vocational employment configurations based on ten competing criteria. The quantitative and qualitative results indicate that our methodology offers a systematic and reliable decision-support tool, that effectively optimizes trade-offs as well as uncertain judgments from policymakers, career advisors, and industry recruiters. The proof-of-concept analysis highlighted the best-ranked alternative while offering intuitions about the sensitivity of the decision process.

Keywords: Complex Neutrosophic, Interval Neutrosophic, Uncertainty, Employment Competence, Higher Vocational College Students.

1. Introduction

In today's dynamic labor market, the employment landscape for higher vocational college students is undergoing significant transformation [1]. The rise of automation, digitalization, and evolving industry demands have intensified the need for job-ready skills among graduates. Apart from common academic pathways, vocational education focuses on practical skill development and industry-specific training [2], [3]. Nevertheless, evaluating employment competence in vocational students turned out to be a complex challenge due to varied uncertainties in skill assessment, job opportunities, as well as career decision-making [4], [5].

Employment competence encompasses a broad spectrum of factors, including technical skills, soft skills, adaptability, and decision-making abilities [6]. However, these attributes are often difficult to quantify and predict using conventional evaluation techniques. Traditional competency assessment models rely on fixed criteria, rigid evaluation metrics, and deterministic decision-making, which fail to account for the uncertainty and subjectivity inherent in real-world employment scenarios [3].

Employment competence encounters many uncertainty challenges. First, modern industry usually meets rapid changes that make the acquired skills obsolete shaping emergent demand for new competencies [7]. Second, the employment landscape varies because of technological progress, economic downturns, as well as fluctuating workforce demands. Third, the vocational students vary in their unique learning capabilities, interests, and career aspirations, which made it difficult to apply standardized evaluation metrics. With these challenges in mind, there is a pressing need for a robust mathematical framework that can handle uncertainty in employment competence assessment [8].

Neutrosophic sets came into sight to provide a promising solution to handle uncertainty, thereby offering a solution for optimizing vocational student employment under realistic higher education [9], [10]. The neutrosophic approach was developed as an extension of classical logic and fuzzy set theory, explicitly designed to handle indeterminacy, incompleteness, and inconsistency in decision-making. Neutrosophic sets, apart from classical models, presented three independent components: Truth, Indeterminacy, as well as Falsity memberships [11]. Inspired by the successful application of Neutrosophic logic in employment market analytics, improving employability practices as well as personnel corresponding, which make it a suitable choice for addressing uncertainty management in vocational student employment.

In this work, we aim to investigate and develop a Multi-Criteria Complex Neutrosophic Approach to optimize and manage uncertainties in vocational student employment. Our approach aims to include multiple contradictory criteria with different ambiguity factors into an organized decision framework. Then, we aim to explore new applications of our approach to real-world vocational training scenarios, demonstrating its implications for refining employability predictions and workforce matching. By performing comparative analysis against competing methods from the literature, we demonstrate the advantage of our approach, demonstrating its practical feasibility and benefits in real-time labor market analytics, improving employability predictions and workforce matching [12].

The remaining part of this research is decomposed into five main sections. The fundamental and core concepts are reviewed in section 2. Section 3 describes the proposed method. Section 4 presents and discusses the experimental results. Section 5 concludes the findings.

2. Essentials and Fundamentals

Herin, we provide a summary of essential concepts and definitions of Neutrosophic sets and related methods.

Definition 1: Given U the universe of discourse, a neutrosophic set constituted of three membership functions, which is expressed as follows [9]**.**

$$\overline{S} = \{(u, T_{\overline{S}}(u), I_{\overline{S}}(u), F_{\overline{S}}(u)) : u \in U\}, \quad (1)$$

such that $T_{\overline{S}}, I_{\overline{S}}, F_{\overline{S}}$ belongs to the interval [0,1], and they are summed under condition $0 \leq supT_{\overline{S}}(u) + supI_{\overline{S}}(u) + supF_{\overline{S}}(u) \leq 3$ [13].

Definition 2: The Complex Fuzzy Set (CFS) extends the classical fuzzy set to incorporate complex-valued membership to better deal with oscillatory behavior. Given U is a universal set [14], the definition of CFS can be expressed as follows:

$$\overline{S} = \{(u, \eta_{\overline{S}}(u)) : u \in U\}. \tag{2}$$

such that

$$\eta_{\overline{S}}(u) = r_{\overline{S}(u)} e^{i\theta_{\overline{S}(u)}} \tag{3}$$

Definition 3: Complex Intuitionistic Fuzzy Set (CIFS) was introduced as an extension to CFS by introducing complex-valued a membership function, $\eta_{\overline{S}}(u)$, as well as a non-membership function $\zeta_{\overline{S}}(u)$ to empower the modeling of decision ambiguity.

$$\overline{S} = \{(u, \eta_{\overline{S}}(u), \zeta_{\overline{S}}(u)) : u \in U\}. \tag{4}$$

Such that:

$\eta_{\overline{S}}(u) = p_{\overline{S}}(u) \cdot e^{j \cdot \mu_{\overline{S}}(u)}$ and $\zeta_{\overline{S}}(u) = r_{\overline{S}}(u) \cdot e^{j \cdot \mu_{\overline{S}}(u)}$

Definition 4: A Single-Valued Complex Neutrosophic Set (SVCNS) [15] was proposed to re-design and replace the real value membership in the original NS with complex-valued truth, indeterminacy, and falsity membership functions. The SVCNS can be expressed as follows:

$$\overline{S} = \left\{\left(u, T_{\overline{S}}(u), I_{\overline{S}}(u), F_{\overline{S}}(u)\right) : u \in U\right\}.$$

such that

$$T_{\overline{S}}(u) = p_{\overline{S}}(u) \cdot e^{j\mu_{\overline{S}}(u)}, I_{\overline{S}}(u) = q_{\overline{S}}(u) \cdot e^{j\nu_{\overline{S}}(u)} and F_{\overline{S}}(u) = r_{\overline{S}}(u) \cdot e^{j\omega_{\overline{S}}(u)}, \tag{5}$$

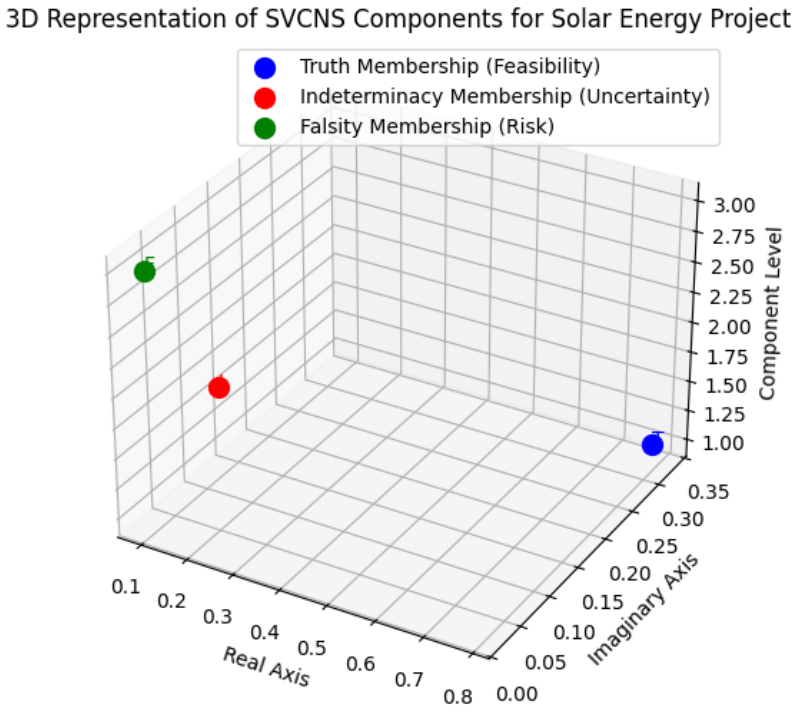

Figure 1. illustration of example SVCNS representation for the Student X $\bar{S}_X = (X, 0.85e^{j0.4}, 0.2e^{j0.3}, 0.1e^{j0.2})$

where $\nu_{\overline{S}}(u)$, $\mu_{\overline{S}}(u)$, $\mu_{\overline{S}}(u)$ are real values, while $p_{\overline{S}}(u), q_{\overline{S}}(u), r_{\overline{S}}(u) \in [0,1]$. In Figure 1, we provided a visualization of the example of SVCNS in the 3D coordinate system.

Definition 5: Given an SVCNS $\overline{S} = \left\{\left(u, T_{\overline{S}}(u), I_{\overline{S}}(u), F_{\overline{S}}(u)\right) : u \in U\right\}$, the complement of this set can be defined as:

$$\begin{aligned}
&\overline{S}^c = \{(u, T_{\overline{S}^c}(u), I_{\overline{S}^c}(u), F_{\overline{S}^c}(u)) : u \in U\}, \\
&\textit{such that} \\
&T_{\overline{S}^c}(u) = p_{\overline{S}^c}(u) \cdot e^{j\cdot\mu_{\overline{S}^c}(u)} \rightarrow p_{\overline{S}^c}(u) = r_{\overline{S}}(u), \mu_{\overline{S}^c}(u) = \mu_{\overline{S}}(u), 2\pi - \mu_{\overline{S}}(u) \textit{ or } \\
&\qquad\qquad \mu_{\overline{S}}(u) + \pi \\
&I_{\overline{S}^c}(u) = q_{\overline{S}^c}(u) \cdot e^{j\cdot\nu_{\overline{S}^c}(u)} \rightarrow q_{\overline{S}^c}(u) = 1 - q_{\overline{S}}(u)\ \nu_{\overline{S}^c}(u) = \nu_{\overline{S}}(u), 2\pi - \nu_{\overline{S}^c}(u) \textit{or } \nu_{\overline{S}^c}(u) + \pi \\
&F_{\overline{S}^c}(u) = r_{\overline{S}^c}(u) \cdot e^{j\cdot\mu_{\overline{S}^c}(u)} \rightarrow r_{\overline{S}^c}(u) = p_{\overline{S}}(u), \mu_{\overline{S}^c}(u) = \mu_{\overline{S}}(u), 2\pi - \mu_{\overline{S}}(u) \textit{ or } \mu_{\overline{S}}(u) + \pi
\end{aligned} \tag{6}$$

Definition 6: Given two SVCNSs $\overline{P} = \left\{\left(u, T_{\overline{P}}(u), I_{\overline{P}}(u), F_{\overline{P}}(u)\right) : u \in U\right\}$, and $\overline{Z} = \left\{\left(u, T_{\overline{Z}}(u), I_{\overline{Z}}(u), F_{\overline{Z}}(u)\right) : u \in U\right\}$, the union of these two sets can be defined as:

$$\begin{aligned}
&\overline{P} \cup \overline{Z} = \left\{\left(u, T_{\overline{P}\cup\overline{Z}}(u), I_{\overline{P}\cup\overline{Z}}(u), F_{\overline{P}\cup\overline{Z}}(U)\right) : u \in U\right\}, \\
&\textit{Such that}
\end{aligned} \tag{7}$$

$$T_{\overline{P}\cup\overline{Z}}(u) = \left[max\left(T_{\overline{P}}(u), T_{\overline{Z}}(u)\right)\right]\cdot e^{j\cdot\mu_{T_{\overline{P}\cup\overline{Z}}}(u)},$$
$$I_{\overline{P}\cup\overline{Z}}(u) = \left[min\left(I_{\overline{P}}(u), I_{\overline{Z}}(u)\right)\right]\cdot e^{j\cdot\nu_{I_{\overline{P}\cup\overline{Z}}}(u)},$$
$$F_{\overline{P}\cup\overline{Z}}(u) = \left[min\left(F_{\overline{P}}(u), F_{\overline{Z}}(u)\right)\right]\cdot e^{j\cdot\omega_{F_{\overline{P}\cup\overline{Z}}}(u)}$$

Definition 7: Given two SVCNSs $\overline{P} = \left\{\left(u, T_{\overline{P}}(u), I_{\overline{P}}(u), F_{\overline{P}}(u)\right) : u \in U\right\}$, and $\overline{Z} = \left\{\left(u, T_{\overline{Z}}(u), I_{\overline{Z}}(u), F_{\overline{Z}}(u)\right) : u \in U\right\}$, the intersection between them can be defined as:

$$\overline{P}\cap\overline{Z} = \left\{\left(u, T_{\overline{P}\cap\overline{Z}}(u), I_{\overline{P}\cap\overline{Z}}(u), F_{\overline{P}\cap\overline{Z}}(U)\right) : u \in U\right\},$$
such that
$$T_{\overline{P}\cap\overline{Z}}(u) = \left[min\left(T_{\overline{P}}(u), T_{\overline{Z}}(u)\right)\right]\cdot \mathrm{e}^{j\cdot\mu_{T_{\overline{P}\cap\overline{Z}}}(u)},$$
$$I_{\overline{P}\cap\overline{Z}}(u) = \left[max\left(I_{\overline{P}}(u), I_{\overline{Z}}(u)\right)\right]\cdot \mathrm{e}^{j\cdot\nu_{I_{\overline{P}\cap\overline{Z}}}(u)}, \quad (8)$$
$$F_{\overline{P}\cap\overline{Z}}(u) = \left[max\left(F_{\overline{P}}(u), F_{\overline{Z}}(u)\right)\right]\cdot \mathrm{e}^{j\cdot\omega_{F_{\overline{P}\cap\overline{Z}}}(u)}$$

Definition 8: Interval Complex Neutrosophic Set (ICNS) was proposed to extend the SVCNS by re-designating the truth, indeterminacy, and falsity membership functions using complex numbers with interval-valued magnitudes.

$$\overline{S} = \left\{\left\langle \begin{array}{l} T_{\overline{S}}(u) = \left[t_{\overline{S_L}}(u), t_{\overline{S_U}}(u)\right]\cdot e^{j\alpha\left[\mu_{\overline{S_L}}(u), \mu_{\overline{S_U}}(u)\right]}, \\ I_{\overline{S}}(u) = \left[i_{\overline{S_L}}(u), i_{\overline{S_U}}(u)\right]\cdot e^{j\beta\left[\nu_{\overline{S_L}}(u), \nu_{\overline{S_U}}(u)\right]}, \\ F_{\overline{S}}(u) = \left[f_{\overline{S_L}}(u), f_{\overline{S_U}}(u)\right]\cdot e^{j\gamma\left[\omega_{\overline{S_L}}(u), \omega_{\overline{S_U}}(u)\right]} \end{array}\right\rangle : u \in U\right\} \quad (9)$$

Such that α, β, γ symbolizes the scaling terms and belongs to the interval $(0, 2\pi]$. In Figure 2, we provided a visualization of the example of ICNS in the 3D coordinate system.

Definition 9: Given two ICNSs $\overline{P} = \left\{\left\langle T_P(u) = t_{\overline{P}}(u)\cdot e^{j\alpha\mu_{\overline{P}}(u)}, I_{\overline{P}}(u) = i_{\overline{P}}(u)\cdot e^{j\beta\nu_{\overline{P}}(u)}, F_{\overline{P}}(u) = f_{\overline{P}}(u)\cdot e^{j\gamma\omega_{\overline{P}}(u)}\right\rangle : u \in U\right\}$, and $\overline{Z} = \left\{\left\langle T_{\overline{Z}}(u) = t_{\overline{Z}}(u)\cdot e^{j\alpha\mu_{\overline{Z}}(u)}, I_{\overline{Z}}(u) = i_{\overline{P}}(u)\cdot e^{j\beta\nu_{\overline{S}}(u)}, F_{\overline{Z}}(u) = f_{\overline{Z}}(u)\cdot e^{j\gamma\omega_{\overline{Z}}(u)}\right\rangle : u \in U\right\}$, then the union between them is defined as follows

$$\overline{S} = \left\{\left\langle \begin{array}{l} T_{\overline{P}\cup\overline{Z}}(u) = [inft_{\overline{P}\cup\overline{Z}}(u), supt_{\overline{P}\cup\overline{Z}}(u)]\cdot e^{j\pi\mu_{\overline{P}\cup\overline{Z}}(u)}, \\ I_{\overline{P}\cup\overline{Z}}(u) = [infi_{\overline{P}\cup\overline{Z}}(u), supi_{\overline{P}\cup\overline{Z}}(u)]\cdot e^{j\pi\nu_{\overline{P}\cup\overline{Z}}(u)}, \\ F_{\overline{P}\cup\overline{Z}}(u) = [inff_{\overline{P}\cup\overline{Z}}(u), supf_{\overline{P}\cup\overline{Z}}(u)]\cdot e^{j\pi\omega_{\overline{P}\cup\overline{Z}}(u)} \end{array}\right\rangle : u \in U\right\} \quad (10)$$

Such that:

$$inft_{\overline{P}\cup\overline{Z}}(u) = mau\left(inft_{\overline{P}}(u), inft_{\overline{Z}}(u)\right), supt_{\overline{P}\cup\overline{Z}}(u) = mau\left(supt_{\overline{P}}(u), supt_{\overline{Z}}(u)\right);$$
$$infi_{\overline{P}\cup\overline{Z}}(u) = min\left(infi_{\overline{P}}(u), infi_{\overline{Z}}(u)\right), supi_{\overline{P}\cup\overline{Z}}(u) = min\left(supi_{\overline{P}}(u), supi_{\overline{Z}}(u)\right);$$
$$inff_{\overline{P}\cup\overline{Z}}(u) = min\left(inff_{\overline{P}}(u), inff_{\overline{Z}}(u)\right), supf_{\overline{P}\cup\overline{Z}}(u) = min\left(supf_{\overline{P}}(u), supf_{\overline{Z}}(u)\right);$$

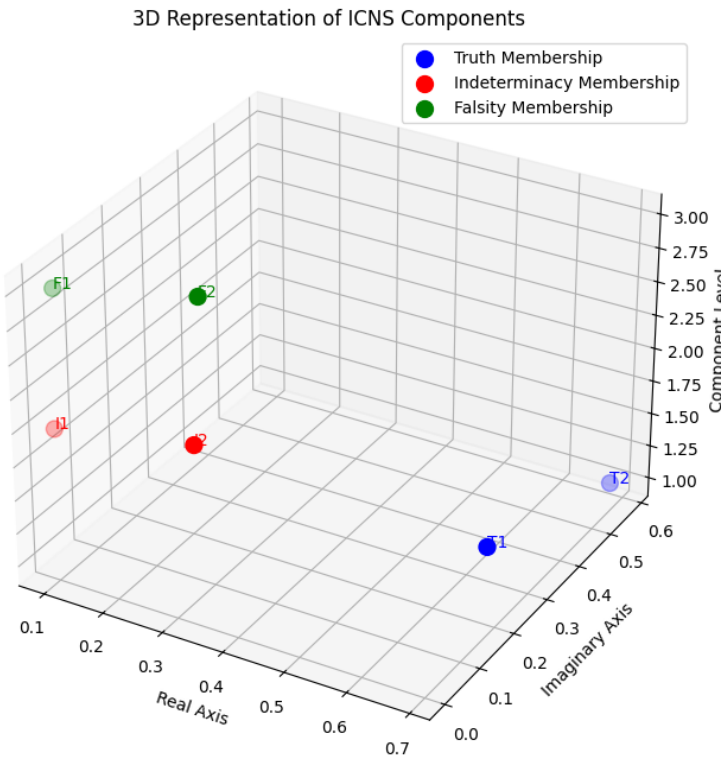

Figure 2. illustration of example IVCNS representation for student X project $\bar{S}_X = \left(\text{X}, \begin{matrix}[0.7,0.9]e^{j\alpha[0.5,0.7]}, [0.1,0.3]e^{j\beta[0.0,0.2]}, \\ [0.1,0.3]e^{j\gamma[0.1,0.3]}\end{matrix}\right)$

Definition 10: Given two ICNSs $\overline{P} = \{\langle T_P(u) = t_{\overline{P}}(u) \cdot e^{j\alpha\mu_{\overline{P}}(u)}, I_{\overline{P}}(u) = i_{\overline{P}}(u) \cdot e^{j\beta\nu_{\overline{P}}(u)}, F_{\overline{P}}(u) = f_{\overline{P}}(u) \cdot e^{j\gamma\omega_{\overline{P}}(u)}\rangle : u \in U\}$, and $\overline{Z} = \{\langle T_{\overline{Z}}(u) = t_{\overline{Z}}(u) \cdot e^{j\alpha\mu_{\overline{Z}}(u)}, I_{\overline{Z}}(u) = i_{\overline{P}}(u) \cdot e^{j\beta\nu_{\overline{S}}(u)}, F_{\overline{Z}}(u) = f_{\overline{Z}}(u) \cdot e^{j\gamma\omega_{\overline{Z}}(u)}\rangle : u \in U\}$, then the intersection between them is defined as follows

$$\overline{S} = \left\{ \left\langle \begin{matrix} T_{\overline{P}\cap\overline{Z}}(u) = [inft_{\overline{P}\cap\overline{Z}}(u), supt_{\overline{P}\cap\overline{Z}}(u)] \cdot e^{j\pi\mu_{\overline{P}\cap\overline{Z}}(u)}, \\ I_{\overline{P}\cap\overline{Z}}(u) = [infi_{\overline{P}\cap\overline{Z}}(u), supi_{\overline{P}\cap\overline{Z}}(u)] \cdot e^{j\pi\nu_{\overline{P}\cap\overline{Z}}(u)}, \\ F_{\overline{P}\cap\overline{Z}}(u) = [inff_{\overline{P}\cap\overline{Z}}(u), supf_{\overline{P}\cap\overline{Z}}(u)] \cdot e^{j\pi\omega_{\overline{P}\cap\overline{Z}}(u)} \end{matrix} \right\rangle : u \in U \right\} \quad (11)$$

where

$$inft_{\overline{P}\cap\overline{Z}}(u) = min\left(inft_{\overline{P}}(u), inft_{\overline{Z}}(u)\right), supt_{\overline{P}\cap\overline{Z}}(u) = min\left(supt_{\overline{P}}(u), supt_{\overline{Z}}(u)\right),$$
$$infi_{\overline{P}\cap\overline{Z}}(u) = max\left(infi_{\overline{P}}(u), infi_{\overline{Z}}(u)\right), supi_{\overline{P}\cap\overline{Z}}(u) = max\left(supi_{\overline{P}}(u), supi_{\overline{Z}}(u)\right),$$
$$inff_{\overline{P}\cap\overline{Z}}(u) = max\left(inff_{\overline{P}}(u), inff_{\overline{Z}}(u)\right), supf_{\overline{P}\cap\overline{Z}}(u) = max\left(supf_{\overline{P}}(u), supf_{\overline{Z}}(u)\right),$$

Definition 11: Given an ICNS $\overline{S} = \{\langle T_{\overline{S}}(u) = t_{\overline{P}}(u) \cdot e^{j\alpha\mu_{\overline{P}}(u)}, I_{\overline{P}}(u) = i_{\overline{S}}(u) \cdot e^{j\beta\nu_{\overline{P}}(u)}, F_{\overline{P}}(u) = f_{\overline{P}}(u) \cdot e^{j\gamma\omega_{\overline{P}}(u)}\rangle : u \in U\}$, then the complement of $\overline{P}$ is denoted as $\overline{P}^c$, and is defined as:

$$\overline{P}^c = \left\{ \left\langle T_{\overline{P}^c}(u) = t_{\overline{P}^c}(u) \cdot e^{j\pi\mu_{\overline{P}^c}(u)}, I_{\overline{P}^c}(u) = i_{\overline{P}^c}(u) \cdot e^{j\pi\nu_{\overline{P}^c}(u)}, F_{\overline{P}^c}(u) = f_{\overline{P}^c}(u) \cdot e^{j\pi\omega_{\overline{P}^c}(u)} \right\rangle : u \in U \right\}, \quad (12)$$

Such that

$t_{\overline{P}^c}(u) = f_{\overline{P}}(u)$, $\mu_{\overline{P}^c}(u) = 2\pi - \mu_{\overline{P}}(u)$ *or* $\mu_{\overline{P}^c}(u) = 2\pi - \mu_{\overline{P}}(u)$,

$i_{\overline{P}^c}(u) = \left(1 - supi_{\overline{P}}(u), 1 - inf i_{\overline{P}}(u)\right)$, $\nu_{\overline{P}^c}(u) = 2\pi - \nu_{\overline{P}}(u)$ *or* $\nu_{\overline{P}}(u) + \pi$

$f_{\overline{P}^c}(u) = i_{\overline{P}^c}(u)$, $\omega_{\overline{P}^c}(u) = 2\pi - \omega_{\overline{P}}(u)$ *or* $\omega_{\overline{P}}(u) + \pi$

Definition 12: Given two ICNSs $\overline{P} = \left\{\left\langle T_P(u) = t_{\overline{P}}(u) \cdot e^{j\alpha\mu_{\overline{P}}(u)}, I_{\overline{P}}(u) = i_{\overline{P}}(u) \cdot e^{j\beta\nu_{\overline{P}}(u)}, F_{\overline{P}}(u) = f_{\overline{P}}(u) \cdot e^{j\gamma\omega_{\overline{P}}(u)}\right\rangle : u \in U\right\}$, and $\overline{Z} = \left\{\left\langle T_{\overline{Z}}(u) = t_{\overline{Z}}(u) \cdot e^{j\alpha\mu_{\overline{Z}}(u)}, I_{\overline{Z}}(u) = i_{\overline{P}}(u) \cdot e^{j\beta\nu_{\overline{S}}(u)}, F_{\overline{Z}}(u) = f_{\overline{Z}}(u) \cdot e^{j\gamma\omega_{\overline{Z}}(u)}\right\rangle : u \in U\right\}$, then the product between them is defined as follows

$$\begin{aligned}\overline{S} = \Big\{\Big\langle T_{\overline{P}*\overline{Z}}(u) &= \left[t_{\overline{P}}^L(u)t_{\overline{Z}}^L(u), t_{\overline{P}}^U(u)t_{\overline{Z}}^U(u)\right] \cdot e^{j\pi\left[\mu_{\overline{P}*\overline{Z}}^L(u), \mu_{\overline{P}*\overline{Z}}^R(u)\right]}, I_{\overline{P}*\overline{Z}}(u) \\ &= \left[i_{\overline{P}}^L(u) + i_{\overline{Z}}^L(u) - i_{\overline{P}}^L(u)i_{\overline{Z}}^L(u), i_{\overline{P}}^R(u) + i_{\overline{Z}}^R(u) - i_{\overline{P}}^R(u)i_{\overline{Z}}^R(u)\right] \\ &\cdot e^{j\pi\left[\nu_{\overline{P}*\overline{Z}}^L(u), \nu_{\overline{P}*\overline{Z}}^R(u)\right]}, F_{\overline{P}\times\overline{Z}}(u) \\ &= [f_{\overline{P}}^L(u) + f_{\overline{Z}}^L(u) - f_{\overline{P}}^L(u)f_{\overline{Z}}^L(u), f_{\overline{P}}^R(u) + f_{\overline{Z}}^R(u) - f_{\overline{P}}^R(u)f_{\overline{Z}}^R(u)] \\ &\cdot e^{j\pi[\omega_{\overline{P}\times\overline{Z}}^L(u), \omega_{\overline{P}\times\overline{Z}}^R(u)]}\Big\rangle : u \in U\Big\}\end{aligned} \tag{13}$$

Where the phase terms are defined as follows

$$\begin{aligned}\mu_{\overline{P}\times\overline{Z}}^L(u) = \mu_{\overline{P}}^L(u)\mu_{\overline{Z}}^L(u), \mu_{\overline{P}\times\overline{Z}}^U(u) = \mu_{\overline{P}}^U(u)\mu_{\overline{Z}}^U(u) \\ \nu_{\overline{P}\times\overline{Z}}^L(u) = \nu_{\overline{P}}^L(u)\nu_{\overline{Z}}^L(u), \nu_{\overline{P}\times\overline{Z}}^U(u) = \nu_{\overline{P}}^U(u)\nu_{\overline{Z}}^U(u) \\ \omega_{\overline{P}\times\overline{Z}}^L(u) = \omega_{\overline{P}}^L(u)\omega_{\overline{Z}}^L(u), \omega_{\overline{P}\times\overline{Z}}^U(u) = \omega_{\overline{P}}^U(u)\omega_{\overline{Z}}^U(u).\end{aligned}$$

Definition 13: Given two ICNSs $\overline{P} = \left\{\left\langle T_P(u) = t_{\overline{P}}(u) \cdot e^{j\alpha\mu_{\overline{P}}(u)}, I_{\overline{P}}(u) = i_{\overline{P}}(u) \cdot e^{j\beta\nu_{\overline{P}}(u)}, F_{\overline{P}}(u) = f_{\overline{P}}(u) \cdot e^{j\gamma\omega_{\overline{P}}(u)}\right\rangle : u \in U\right\}$, and $\overline{Z} = \left\{\left\langle T_{\overline{Z}}(u) = t_{\overline{Z}}(u) \cdot e^{j\alpha\mu_{\overline{Z}}(u)}, I_{\overline{Z}}(u) = i_{\overline{P}}(u) \cdot e^{j\beta\nu_{\overline{S}}(u)}, F_{\overline{Z}}(u) = f_{\overline{Z}}(u) \cdot e^{j\gamma\omega_{\overline{Z}}(u)}\right\rangle : u \in U\right\}$, then the addition of them is defined as follows

$$\begin{aligned}\overline{S} = \Big\{\Big\langle &\left[t_{\overline{P}}^L(u) + t_{\overline{Z}}^L(u) - t_{\overline{P}}^L(u)t_{\overline{Z}}^L(u), t_{\overline{P}}^U(u) + t_{\overline{Z}}^U(u) - t_{\overline{P}}^U(u)t_{\overline{Z}}^U(u)\right] \cdot e^{j\pi\left[\mu_{\overline{P}+\overline{Z}}^L(u), \mu_{\overline{P}+\overline{Z}}^L(u)\right]}, I_{\overline{P}+\overline{Z}}(u) \\ &= \left[i_{\overline{P}}^L(u)i_{\overline{Z}}^L(u), i_{\overline{P}}^U(u)i_{\overline{Z}}^U(u)\right] \cdot e^{j\pi\left[\nu_{\overline{P}+\overline{Z}}^L(u), \nu_{\overline{P}+\overline{Z}}^R(u)\right]}, F_{\overline{P}+\overline{Z}}(u) \\ &= \left[f_{\overline{P}}^L(u)f_{\overline{Z}}^L(u), f_{\overline{P}}^R(u)f_{\overline{Z}}^R(u)\right] \cdot e^{j\pi\left[\omega_{\overline{P}+\overline{Z}}^L(u), \omega_{\overline{P}+\overline{Z}}^R(u)\right]}\Big\rangle : u \in U\Big\}\end{aligned} \tag{14}$$

Where the phase terms are defined as follows

$$\begin{aligned}\mu_{\overline{P}+\overline{Z}}^L(u) = \mu_{\overline{P}}^L(u) + \mu_{\overline{Z}}^L(u), \mu_{\overline{P}+\overline{Z}}^U(u) = \mu_{\overline{P}}^U(u) + \mu_{\overline{Z}}^U(u), \nu_{\overline{P}+\overline{Z}}^L(u) = \nu_{\overline{P}}^L(u) + \nu_{\overline{Z}}^L(u), \nu_{\overline{P}+\overline{Z}}^U(u) \\ = \nu_{\overline{P}}^U(u) + \nu_{\overline{Z}}^U(u), \omega_{\overline{P}+\overline{Z}}^L(u) = \omega_{\overline{P}}^L(u) + \omega_{\overline{Z}}^L(u), \omega_{\overline{P}+\overline{Z}}^U(u) = \omega_{\overline{P}}^U(u) + \omega_{\overline{Z}}^U(u)\end{aligned}$$

Definition 14: Given ICNS $\overline{P} = \left\{\left\langle T_P(u) = t_{\overline{P}}(u) \cdot e^{j\alpha\mu_{\overline{P}}(u)}, I_{\overline{P}}(u) = i_{\overline{P}}(u) \cdot e^{j\beta\nu_{\overline{P}}(u)}, F_{\overline{P}}(u) = f_{\overline{P}}(u) \cdot e^{j\gamma\omega_{\overline{P}}(u)}\right\rangle : u \in U\right\}$, the scalar multiplication is defined as follows:

$$\begin{aligned}T_{\overline{C}}(u) = [1 - (1 - t_{\overline{P}}^L(u))^k, 1 - (1 - t_{\overline{P}}^R(u))^k] \cdot e^{j\pi\left[\mu_{\overline{C}}^L(u), \mu_{\overline{C}}^R(u)\right]}, I_{\overline{C}}(u) = [(i_{\overline{P}}^L(u))^k, (i_{\overline{P}}^R(u))^k] \cdot \\ e^{j\pi[\nu_{\overline{C}}^L(u), \nu_{\overline{C}}^R(u)]}, I_{\overline{C}}(u) = [(i_{\overline{P}}^L(u))^k, (i_{\overline{P}}^R(u))^k] \cdot e^{j\pi[\nu_{\overline{C}}^L(u), \nu_{\overline{C}}^R(u)]}\end{aligned} \tag{15}$$

Where the phase terms are defined as follows

$$\begin{aligned}\mu_{\overline{C}}^L(u) = \mu_{\overline{P}}^L(u) \cdot k; \mu_{\overline{C}}^R(u) = \mu_{\overline{P}}^R(u) \cdot k, \nu_{\overline{C}}^L(u) = \nu_{\overline{P}}^L(u) \cdot k; \nu_{\overline{C}}^R(u) = \nu_{\overline{P}}^R(u) \cdot k, \omega_{\overline{C}}^L(u) = \omega_{\overline{P}}^L(u) \cdot k; \omega_{\overline{C}}^R(u) \\ = \omega_{\overline{P}}^R(u) \cdot k\end{aligned}$$

3. Proposed Method

In this section, we introduce the proposed methodology for optimizing and managing uncertainty in vocational student employment using a Multi-Criteria Complex Neutrosophic Approach. The method integrates ICNS with the MCDM technique, which enables a robust decision framework. Our approach consists of several key phases, including ICNS transformation of linguistic evaluations into ICNS numbers, weight determination, as well as alternative evaluation and ranking. In the following, we propose to explain the methodological steps involved in the proposed approach.

In step 1, we calculate the weights of different components in ICNS based on the relative dominance and uncertainty of each component. w_T , w_I , and w_F. Relative dominance implies that components with a higher average magnitude should assigned higher weight. Uncertainty implies that the component with lower uncertainty (narrower interval) should have a higher weight. The weights are computed as follows:

$$\begin{aligned} w_T &= \frac{t_S(u)\cdot(1-U_T(u))}{t_S(u)\cdot(1-U_T(u))+i_S(u)\cdot(1-U_I(u))+f_S(u)\cdot(1-U_F(u))}, \\ w_I &= \frac{i_S(u)\cdot(1-U_I(u))}{t_{\bar{S}}(u)\cdot(1-U_T(u))+i_{\bar{S}}(u)\cdot(1-U_I(u))+f_{\bar{S}}(u)\cdot(1-U_F(u))}, \\ w_F &= \frac{f_{\bar{S}}(u)\cdot(1-U_F(u))}{t_S(u)\cdot(1-U_T(u))+i_S(u)\cdot(1-U_I(u))+f_S(u)\cdot(1-U_F(u))}, \end{aligned} \tag{16}$$

where $U_T(u), U_I(u)$, and $U_F(u)$ denote the normalized uncertainties:

$$\begin{aligned} U_T(u) &= \frac{t_{\bar{S}_U}(u)-t_{\bar{S}_L}(u)}{t_{\bar{S}_U}(u)+i_{\bar{S}_U}(u)+f_{\bar{S}_U}(u)}, U_I(u) = \frac{i_{S_U}(u)-i_{S_L}(u)}{t_{\bar{S}_U}(u)+i_{\bar{S}_U}(u)+f_{\bar{S}_U}(u)} \\ U_F(u) &= \frac{f_{S_U}(u)-f_{S_L}(u)}{t_{S_U}(u)+i_{S_U}(u)+f_{S_U}(u)} \end{aligned} \tag{17}$$

In step 2, we apply these weights to compute the weighted average magnitude and weighted average phase.

$$\bar{t}(u) = w_T\cdot\frac{t_{S_L}(u)+t_{S_U}(u)}{2},\ \bar{\imath}(u) = w_I\cdot\frac{i_{S_L}(u)+i_{S_U}(u)}{2},\ \bar{f}(u) = w_F\cdot\frac{f_{S_L}(u)+f_{S_U}(u)}{2} \tag{18}$$

$$\mu(u) = w_T\cdot\frac{\mu_{\bar{S}_L}(u)+\mu_{\bar{S}_U}(u)}{2},\ \nu(u) = w_I\cdot\frac{\nu_{\bar{S}_L}(u)+\nu_{\bar{S}_U}(u)}{2},\ \omega(u) = w_F\cdot\frac{\omega_{\bar{S}_L}(u)+\omega_{\bar{S}_U}(u)}{2} \tag{19}$$

In step 3, we inject phase information into the scoring mechanism:

$$\begin{gathered} S_T(u) = \bar{t}(u)\cdot\cos\big(\mu(u)\big),\ S_i(u) = \bar{\imath}(u)\cdot\cos\big(\nu(u)\big), \\ S_f(u) = \bar{f}(u)\cdot\cos\big(\omega(u)\big) \end{gathered} \tag{20}$$

In step 4, we combine the phase-sensitive scores and uncertainty measures into a final de-neutrosophication score:

$$\begin{aligned} \mathfrak{D}(P) = \Bigg(& S_T(u)\frac{\big(t_{S_U}(u)+t_{S_L}(u)\big)}{2} + S_i(u)\left(1-\frac{\big(i_{S_U}(u)+i_{S_L}(u)\big)}{2}\right)\big(i_{S_U}(u)\big) \\ & - S_i(u)\left(\frac{\big(f_{S_U}(u)+f_{S_L}(u)\big)}{2}\right)\big(1-f_{S_U}(u)\big)\Bigg) \end{aligned} \tag{21}$$

Following the de-neutrosophication of ICNS values, we introduce pseudocode for the weight calculation algorithm based on AHP [16]. As shown in Algorithm 1, the pseudocode is written in structured mathematical step terms.

Algorithm 1: pseudocode for AHP weighting

Input: Group of $\boldsymbol{n}$ criteria $\boldsymbol{C} = \{\boldsymbol{C_1}, \boldsymbol{C_2}, \dots, \boldsymbol{C_n}\}$, pairwise comparisons from DMs.

Output: A weight vector $\boldsymbol{w} = \{\boldsymbol{w_1}, \boldsymbol{w_2}, \dots, \boldsymbol{w_n}\}$, where $\boldsymbol{w_i}$ Represents the weight of $\boldsymbol{i-th}$ criterion.

Aggregate pairwise comparison matrices

For each pair of criteria (C_i, C_j):

  If $i = j$

    set $\mathbb{A}_{ij} = 1$

  else

    $a_{ij} \leftarrow$ DM value on a scale of 1 to 9

  $a_{ij} = 1$: C_i is equally important to C_j.

  $a_{ij} = 9$: C_i is extremely more important than C_j

  Set $\mathbb{A}_{ji} = \frac{1}{a_{ij}}$ (reciprocal value).

$$\mathbb{A} = \begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \dots & 1 \end{bmatrix}$$

Get column sums: $S_j = \sum_{i=1}^{n} \mathbb{A}_{ij}$, for $j = 1,2, \dots, n$

Normalize each element of $\mathbb{A}$ $N_{ij} = \frac{\mathbb{A}_{ij}}{S_j}$, for $i, j = 1,2, \dots, n$ $N = \begin{bmatrix} N_{11} & N_{12} & \dots & N_{1n} \\ N_{21} & N_{22} & \dots & N_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ N_{n1} & N_{n2} & \dots & N_{nn} \end{bmatrix}$

Calculate the Weights $w_i = \frac{1}{n}\sum_{j=1}^{n} N_{ij}$, for $i = 1,2, \dots, n$

Return $\boldsymbol{W} = \{\boldsymbol{w_1}, \boldsymbol{w_2}, \dots, \boldsymbol{w_n}\}$

To make the multi-criteria decision, we apply VIKOR (VIšekriterijumsko KOmpromisno Rangiranje) [17] to identify a compromise solution that balances conflicting criteria. It is particularly useful in our case as we aim to evaluate alternatives based on multiple, often competing, objectives. Below is a step-by-step description of the VIKOR method:

In step 1, we obtain the decision matrix $U_{m\times n}$, based on equation (26), along with the weight vector $W_{1\times n} = \{w_1, w_2, \dots, w_n\}$ Computed by algorithm 1. Also, we take into account the type of criterion as annotated by decision-makers. The VIKOR method uses strategy coefficient v : A scalar value between 0 and 1 (default = 0.5).

In step 2, we normalize the matrix $M_{m\times n}$ as follows. For each criterion j, determine the best (f_j^*) and worst (f_j^-) values:

$$f_j^* = \begin{cases} \max(M_{ij}) & \text{if } c_j =' \max \\ \min(M_{ij}) & \text{if } c_j =' \min \end{cases} f_j^- = \begin{cases} \min(M_{ij}) & \text{if } c_j =' \max \\ \max(M_{ij}) & \text{if } c_j =' \min \end{cases} \quad (22)$$

Then, we calculate the normalized decision matrix S :

$$S_{ij} = w_j \cdot \frac{\left|f_j^* - M_{ij}\right|}{\left|f_j^* - f_j^-\right| + \epsilon} \tag{23}$$

Where ϵ is a small constant (e.g., 10^{-16}) to avoid division by zero.

In step 3, we calculate the usefulness measure S_i, as well as the regret measure R_i for each i-th alternative:

$$S_i = \sum_{j=1}^{n} S_{ij} \; R_i = \max\left(S_{ij}\right) \tag{24}$$

In step 4, we computed the VIKOR Index $(Q_i)$to decide on the best and worst values for S_i and R_i:

$$\begin{aligned} S^* &= \min(S_i), \quad S^- = \max(S_i) \\ R^* &= \min(R_i), \quad R^- = \max(R_i) \end{aligned} \tag{25}$$

In step 5, we compute the VIKOR index Q_i for i-th alternative:

$$Q_i = v \cdot \frac{S_i - S^*}{S^- - S^*} + (1 - v) \cdot \frac{R_i - R^*}{R^- - R^*} \tag{26}$$

Finally, we rank the alternatives according to the values of Q_i in ascending order. The alternative with the smallest Q_i is the best compromise solution. We use the following conditions to ensure the ranking is acceptable:

$$Q_{(2)} - Q_{(1)} \geq \frac{1}{m-1} \tag{27}$$

where $Q_{(1)}$ and $Q_{(2)}$ are the first and second smallest Q_i Values, respectively.

4. Case Study & Results

This study introduces a case study of a Chinese university campus that plans to implement vocational student employment to enable local industry to evaluate and rank five graduating students for a technical specialist role in a manufacturing and technology firm. The employment process usually includes both qualitative and quantitative valuations, in which uncertainty in employment competence occurs because of subjective and contradictory criteria. The case study contains many conflicting criteria that govern the decision-making, some of them need to be maximized and other needs to be minimized or avoided. In Table 1, we present ten evaluation criteria used to assess vocational college students' employment competence. To facilitate a systematic and structured evaluation, Table 1 defines six linguistic terms that describe candidate performance in each criterion and maps them to ICNS, which represent uncertainty, hesitation, and imprecision in assessment are properly captured.

Table 1. Summary of vocational employment criteria and related linguistic rates

Criteria	ID	Type	Linguistic rating	
Leadership & Initiative	C1	Benefit	**Linguistic Term**	**ICNS terms**

Communication & Teamwork	C2	Benefit	Very Poor (VP)	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]}, [0.7,0.9]e^{j\beta[0.4,0.6]}, \\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$
Adaptability & Problem-Solving	C3	Benefit	Poor (P)	$\left\{\left\langle \begin{matrix}[0.1,0.3]e^{j\alpha[0.2,0.4]}, [0.6,0.8]e^{j\beta[0.3,0.5]}, \\ [0.7,0.9]e^{j\gamma[0.4,0.6]}\end{matrix}\right\rangle\right\}$
Training Costs Incurred	C4	Cost	Fair (F)	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]}, [0.5,0.7]e^{j\beta[0.2,0.4]}, \\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$
Internship Performance Score	C5	Benefit	Good (G)	$\left\{\left\langle \begin{matrix}[0.5,0.7]e^{j\alpha[0.4,0.6]}, [0.3,0.5]e^{j\beta[0.1,0.3]}, \\ [0.3,0.5]e^{j\gamma[0.2,0.4]}\end{matrix}\right\rangle\right\}$
Response Time	C6	Cost	Very Good (VG)	$\left\{\left\langle \begin{matrix}[0.7,0.9]e^{j\alpha[0.5,0.7]}, [0.1,0.3]e^{j\beta[0.0,0.2]}, \\ [0.1,0.3]e^{j\gamma[0.1,0.3]}\end{matrix}\right\rangle\right\}$
Error Rate in Practical Tasks	C7	Cost	Excellent (E)	$\left\{\left\langle \begin{matrix}[0.9,1.0]e^{j\alpha[0.6,0.8]}, [0.0,0.2]e^{j\beta[0.0,0.1]}, \\ [0.0,0.2]e^{j\gamma[0.0,0.2]}\end{matrix}\right\rangle\right\}$
Employment Competence Indeterminacy Score	C8	Cost		
Technical Skill Level	C9	Benefit		
Industry Certifications	C10	Benefit		

In our case study, the decision-maker must evaluate different higher vocational students. In this case study, four policymakers are involved in evaluating the above alternatives based on the 10 criteria. Each policy maker (e.g., PM1: HR professionals, PM2: vocational training instructors, PM3: industry experts, PM4: academic counselor) provides their subjective judgment for criteria of each alternative using a linguistic rating set, which is later transformed into ICSN for decision-making. in Table 2, we present the aggregated ICNS decision matrix that consolidated assessments from the policymakers.

Table 2. Summary of aggregated ICNS decision matrix from different DMs.

Criteria	**C1**	**C2**
A1	$\left\{\left\langle \begin{matrix}[0.7,0.9]e^{j\alpha[0.5,0.7]}, [0.1,0.3]e^{j\beta[0.0,0.2]}, \\ [0.1,0.3]e^{j\gamma[0.1,0.3]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.1,0.3]e^{j\alpha[0.2,0.4]}, [0.6,0.8]e^{j\beta[0.3,0.5]}, \\ [0.7,0.9]e^{j\gamma[0.4,0.6]}\end{matrix}\right\rangle\right\}$
A2	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]}, [0.5,0.7]e^{j\beta[0.2,0.4]}, \\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]}, [0.7,0.9]e^{j\beta[0.4,0.6]}, \\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$
A3	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]}, [0.7,0.9]e^{j\beta[0.4,0.6]}, \\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]}, [0.7,0.9]e^{j\beta[0.4,0.6]}, \\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$

A4	$\left\{\left\langle \begin{matrix}[0.1,0.3]e^{j\alpha[0.2,0.4]},[0.6,0.8]e^{j\beta[0.3,0.5]},\\ [0.7,0.9]e^{j\gamma[0.4,0.6]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]},[0.7,0.9]e^{j\beta[0.4,0.6]},\\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$
A5	$\left\{\left\langle \begin{matrix}[0.7,0.9]e^{j\alpha[0.5,0.7]},[0.1,0.3]e^{j\beta[0.0,0.2]},\\ [0.1,0.3]e^{j\gamma[0.1,0.3]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]},[0.5,0.7]e^{j\beta[0.2,0.4]},\\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$

Table 2. Summary of aggregated ICNS decision matrix from different DMs (cont.…).

Criteria	C3	C4
A1	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]},[0.7,0.9]e^{j\beta[0.4,0.6]},\\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]},[0.5,0.7]e^{j\beta[0.2,0.4]},\\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$
A2	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]},[0.7,0.9]e^{j\beta[0.4,0.6]},\\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]},[0.7,0.9]e^{j\beta[0.4,0.6]},\\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$
A3	$\left\{\left\langle \begin{matrix}[0.5,0.7]e^{j\alpha[0.4,0.6]},[0.3,0.5]e^{j\beta[0.1,0.3]},\\ [0.3,0.5]e^{j\gamma[0.2,0.4]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]},[0.5,0.7]e^{j\beta[0.2,0.4]},\\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$
A4	$\left\{\left\langle \begin{matrix}[0.5,0.7]e^{j\alpha[0.4,0.6]},[0.3,0.5]e^{j\beta[0.1,0.3]},\\ [0.3,0.5]e^{j\gamma[0.2,0.4]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]},[0.7,0.9]e^{j\beta[0.4,0.6]},\\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$
A5	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]},[0.5,0.7]e^{j\beta[0.2,0.4]},\\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]},[0.5,0.7]e^{j\beta[0.2,0.4]},\\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$

Table 2. Summary of aggregated ICNS decision matrix from different DMs (cont.…).

Criteria	C5	C6
A1	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]},[0.7,0.9]e^{j\beta[0.4,0.6]},\\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]},[0.5,0.7]e^{j\beta[0.2,0.4]},\\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$
A2	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]},[0.7,0.9]e^{j\beta[0.4,0.6]},\\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.5,0.7]e^{j\alpha[0.4,0.6]},[0.3,0.5]e^{j\beta[0.1,0.3]},\\ [0.3,0.5]e^{j\gamma[0.2,0.4]}\end{matrix}\right\rangle\right\}$
A3	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]},[0.5,0.7]e^{j\beta[0.2,0.4]},\\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]},[0.5,0.7]e^{j\beta[0.2,0.4]},\\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$
A4	$\left\{\left\langle \begin{matrix}[0.5,0.7]e^{j\alpha[0.4,0.6]},[0.3,0.5]e^{j\beta[0.1,0.3]},\\ [0.3,0.5]e^{j\gamma[0.2,0.4]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]},[0.5,0.7]e^{j\beta[0.2,0.4]},\\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$
A5	$\left\{\left\langle \begin{matrix}[0.7,0.9]e^{j\alpha[0.5,0.7]},[0.1,0.3]e^{j\beta[0.0,0.2]},\\ [0.1,0.3]e^{j\gamma[0.1,0.3]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.7,0.9]e^{j\alpha[0.5,0.7]},[0.1,0.3]e^{j\beta[0.0,0.2]},\\ [0.1,0.3]e^{j\gamma[0.1,0.3]}\end{matrix}\right\rangle\right\}$

Table 2. Summary of aggregated ICNS decision matrix from different DMs (cont.…).

Criteria	C7	C8
A1	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]},[0.7,0.9]e^{j\beta[0.4,0.6]},\\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]},[0.5,0.7]e^{j\beta[0.2,0.4]},\\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$
A2	$\left\{\left\langle \begin{matrix}[0.7,0.9]e^{j\alpha[0.5,0.7]},[0.1,0.3]e^{j\beta[0.0,0.2]},\\ [0.1,0.3]e^{j\gamma[0.1,0.3]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.7,0.9]e^{j\alpha[0.5,0.7]},[0.1,0.3]e^{j\beta[0.0,0.2]},\\ [0.1,0.3]e^{j\gamma[0.1,0.3]}\end{matrix}\right\rangle\right\}$
A3	$\left\{\left\langle \begin{matrix}[0.1,0.3]e^{j\alpha[0.2,0.4]},[0.6,0.8]e^{j\beta[0.3,0.5]},\\ [0.7,0.9]e^{j\gamma[0.4,0.6]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]},[0.5,0.7]e^{j\beta[0.2,0.4]},\\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$

A4	$\left\{\left\langle \begin{matrix}[0.1,0.3]e^{j\alpha[0.2,0.4]}, [0.6,0.8]e^{j\beta[0.3,0.5]}, \\ [0.7,0.9]e^{j\gamma[0.4,0.6]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]}, [0.7,0.9]e^{j\beta[0.4,0.6]}, \\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$
A5	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]}, [0.5,0.7]e^{j\beta[0.2,0.4]}, \\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]}, [0.5,0.7]e^{j\beta[0.2,0.4]}, \\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$

Table 2. Summary of aggregated ICNS decision matrix from different DMs (cont.…).

Criteria	C9	C10
A1	$\left\{\left\langle \begin{matrix}[0.7,0.9]e^{j\alpha[0.5,0.7]}, [0.1,0.3]e^{j\beta[0.0,0.2]}, \\ [0.1,0.3]e^{j\gamma[0.1,0.3]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.1,0.3]e^{j\alpha[0.2,0.4]}, [0.6,0.8]e^{j\beta[0.3,0.5]}, \\ [0.7,0.9]e^{j\gamma[0.4,0.6]}\end{matrix}\right\rangle\right\}$
A2	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]}, [0.7,0.9]e^{j\beta[0.4,0.6]}, \\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.3,0.5]e^{j\alpha[0.3,0.5]}, [0.5,0.7]e^{j\beta[0.2,0.4]}, \\ [0.5,0.7]e^{j\gamma[0.3,0.5]}\end{matrix}\right\rangle\right\}$
A3	$\left\{\left\langle \begin{matrix}[0.0,0.2]e^{j\alpha[0.1,0.3]}, [0.7,0.9]e^{j\beta[0.4,0.6]}, \\ [0.8,1.0]e^{j\gamma[0.5,0.7]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.5,0.7]e^{j\alpha[0.4,0.6]}, [0.3,0.5]e^{j\beta[0.1,0.3]}, \\ [0.3,0.5]e^{j\gamma[0.2,0.4]}\end{matrix}\right\rangle\right\}$
A4	$\left\{\left\langle \begin{matrix}[0.5,0.7]e^{j\alpha[0.4,0.6]}, [0.3,0.5]e^{j\beta[0.1,0.3]}, \\ [0.3,0.5]e^{j\gamma[0.2,0.4]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.5,0.7]e^{j\alpha[0.4,0.6]}, [0.3,0.5]e^{j\beta[0.1,0.3]}, \\ [0.3,0.5]e^{j\gamma[0.2,0.4]}\end{matrix}\right\rangle\right\}$
A5	$\left\{\left\langle \begin{matrix}[0.7,0.9]e^{j\alpha[0.5,0.7]}, [0.1,0.3]e^{j\beta[0.0,0.2]}, \\ [0.1,0.3]e^{j\gamma[0.1,0.3]}\end{matrix}\right\rangle\right\}$	$\left\{\left\langle \begin{matrix}[0.5,0.7]e^{j\alpha[0.4,0.6]}, [0.3,0.5]e^{j\beta[0.1,0.3]}, \\ [0.3,0.5]e^{j\gamma[0.2,0.4]}\end{matrix}\right\rangle\right\}$

To facilitate decision-making under uncertainty, we apply the de-neutrosophication process to encode ICNS values into crisp numerical values. This can ensure that the decision matrix is more interpretable and compatible with MCDM methods. The de-neutrosophicated values are presented in Table 3 for each alternative across all criteria.

Table 3. Summary of de-neutrosophication matrix derived from different criteria from ICNS decision matrix.

Alternative	C 1	C 2	C 3	C 4	C 5	C 6	C 7	C 8	C 9	C 10
A1	0.396	0.044	0.063	0.062	0.063	0.062	0.063	0.062	0.396	0.044
A2	0.062	0.063	0.063	0.063	0.063	0.162	0.396	0.396	0.063	0.062
A3	0.063	0.063	0.162	0.062	0.062	0.062	0.044	0.062	0.063	0.162
A4	0.044	0.063	0.162	0.063	0.162	0.062	0.044	0.063	0.162	0.162
A5	0.396	0.062	0.062	0.062	0.396	0.396	0.062	0.062	0.396	0.162

The AHP-based weight computation derived from Algorithm 1 determines the relative importance of each criterion. These weights, obtained using the geometric mean method, are illustrated in Figure 3.

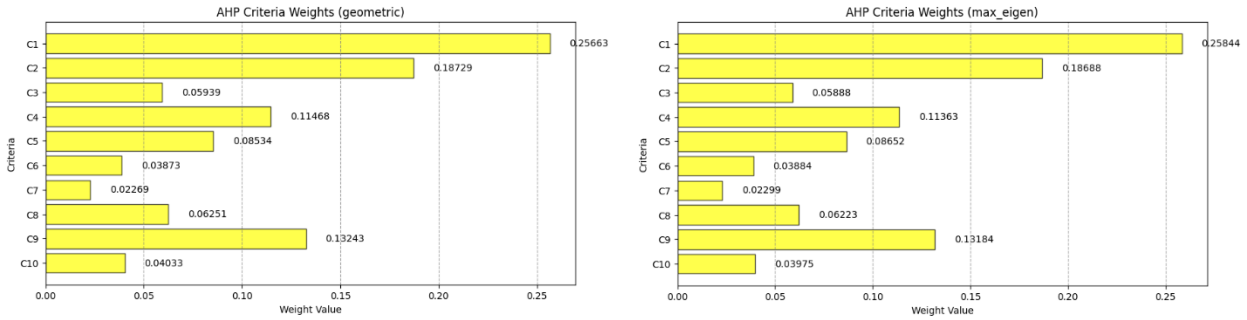

Figure 3. Qualitative results of AHP weights for different criteria from de-neutrosophied ICNS decision matrix.

In the following, we provide a visualization of ICNS-VIKOR analytical results by displaying S, R, and Q for each alternative in Figure 4. As shown, the plotted S values reflect the overall closeness of each alternative to the ideal solution, meanwhile, the R values symbolize the maximum regret of alternative w.r.t others. The Q values explain the final VIKOR ranking index. The implications of this analysis visually assess the relative performance of alternatives and support informed optimization of vocational student employment.

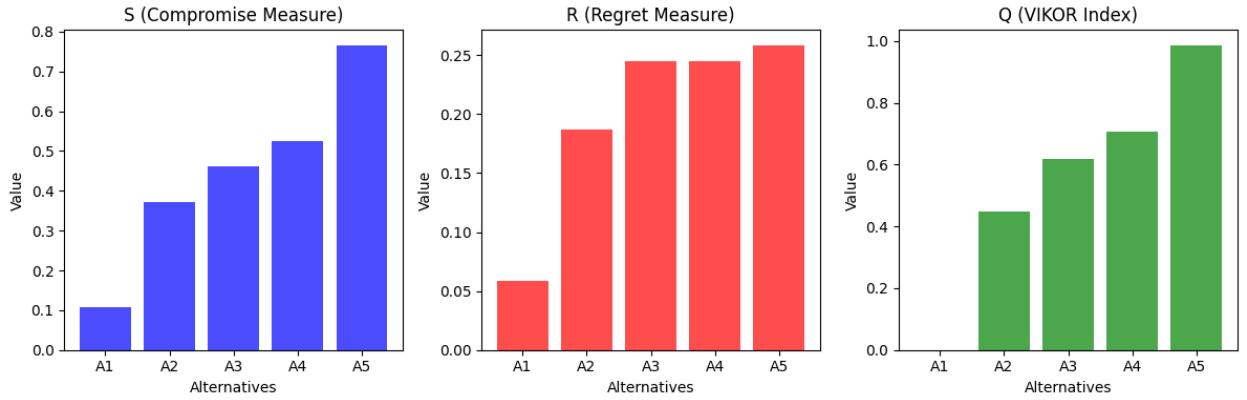

Figure 4. Qualitative results of comparing Compromise Measure, Regret Measure, and Q Index) for alternative vocational employment.

To validate the robustness of the decision-making process, we compare the AHP weights for different alternative rankings. The alternative ranking trends -under different values of the strategy coefficient - are visualized in Figure 5, which presents the prioritization of vocational student employment. This comparison ensures that our proposed ranking approach aligns with practical decision-making expectations.

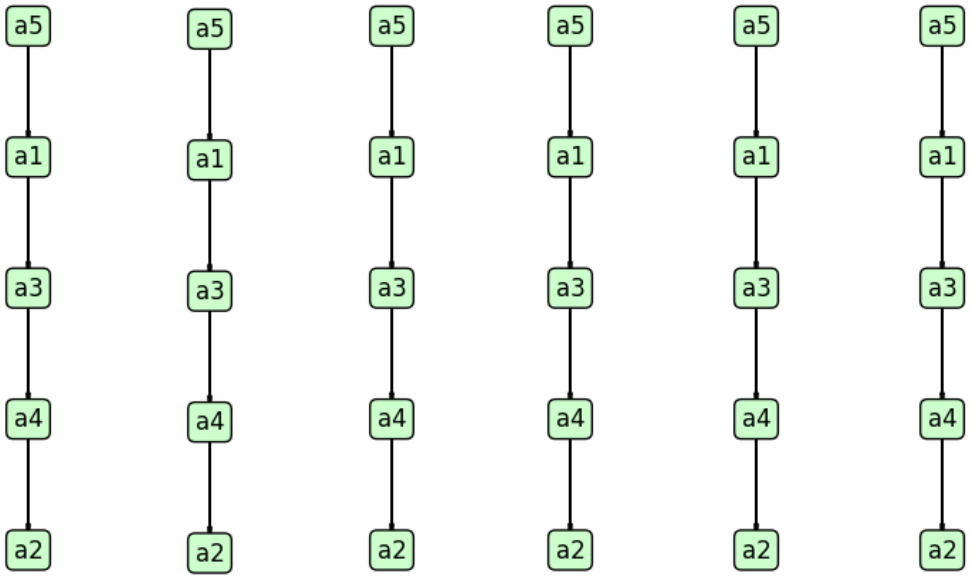

Figure 5. Qualitative ranking of alternatives from the proposed approach.

A comparative analysis of the proposed method against other MCDM techniques (such as ICNS-COPELAND [18], ICNS-CRADIS[19], and ICNS-LMAW [20]) is conducted to verify its effectiveness. The results, summarized in Table 4, demonstrate how the proposed ICNS-VIKOR framework performs compared to other approaches. The analysis highlights the accuracy, reliability, and adaptably of the model in uncertainty-driven decision environments.

Table 4: Comparative analysis of the proposed ICNS-VIKOR with previous methods for vocational student employment ranking.

	ICNS-VIKOR	**ICNS-COPELAND**	**ICNS-CRADIS**	**ICNS-LMAW**
Rank 1	A5	A3	A5	A5
Rank 2	A1	A5	A1	A1
Rank 3	A3	A4	A4	A4
Rank 4	A4	A1	A3	A3
Rank 5	A2	A2	A2	A2

To assess the stability of the decision-making process, a sensitivity analysis is performed. This analysis evaluates how variations in criteria weights affect the final ranking of alternatives. Table 5 presents the results of different weight perturbation scenarios, allowing us to determine the robustness of the ranking outcomes. The finding ensured that small changes in input parameters do not meaningfully alter the final decision, making the approach highly reliable for real-world applications.

Table 5: Sensitivity analysis showing the effect of varying criteria weights on alternative rankings.

	S-ranked	Q-ranked
+5%	a5 → a3 → a1 → a4 → a2	a5 → a1 → a3 → a4 → a2
+25%	a5 → a3 → a1 → a4 → a2	a5 → a1 → a3 → a4 → a2
+45%	a5 → a3 → a1 → a4 → a2	a5 → a1 → a3 → a4 → a2
-5%	a5 → a3 → a1 → a4 → a2	a5 → a1 → a3 → a4 → a2
-25%	a5 → a3 → a1 → a4 → a2	a5 → a1 → a3 → a4 → a2
-45%	a5 → a3 → a1 → a4 → a2	a5 → a1 → a3 → a4 → a2

Table 5: Sensitivity analysis showing the effect of varying criteria weights on alternative rankings (cont.…).

	R-ranked
+5%	a5 → a1 → a3 → a2 → a4
+25%	a5 → a1 → a3 → a2 → a4
+45%	a5 → a1 → a3 → a2 → a4
-5%	a5 → a1 → a3 → a2 → a4
-25%	a5 → a1 → a3 → a2 → a4
-45%	a5 → a1 → a3 → a2 → a4

5. Conclusion

This study proposed a Multi-Criteria Complex Neutrosophic Approach integrated for optimizing vocational student employment selection under uncertainty. With the integration of ICNS, we effectively managed uncertainty in optimizing the decision-making. A new de-neutrosophication process is presented to drive a crisp representation of uncertain data, while analytical weights are derived to ensure a structured prioritization of criteria. The results demonstrated the contribution of our framework, confirming its

efficiency, sustainability, and reliability. Comparative and sensitivity analyses validated the robustness and stability of the proposed approach against traditional MCDM techniques.

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Statistical Characterization of the Neutrosophic Moyal Distribution

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Abstract: The Moyal distribution is widely utilized in various fields due to its effectiveness in modeling heavy-tailed processes, such as ionization losses and certain risk assessments. In this work, we present a neutrosophic version of the Moyal distribution that extends the classical Moyal distribution while incorporating uncertainty and imprecision associated with its parameters. To better explore the neutrosophic structure, we provide the detailed derivation of some essential mathematical characteristics concerning all the properties that this distribution possesses. The proposed neutrosophic Moyal (NM) model is the one that allows it to work very well in situations where data is vague or undetermined. The proposed neutrosophic model can be effectively adopted when the Moyal distribution parameters are known and lie in the range or interval. We also develop the quantile function of the proposed model under an uncertain environment. A simulation study has been carried out to assess the adequacy of neutrosophic quantile function. Numerical results demonstrate that quantile function is quite efficient in terms of generating neutrosophic random data from the proposed model.

Keywords: Moyal model, neutrosophic probability, neutrosophic distribution, estimation, simulation

1. Introduction

Survival analysis and reliability are based on probability distributions, which describe the time-to-event data and assist researchers and engineers to calculate lifetime and survival probabilities of a system [1]. Statistical distributions have a long history in reliability and survival analysis [2]. The simplest and one of the most popular distributions used in reliability analysis is the exponential distribution [3]. It is primarily used for modelling electronic components and mechanical systems with a constant failure rate. Another basic model is the Weibull distribution which is commonly used in survival analysis, allowing for failure rates to be increasing, decreasing or constant by which its shape parameter [4]. This is particularly helpful when we model lifetimes of products or the progression of diseases. The gamma distribution is an extension of the exponential model, increasing failure time variability with the inclusion of a shape parameter that can allow for aging effects to be modeled in the system [5].

The log-normal distribution is a common model when the times to failure are influenced by multiple independent elements, which multiply their contributions to the total risk of failure [6]. It is commonly used in medical and biological survival studies where the data is right-skewed. Another right-skewed model that is especially useful in reliability testing during which failure rates accelerate exponentially over time is the inverse Gaussian distribution [7]. The normal distribution, while not used directly in survival analysis, provides an approximation of failure time distributions when lifetimes are symmetrically distributed [8]. This is inspired by the Gumbel distribution, part of the extreme value family; it best represents the breakdown of the weakest link in a system, where a component falling down initiates the collapse. Also, the Burr distribution is another candidate for heavy-tailed survival-data, meaning it can be used for high-variance lifetimes [9]. For survival data with non-monotonic hazard functions, that is when there is an initial increased risk of failure followed by a decreasing risk over time, we find the log-logistic distribution to be a generalizable option for modeling [10]. The gamma and inverse gamma distributions are common conjugate prior distributions for Bayesian survival analysis, allowing for the updating of prior beliefs about the rate of failure with new data. More sophisticated applications might use mixture distributions, in which multiple probability distributions are mixed to model highly heterogeneous populations with diverse failure modes. In reliability and survival analysis, choosing appropriate probability distributions is critical as it influences risk assessment, maintenance planning, and decision-making processes in engineering, health, and industrial applications.

The Moyal distribution, as described by Moyal [11], is a continuous probability distribution with particular applications in reliability analysis, stochastic processes, and survival studies. It is specifically applicable for modeling energy in ultrahigh-energy collisions and stochastic deviations, which is unlike the normal and heavy-tailed right distributions. The studies of Landau [12] and Möller [13] investigated the Moyal distribution as an approximation to the Landau distribution, particularly in cases where the computational performance is a crucial point. These studies showed that the Moyal distribution is a reasonable approximation for the tail of energy loss distributions so that it can be a valuable option to consider in particle physics simulations, finance, and engineering while discussing properties with other heavy tailed distributions. Zhang and Li [14] had introduced a generalized Moyal distribution that able to measure energy loss in high-energy particle collisions and yield superior fit to the standard Moyal distribution. Wang and Liu [15] used the Moyal distribution to fit heavy-tailed financial data, improving risk assessment and capturing extreme market events. The asymptotic behavior of the Moyal and Gumbel distributions in the context of extreme value theory was also analyzed by Smith and Johnson in [16], where applications in environmental science were discussed. Rodriguez et al. [17] utilized Moyal distribution to model energy loss in cosmic-ray detectors and compared it to the Landau distribution.

Neutrosophic set, proposed by Smarandache, is an extension of the classical set and fuzzy set, covering three components: truth (T), indeterminacy (I), and falsity (F), each allowed to be independent in the range of [0,1] [18]. It is an efficient abstraction of uncertainty, vagueness, and incomplete information and is applicable to a broad range of fields, including decision-making, artificial intelligence, and optimization problems [19]. Because the uncertainty is inevitable, neutrosophic sets have wide applications including medical diagnosis, image processing, risk assessment and machine learning and so on. Their flexibility allows for more precise reasoning about complex systems than classical paradigms, making them an effective means to handle ambiguous and contradictory data across a wide range of scientific and engineering domains [20].

Neutrosophic probabilistic distributions provide a natural representation encompassing the notion of indeterminacy, which makes them a fitting choice for modeling uncertainties, fluctuating decision-making or incomplete information in practical scenarios [21]. Unlike traditional probability

distributions that assign a precise probability to an event, neutrosophic distributions represent probabilities as estimates represented by triplets (T, I, F), where T stands for the degree of truth, I for the degree of indeterminacy, and F for the degree of falsity [22-24]. This perspective offers greater freedom in thinking about complex systems, more than a mere uncertain process [25]. Some classical probability distributions can be generalized to obtain such distributions such as neutrosophic normal, neutrosophic exponential and neutrosophic Weibull distributions [26]. These distributions are a common choice for various applications. Neutrosophic Weibull and neutrosophic exponential distributions are applied in engineering reliability to accommodate uncertain failure times and lifetimes of systems, allowing for more robust risk assessments [27]. In medical and survival analysis, neutrosophic distributions can be used to analyze the uncertain recovery time of patients and uncertain disease progression, and which will be beneficial in personalized treatment planning. In finance and economics, neutrosophic probability models incorporate the uncertainties of market fluctuations, investment risks, and economic conditions, providing a more realistic approach to decision making [28]. As applied to artificial intelligence and machine learning, these distributions assist probabilistic models by introducing the concept of undetermined values for instances that have missing data, driving towards better decision-making across large spaces and incomplete datasets [29]. Neutrosophic p-control and other statistical process control charts can achieve diversity in production management by reflecting uncertainty in quality control and industrial applications [30]. Neutrosophic distributions have applications within the environmental sciences, including climate modeling, where uncertainty in temperature trends, pollution levels, etc., can be better represented. When it comes to dealing with data that has not only uncertainty but also vagueness, neutrosophic probability distributions play a significant role in addressing a wide range of issues.

In this article, we introduce a new generalization of the Moyal distribution with the follow-up objective of broadening its field of applicability in areas like reliability description, survival analysis, and risk evaluation. The concept for introducing this generalization derives from the idea of neutrosophy, extending applications for dealing with indeterminate, inconsistent or vague data. The uncertainty inherent in study parameters is a key feature that cannot be overlooked in concrete analyzes and needs to be housed within a mathematical framework describing a system. To the best of our knowledge, there are no research articles published in the context of the Moyal model. This difference is a primary motivation for our work.

2. Classical Structure

In this section we will discuss the classical structure of the Moyal model. The Moyal distribution is a continuous probability distribution based on the work of José Enrique Moyal. It has the most utility in particle physics (high-energy particle physics, statistical mechanics) as a good model of energy loss and momentum distributions. It is a generalization of the normal distribution but has fatter tails.
The PDF and CDF of the Moyal distribution with given parameters can be defined as:

$$f(t;\alpha,\beta)=\frac{1}{\sqrt{2\pi\beta}}\exp\left(-\frac{1}{2}\left(\frac{t-\alpha}{\beta}+e^{-(t-\alpha)/\beta}\right)\right) \quad (1)$$

$$F(t;\alpha,\beta)=F(T)=\int_{-\infty}^{t}f(t)dt \quad (2)$$

Using the appropriate transformation, Eq(2) can be rewritten as:

$$F(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{T}\exp\left(-\frac{1}{2}(t+e^{-t})\right)dt. \quad (3)$$

Applying substitution Eq (3) further can be written as:

$$F(t)=\frac{1}{\sqrt{\pi}}\int_{e^{-T/2}}^{\infty}\frac{e^{-y}}{\sqrt{y}}dy$$

Recognizing this expression as incomplete gamma function we can further write as:

$$F(t) = \frac{\Gamma\left(\frac{1}{2}, \frac{e^{-t}}{2}\right)}{\Gamma\left(\frac{1}{2}\right)},$$

which is upper incomplete gamma so we can write Eq (3) as:

$$F(t) = 1 - \gamma\left(\frac{1}{2}, \frac{e^{-t}}{2}\right)$$

where $t = \frac{x-\alpha}{\beta}$ and $\gamma(a, x)$ is upper incomplete gamma distribution.

There is not a specific closed form for the CDF. Now the R program can be used to evaluate the value of CDF. The parameter α in the Moyal distribution is the location parameter, which translates the distribution left or right on the horizontal axis and determines where its mass is centered, although the mean is not equal to this value (due to the skewness). The parameter β regulates the distribution's scale the higher the β the more spread out the distribution, and lower values concentrate it closer to α. Combined, these parameters define the behavior of the distribution, which is why it suits well the modeling of asymmetric and heavy-tailed data in a wide range of scientific applications.

Figure 1 PDF and CDF of the Moyal distribution for certain distributional parameters

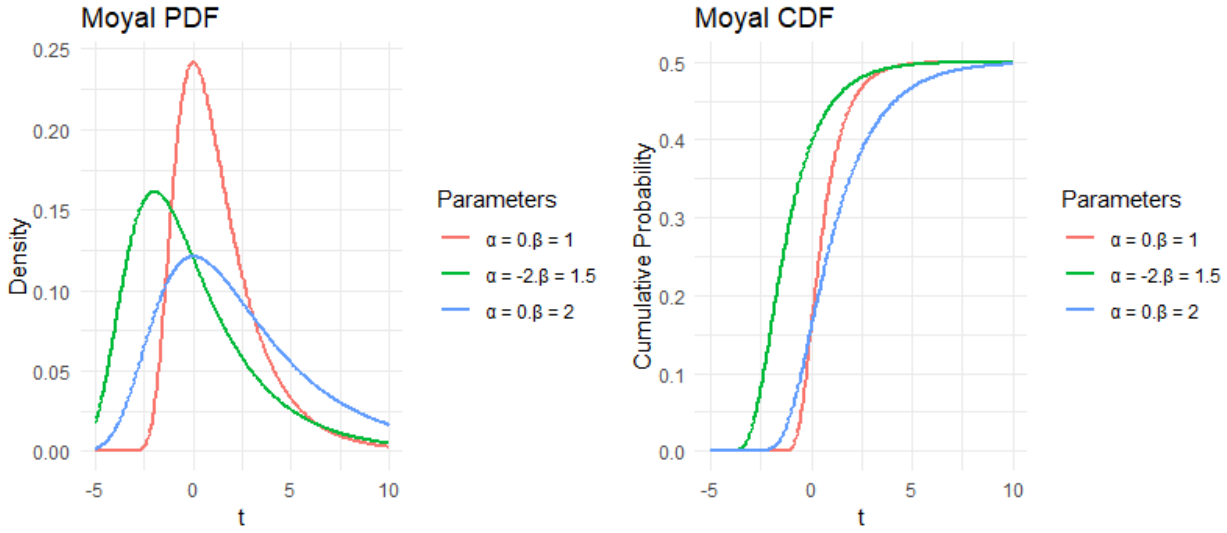

Figure 1 shows the basic plots of PDF and CDF of the Moyal distribution. The Moyal distribution is a significant probability distribution in various fields like physics, statistics, and engineering. Due to its ability to model varieties of asymmetric behavior and heavy tails, it can be useful in representing energy loss distributions in high energy physics, turbulent fluid flows, and some types of economic models. Because the probability density function (PDF) of the Moyal distribution has exponential and Gaussian-like properties, it is able to capture a number of aspects of skewed and long-tailed data that the normal distribution would otherwise be unfit to describe. While the CDF has no simple closed-form solution, it lets us understand the chances of the value being in a certain part of the space. Its flexibility encourages several applications in quantum mechanics, meteorology, and other fields, where the modeling of extreme events or non-Gaussian behaviors is required.

Similarly mean and variance of the Moyal distribution in classical context are given by:

$$E[T] = \alpha + \beta(\gamma + \ln 2) \quad (4)$$

where ($\gamma \approx 0.5772$) is the Euler-Mascheroni constant.

$$\mathrm{Var}(T) = \frac{\pi^2}{2}\beta^2 \quad (5)$$

The mean and variance of the distribution define key insights into the statistical properties of the Moyal distribution. Eq (3) indicates how the mean is displaced from the location parameter α. According to Eq(4), the variance scales quadratically with β. The appearance of $\frac{\pi^2}{2}$ in this variance is also indicative of the heavy-tailed character of the distribution, making it suitable for modeling skewed as well as extreme-value data found in a wide variety of different applications in physics and reliability analysis. More about shape of the Moyal distribution we need to study about skewness coefficient of the model which is given by:

$$\text{Skewness} = \frac{2\sqrt{2}\zeta(3)}{\pi^3} \approx 0.608 \quad (6)$$

where $\zeta(3)$ is the is Apéry's constant.

Kurtosis coefficient is another important measure about tail structure of the distribution which is given by:

$$\text{Kurtosis} = \frac{12}{5} \approx 2.4 \quad (7)$$

Now if assume the different values of Moyal's parameters we can calculate its basic characteristics which are given in Table 1.

Table 1 Basic characteristics of the Moyal distribution for different parameters setting

α	β	Mean	Variance	Skewness	Kurtosis
0	1	1.693	4.934	0.608	2.4
1	2	2.693	19.736	0.608	2.4
-2	1.5	1.433	11.153	0.608	2.4
3	0.5	2.265	0.493	0.608	2.4
5	2	4.693	19.736	0.608	2.4

Table 1 provides some basic properties of the Moyal model. The main statistical properties of the Moyal distribution for five pairs of α (location) and β parameters. For each pair, it computes a table with mean, variance, skewness, and kurtosis The mean is a function of α and β, while the variance is proportional to β. Moyal distribution parameters do not alter skewness and kurtosis values and maintain shape characteristics: a skewed shape with large tails useful in extreme value modeling. Now in the next section, we would overview the parallel properties of the Moyal distribution in neutrosophic structure.

3. Neutrosophic Moyal Model

In this section we will discuss the neutrosophic framework of the Moyal model. Let x be a random variable based on the neutrosophic Moyal standard probability density function (pdf) provided by:

$$f(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x + e^{-x})\right), \quad -\infty < x < \infty. \quad (8)$$

Using a location parameter $\alpha_n = [\alpha_l, \alpha_u]$ and a scale factor $\beta_n = [\beta_l, \beta_u]$, the random variable $x = \frac{z-\alpha_n}{\beta_n}$ has a neutrosophic Moyal (NM) distribution, denoted as $\mathcal{M}_O(\alpha_n, \beta_n)$

$$f(z) = \frac{1}{\sqrt{2\pi\beta_n}}\exp\left(-\frac{1}{2}\left(\frac{z-\alpha_n}{\beta_n} + e^{-\left(\frac{z-\alpha_n}{\beta_n}\right)}\right)\right), -\infty < z, \alpha_n < \infty, \beta_n > 0. \quad (9)$$

The lower incomplete gamma function $\Upsilon(\lambda, z) = \int_0^z t^{\lambda-1}e^{-t}dt$ determines the cumulative distribution function (cdf) that corresponds to (2). It is provided by

$$\mathcal{G}(z) = 1 - \frac{\Upsilon\left(\frac{1}{2}\frac{1}{2}e^{-\left(\frac{z-\alpha_n}{\beta_n}\right)}\right)}{\Gamma\left(\frac{1}{2}\right)}, \tag{10}$$

$$= 1 - \mathcal{P}\left(\frac{1}{2}, \frac{1}{2}e^{-x}\right) \tag{11}$$

where $\mathcal{P}$ is the incomplete gamma function.

By assuming the different values of neutrosophic parameters of the proposed model , PDF curves can be seen in Figure2.

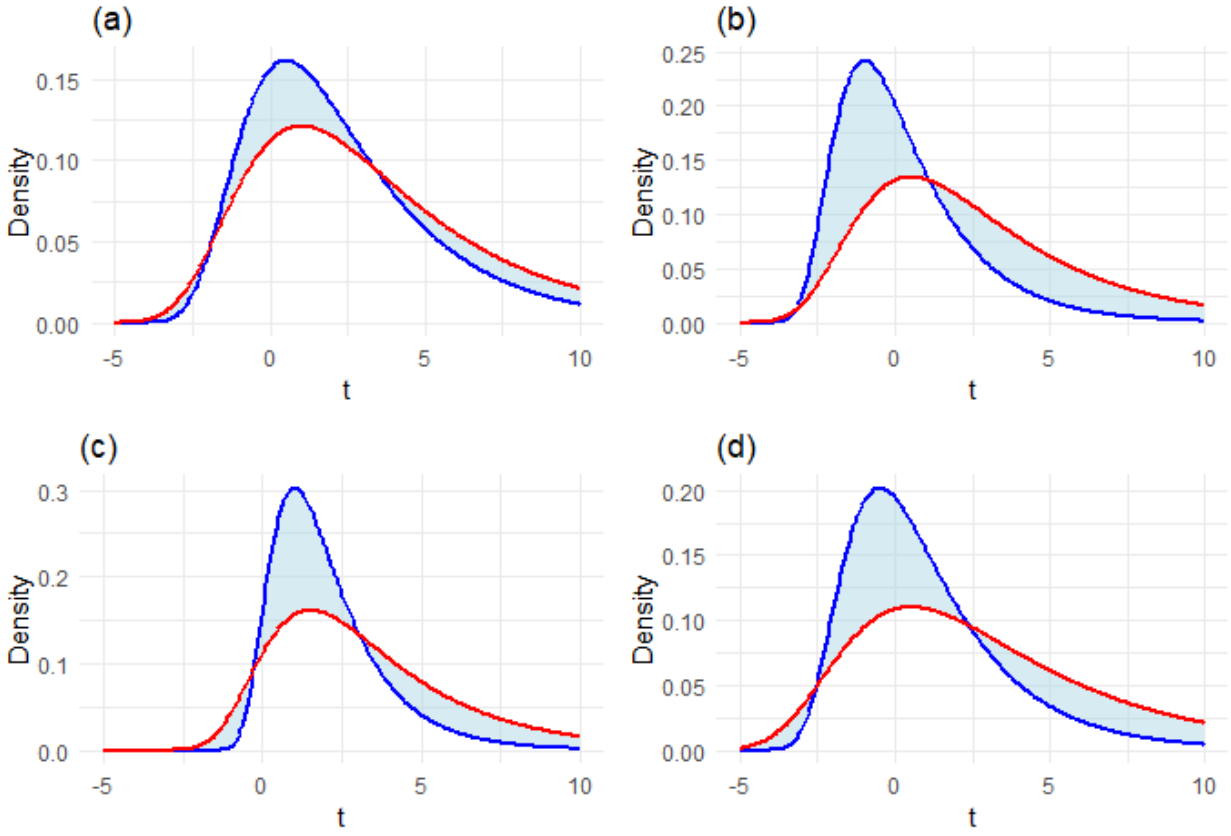

Figure 2 PDF curves of the model with (a) $\alpha_n = [0.5,1]$, $\beta_n = [1.5,2]$ (b) $\alpha_n = [-1,0.5]$, $\beta_n = [1,1.8]$(c) $\alpha_n = [1,1.5]$, $\beta_n = [0.8,1.5]$and (d) $\alpha_n = [-0.5,0.5]$, $\beta_n = [1.2,2.2]$

Figure 2 illustrates the PDF curves of the proposed model using the given neutrosophic values for the distributional parameters. Due to imprecision in the assumed parameter values, the curves appear as layers instead of single lines. The indeterminate region is highlighted with a shaded area between the curves. Similarly for same parameters setting CDF curves of the proposed model is shown in Figure 3

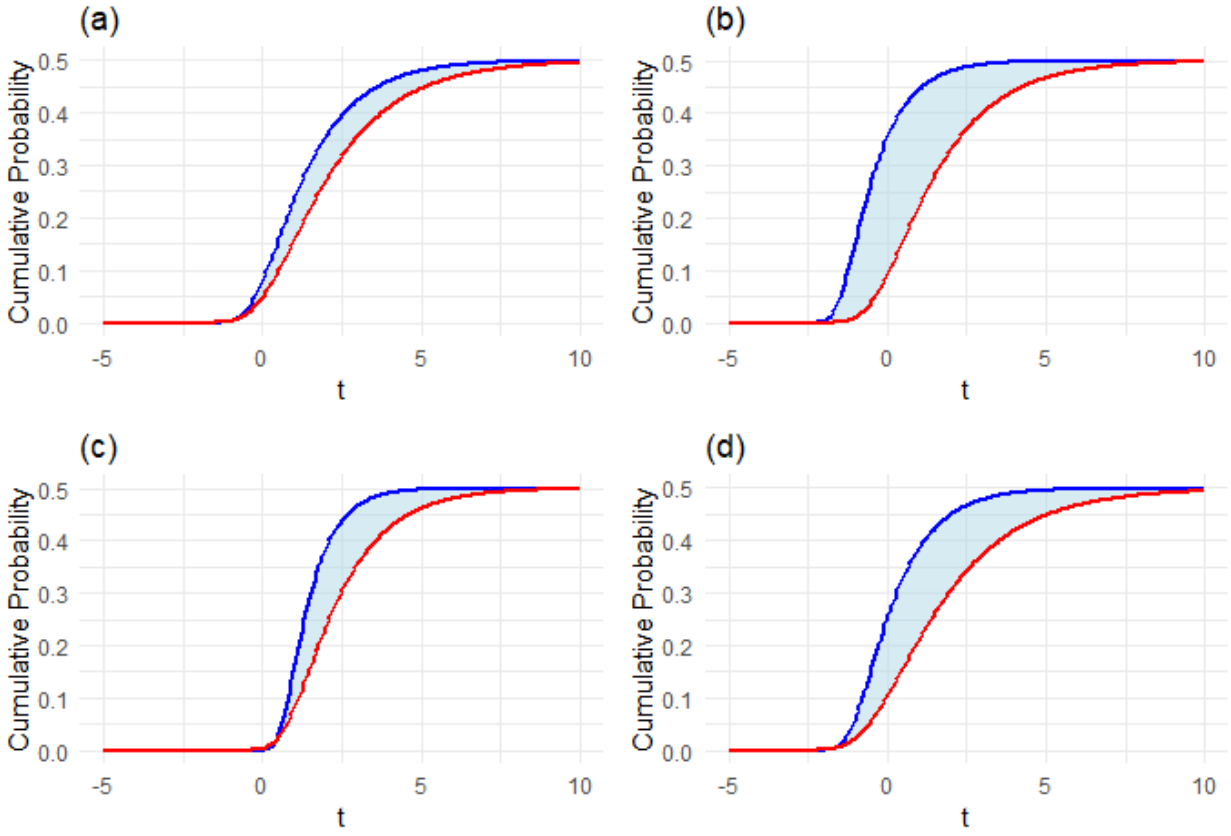

Figure 3 CDF curves of the model with (a) $\alpha_n = [0.5,1]$, $\beta_n = [1.5,2]$ (b) $\alpha_n = [-1,0.5]$, $\beta_n = [1,1.8]$(c) $\alpha_n = [1,1.5]$, $\beta_n = [0.8,1.5]$and (d) $\alpha_n = [-0.5,0.5]$, $\beta_n = [1.2,2.2]$

The CDF curves of the neutrosophic Moyal distribution are illustrated in Figure 3, demonstrating how the uncertainty in the distributional parameters affects the CDF curves. The CDF is not a single curve but a shaded region between two boundary curves. The presented neutrosophic region is covered by the wide shaded area, which means that we are not sure about the actual value, due to its neutrosophic uncertainty. The bounds are established by the lower curve and the upper curve, so the actual CDF lies between the two curves.

Definition 1. The mean of NMD is $\gamma + ln2$.

Proof: By definition the NMD is

$$E[x] = \int_{-\infty}^{\infty} xf(x)dx \tag{12}$$

$$E[x] = \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x + e^{-x})\right). \tag{13}$$

Define

$$z = e^{-x} \Rightarrow dz = -e^{-x}dx = -zdx$$

Thus,

$$dx = -\frac{dz}{z}$$

Rewrite x in term of z:

$$x = -lnz.$$

$$E[z] = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} -lnz\, e^{-\frac{1}{2}(-lnz+z)}\left(-\frac{dz}{z}\right)$$

Simplify

$E[z] = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} lnz\, e^{-\frac{1}{2}(-lnz+z)} \left(\frac{dz}{z}\right) = \sqrt{2\pi}(\gamma + ln2),$

where γ is the Euler-Mascheroni constant (0.5772).

Thus,

$E[z] = (\gamma + ln2)$. Hence proved.

Definition 2. The variance of NM distributino is $\frac{\pi^2}{2}.{\beta_n}^2$

Proof: By definition the NMD is $\frac{\pi^2}{2}$

$V(x) = E[x^2] - (E[x])^2$ (14)

First we compute $E[x^2]$:

$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x) dx$ (15)

Using integral technique

$E[x^2] = \frac{\pi^2}{2}{\beta_n}^2 + (\gamma + ln2)^2.$

Thus, the variance is;

$V(x) = \frac{\pi^2}{2}{\beta_n}^2 + (\gamma + ln2)^2 - (\gamma + ln2)^2$

$V(x) = \frac{\pi^2}{2}{\beta_n}^2\ hence\ proved.$

Definition 3: Show that the k^{th} raw moment of the NMD is $(n-1)!\,(2^n - 1)\xi_n$

Proof: Observing that the k^{th} moment of the NMD is:

$m_{\mathbf{1}} = -ln2 - \phi\left(\frac{1}{2}\right) = ln2 + \gamma$ (16)

$m_k = (-1)^k \phi^{(k-1)}\left(\frac{1}{2}\right) = (k-1)!\,(2^k - 1)\xi_k, for\ k \geq 2$ (17)

Using the cumulants we find the lower order moments and the coefficient of skewness and kurtosis to be

$\mu_1'' = E[x] = m_{\mathbf{1}} = ln2 + \gamma$

$\mu_2'' = V[x] = m_{\mathbf{2}} = \phi^{(1)}\left(\frac{1}{2}\right) = \frac{\pi^2}{2}$

$\mu_3'' = m_{\mathbf{3}} = -\phi^{(2)}\left(\frac{1}{2}\right) = 14\xi_3$

$\mu_4'' = m_{\mathbf{4}} + 3m_2^2 = \phi^{(3)}\left(\frac{1}{2}\right) + 3\phi^{(1)}\left(\frac{1}{2}\right)^2 = \frac{7\pi^4}{4}$

Basic on these moments skewness and kurtosis coefficients can be derived. However, we note that skewness and kurtosis coefficinets are independent of shape and location parameters so remain same as given above. Similarly other properties of the NM distribution under neutrosophic framework can be developed.

4 Quantile Function

In this section, we discuss the random generation and estimation procedure of the proposed model. The inverse cdf or the inverse transform method is a widely used technique to generate random samples from a specified probability distribution. It uses that fact that for any continuous random variable X with some F(x) there exists a way to sample from it, by drawing a uniform random variable and then applying the inverse cdf. This approach will work because the cdf "transforms" the probability range [0, 1] to the values of the random variable. First, uniform random values are generated and then, the inverse cdf is applied to transform these random values into the target distribution.

In this section, we would study about the quantile function of the proposed model and how it can be use to draw the random samples. If you use a quantile function, we can generate random samples from a proposed model. Based on inverse transform sampling, it gives the possibilities of converting random variables of uniform distribution into random variables of given distribution (in our case, with

inverse transform sampling, the Moyal distribution). This is crucial for Monte Carlo simulations, bootstrapping methods, and other methods that use random sampling to estimate statistical properties. The quantile function enables sampling from the distribution in an efficient manner which is crucial for simulations that do not have conveniently computable analytical solutions. Monte Carlo simulation is also commonly applied in risk analysis, financial modeling, and reliability engineering to model and simulate outcomes based on various probabilistic assumptions. The quantile function of the proposed model can be written as:

$$Q(p;\alpha_n,\beta_n) = \alpha_n - \beta_n \cdot \ln(2\,p^{-1}(2,1-p)) \quad (18)$$

where p^{-1} is the inverse of incomplete upper gamma function.

Monte Carlo simulation can be used to generate random samples from the quantile function of the neutrosophic Moyal distribution; this can be done by first generating uniform random numbers in the range [0,1]. Now, these uniform values represent corresponding probabilities. We can then take each of these probabilities to apply the quantile function over the Moyal distribution that will correspond to each probability. This means that it correctly transforms uniform random numbers into neutrosophic Moyal distributed ones. This process is repeated as many times as should be sampled, with the values simply stored. Subsequently, these generated samples can be applied in simulations, statistical modeling, or risk analysis in order to reflect the behavior of the Moyal distribution associated with neutrosophic parameters α_n and β_n, which offers a powerful approach to simulate results according to probabilistic assumptions. A program written in R language has been used to draw these random samples. The output of these random samples is provided in Table 2.

Table 2 Random samples generated from the proposed model

Random Samples					
[-0.831,0.043]	[-0.849, 0.813]	[0.04, 1.596]	[-1.688, 0.108]	[0.47, 1.203]	[-0.423, 0.907]
[-0.177, 0.781]	[-0.592, 0.362]	[-0.173, 0.011]	[-1.273, 0.162]	[-1.435, 0.084]	[0.158, 0.967]
[-2.678, 0.526]	[-1.883, 0.415]	[0.405, 0.615]	[-0.157, 0.023]	[-0.319, 0.151]	[0.159, 0.424]
[-1.786, 0.042]	[-0.042, 0.306]	[-1.143, 0.443]	[-1.544, 0.032]	[0.013, 0.706]	[0.331, 0.807]
[-1.635, -0.13]	[-0.821, 0.153]	[0.262, 0.468]	[-0.509, 0.447]	[-0.143, 1.631]	[-0.705, 0.232]

Table 2 indicates the random samples generated from proposed model with neutrosophic values $\alpha_n = [0,0]$ and $\beta_n = [0.5, 1.5]$. Indeterminacy in the neutrosophic parameters causes imprecise random values as shown in Table 2. These random sample values are generated via R program. If we consider zero imprecision in the simulated parameters, results of neutrosophic Moyal distribution match with classical model.

5 Conclusions

In this study, the neutrosophic Moyal (NM) distribution as an extension of the classical Moyal distribution to account for uncertainty and imprecision in its parameters has been presented. By incorporating the neutrosophic logic, essential mathematical properties of the proposed distribution have been established. Mathematical results demonstrated their suitability for modeling data characterized by vagueness or indeterminacy. The quantile function of the NM distribution has been derived, enabling efficient random data generation under uncertainty. A simulation study confirmed the adequacy of the proposed quantile function, highlighting its effectiveness in generating neutrosophic random samples. The findings suggest that the NM distribution can serve as a valuable tool in various applications where the classical Moyal distribution may be insufficient due to inherent data imprecision. Further work on applications and parameter estimation methods of NM model in generic instances would be worth investigating in future research.

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Conflicts of Interest: The authors declare no conflict of interest.

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Early Parkinson's Disease Detection and Classification using Machine Learning and Neutrosophic Sets

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Abstract:

Optimizing therapy and rehabilitation for Parkinson's disease (PD) requires early identification and precise evaluation of the illness's course. However, there is disagreement about the best way to use gait analysis to categorize the severity of motor symptoms and identify early-stage Parkinson's disease. The precision of machine learning (ML) models in identifying early and intermediate stages of Parkinson's disease was assessed in this study. Six ML models are used in this study for the prediction of PD. Different metrics are used to evaluate ML models such as accuracy, precision, recall, f1-score, and AUC score. Then we propose a multi-criteria decision-making methodology (MCDM) to evaluate the ML models and select the best one. Two MCDM methods are used such as CRITIC method to compute the criteria weights and the TOPSIS method to rank the alternatives. These methods are used under the bipolar neutrosophic sets (BNSs) to deal with uncertainty information. The results show the support vector machine is the best ML model for the prediction of PD.

Keywords: Parkinson's Disease; Machine Learning; Neutrosophic Numbers; Early Detection; Uncertainty.

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1. Introduction

Parkinson's Disease (PD) is a progressive neurodegenerative disorder that primarily affects motor functions due to the gradual loss of dopamine-producing neurons in the brain. Early diagnosis is crucial for managing symptoms and slowing disease progression, yet detecting PD in its initial stages remains a significant challenge. Traditional diagnostic methods rely on clinical assessments and neurological examinations, which may not always be accurate or feasible for

detecting the disease before visible motor symptoms appear. This limitation has driven the need for advanced computational techniques, particularly machine learning (ML) and artificial intelligence (AI), to aid in the early detection and classification of Parkinson's Disease[1], [2].

Emerging research has explored various biomarkers for PD detection, including voice abnormalities, handwriting patterns, gait disturbances, and neuroimaging data. These biomarkers provide valuable information that, when analyzed using computational models, can significantly improve diagnostic accuracy. Speech-based assessments, for example, have gained considerable attention due to their non-invasive nature and ease of implementation. Individuals with early-stage PD often exhibit subtle voice changes, such as reduced vocal intensity and increased speech irregularities, which can be quantified using ML algorithms[3], [4].

Machine learning models such as Support Vector Machines (SVM), Random Forest, and deep learning techniques like Convolutional Neural Networks (CNNs) have shown promise in detecting early-stage PD by analyzing structured patient data. These models are trained on datasets containing clinical features, speech patterns, or movement data, allowing them to identify hidden patterns indicative of the disease. Additionally, ensemble learning and hybrid approaches have further enhanced classification accuracy by combining multiple models' strengths to improve predictive performance[5], [6].

The integration of wearable sensors and mobile health technologies has also revolutionized Parkinson's detection by enabling continuous and real-time monitoring of patients. Smart devices equipped with accelerometers and gyroscopes can capture fine motor impairments, such as tremors and bradykinesia, which may go unnoticed in clinical settings. When combined with cloud-based AI analytics, these technologies provide a scalable and accessible approach for early PD detection, particularly in remote or underserved populations[7], [8].

Despite the promising advancements in AI-driven PD detection, several challenges remain. Computational complexity, data privacy concerns, and the need for high-quality labeled datasets are significant barriers to widespread clinical adoption. Moreover, ensuring model interpretability and reliability is critical, as clinicians must trust and understand AI-driven decisions. Future research must focus on refining existing models, incorporating multi-modal data fusion techniques, and validating AI-driven approaches through extensive clinical trials to ensure robust and accurate detection[9].

Early detection and classification of Parkinson's Disease using computational intelligence hold immense potential in transforming neurodegenerative disease diagnosis. By leveraging ML algorithms, wearable technology, and multi-modal data analysis, researchers can develop more effective diagnostic tools to identify PD at its earliest stages. While challenges persist, continued innovation in AI-driven healthcare solutions offers hope for improved patient outcomes and a better quality of life for those at risk of developing Parkinson's Disease[10], [11].

1.1 Neutrosophic Sets

Using a multi-criteria decision-making (MCDM) strategy to rank machine learning models offers a more realistic answer to this issue. When faced with the challenge of identifying or articulating preferences, as well as when decisions must be made based on several conflicting signs or elements that vie for significance, MCDM techniques are very helpful. Then, using the chosen criteria or alternatives, MCDM techniques assist in resolving conflicting issues and selecting the optimal answer. Choosing options with competing criteria is a difficulty that decision-makers encounter frequently. One specific MCDM issue that requires careful consideration of several conflicting intangible factors is the ranking of ML models[12], [13].

To put it briefly, MCDM techniques have been applied more and more to address a variety of decision-making issues, including ranking ML models. Few studies, meanwhile, have been able to develop a definitive MCDM approach, especially for ML model selection[14], [15].

The neutrosophic set is a subfield of neutrosophy, a field of philosophy that examines the nature, origin, and extent of neutralities as well as how they interact with other ideational spectra. Truth membership (T), indeterminacy membership (I), and falsity membership (F) are the three components of neutrosophic[16]. Every member value in the neutrosophic set is a real standard or non-standard. The single-valued neutrosophic set (SVNS), an extension of the neutrosophic set that takes the value from the subset of [0, 1], was then suggested by Wang et al. Numerous studies have focused specifically on the T, I, and F components, which ultimately led to the definition of a specific example of neutrosophic sets[17], [18].

2. Neutrosophic Numbers and Machine Learning

This section shows the steps of the neutrosophic approach to selecting the best ML models to be selected to predict Parkinson's disease. We use two MCDM methods such as CRITIC method to compute the criteria weights and the TOPSIS method to rank the ML models. These methods are used under the bipolar neutrosophic sets (BNSs). We show some definitions of BNSs such as:

Definition 1.

The bipolar neutrosophic sets (BNSs) can be defined as[19], [20]:

$$X = \left\{z, \left(T_X^+(z), I_X^+(z), F_X^+(z), T_X^-(z), I_X^-(z), F_X^-(z)\right) z \in Z\right\} \quad (1)$$

$$T_X^+(z), I_X^+(z), F_X^+(z): Z \to [0,1] \quad (2)$$

$$T_X^-(z), I_X^-(z), F_X^-(z): Z \to [-1,0] \quad (3)$$

Definition 2.

Let two bipolar neutrosophic numbers (BNNs) such as:

$X_1 = \{T_1^+(z), I_1^+(z), F_1^+(z), T_1^-(z), I_1^-(z), F_1^-(z)\}$ and $X_2 = \{T_2^+(z), I_2^+(z), F_2^+(z), T_2^-(z), I_2^-(z), F_2^-(z)\}$

$$X_1 \cup X_2 = \begin{pmatrix} \max\left(T_1^+(z), T_2^+(z)\right) \\ , \frac{I_1^+(z)+I_2^+(z)}{2}, \\ \min\left(F_1^+(z), F_2^+(z)\right), \min \left(T_1^-(z), T_2^-(z)\right) \\ , \frac{I_1^-(z)+I_2^-(z)}{2} \\ \max\left(F_1^-(z), F_2^-(z)\right) \end{pmatrix} \quad (4)$$

Definition 3.

We show operations of BNNs such as:

$$X_1 + X_2 = \begin{pmatrix} T_1^+(z) + T_2^+(z) - T_1^+(z)T_2^+(z), \\ I_1^+(z)I_2^+(z), \\ F_1^+(z)F_2^+(z), \\ -T_1^-(z)T_2^-(z), \\ -\begin{pmatrix} -I_1^-(z) - I_2^-(z) - \\ I_1^-(z)I_2^-(z) \end{pmatrix}, \\ -\begin{pmatrix} -F_1^-(z) - F_2^-(z) \\ -F_1^-(z)F_2^-(z) \end{pmatrix} \end{pmatrix} \quad (5)$$

$$X_1 X_2 = \begin{pmatrix} T_1^+(z)T_2^+(z), \\ I_1^+(z) + I_2^+(z) \\ -I_1^+(z)I_2^+(z) + \\ F_1^+(z) + F_2^+(z) \\ -F_1^+(z)F_2^+(z), \\ -\begin{pmatrix} -T_1^-(z) - T_2^-(z) \\ -T_1^-(z)T_2^-(z) \end{pmatrix}, \\ -I_1^-(z)I_2^-(z), \\ -F_1^-(z)F_2^-(z) \end{pmatrix} \quad (6)$$

$$\Xi X_1 = \begin{pmatrix} \left(1 - \begin{pmatrix} 1 - \\ T_1^+(z) \end{pmatrix}\right)^{\Xi}, \\ \left(I_1^+(z)\right)^{\Xi}, \\ \left(F_1^+(z)\right)^{\Xi}, \\ -\left(-\left(T_1^-(z)\right)^{\Xi}\right), \\ -\left(-\left(I_1^-(z)\right)^{\Xi}\right), \\ -\left(1 - \begin{pmatrix} 1 - \\ F_1^-(z) \end{pmatrix}\right)^{\Xi} \end{pmatrix} \quad (7)$$

$$a_1^{\Xi} = \begin{pmatrix} \left(T_1^+(z)\right)^{\Xi}, \\ \left(1-\left(1-I_1^+(z)\right)\right)^{\Xi}, \\ \left(1-\left(1-F_1^+(z)\right)\right)^{\Xi}, \\ -\left(1-\left(1-T_1^-(z)\right)\right)^{\Xi}, \\ -\left(-\left(I_1^-(z)\right)^{\Xi}\right) \\ -\left(-\left(F_1^-(z)\right)^{\Xi}\right) \end{pmatrix} \tag{8}$$

Definition 4.

The score function of BNSs is:

$$S(X_1) = \frac{\begin{pmatrix} T_1^+(z)+1-I_1^+(z) \\ +1-F_1^+(z)+ \\ 1+T_1^-(z) \\ -I_1^-(z)-F_1^-(z) \end{pmatrix}}{6} \tag{9}$$

We show the steps of the CRITIC method to compute the criteria weights.

Create the decision matrix.

The decision matrix is created as follows:

$$X = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{m1} & \cdots & z_{mn} \end{bmatrix}_{m\times n} ; i = 1, \dots, m; j = 1, \dots, n \tag{10}$$

Normalize the decision matrix

We normalize the decision matrix for beneficial and non-beneficial criteria such as:

$$y_{ij} = \frac{x_{ij}-\min x_{ij}}{\max x_{ij}-\min x_{ij}} \tag{11}$$

$$y_{ij} = \frac{x_{ij}-\max x_{ij}}{\min x_{ij}-\max x_{ij}} \tag{12}$$

Determine the correlation between the criteria F_{jk}

Determine the C index

$$C_j = \xi_j \sum_{k=1}^{n}\left(1-F_{jk}\right) \tag{13}$$

Where ξ_j refers to the standard deviation

Compute the criteria weights

$$W_j = \frac{C_j}{\sum_{j=1}^{n} C_j} \tag{14}$$

Then we show the steps of the TOPSIS method to rank the alternatives. The TOPSIS method starts with the decision matrix.

Normalization decision matrix

$$a_{ij} = \frac{z_{ij}}{\sum_{i=1}^{i} z_{ij}^2} \tag{15}$$

Determine the weighted normalized matrix

$$d_{ij} = w_j a_{ij} \tag{16}$$

Determine ideal and non-ideal solution

$$L^+ = \left\{ \begin{matrix} \max d_{ij} \\ \left(\min d_{ij}\ \right) \end{matrix} \right\} \tag{17}$$

$$L^- = \left\{ \begin{matrix} \min d_{ij} \\ \left(\max d_{ij}\ \right) \end{matrix} \right\} \tag{18}$$

Determine the separation measures

$$B_i^+ = \sum_{j=1}^{n} \left(d_{ij} - d_i^+\right)^2 \tag{19}$$

$$B_i^- = \sum_{j=1}^{n} \left(d_{ij} - d_i^-\right)^2 \tag{20}$$

Determine the relative closeness values

$$S_i = \frac{B_i^-}{B_i^- + B_i^+} \tag{21}$$

3. Implementation

This section shows the implementation of the proposed approach to show the best ML models on the Parkinson dataset.

3.1 Description of the Dataset

The Parkinson dataset is gathered from the Kaggle website to be analyzed in this study. Table 1 shows some information on the dataset. This dataset has 24 features and 195 rows. This dataset has no missing values. We show some statistical analysis of this dataset as shown in Table 2.

Table 1. Some information from the dataset.

	phon_R01_S01_1	phon_R01_S01_2	phon_R01_S01_3	phon_R01_S01_4	phon_R01_S01_5
MDVP:Fo(Hz)	119.992	122.4	116.682	116.676	116.014
MDVP:Fhi(Hz)	157.302	148.65	131.111	137.871	141.781
MDVP:Flo(Hz)	74.997	113.819	111.555	111.366	110.655
MDVP:Jitter(%)	0.00784	0.00968	0.0105	0.00997	0.01284
MDVP:Jitter(Abs)	0.00007	0.00008	0.00009	0.00009	0.00011
MDVP:RAP	0.0037	0.00465	0.00544	0.00502	0.00655
MDVP:PPQ	0.00554	0.00696	0.00781	0.00698	0.00908
Jitter:DDP	0.01109	0.01394	0.01633	0.01505	0.01966

MDVP:Shimmer	0.04374	0.06134	0.05233	0.05492	0.06425
MDVP:Shimmer(dB)	0.426	0.626	0.482	0.517	0.584
Shimmer:APQ3	0.02182	0.03134	0.02757	0.02924	0.0349
Shimmer:APQ5	0.0313	0.04518	0.03858	0.04005	0.04825
MDVP:APQ	0.02971	0.04368	0.0359	0.03772	0.04465
Shimmer:DDA	0.06545	0.09403	0.0827	0.08771	0.1047
NHR	0.02211	0.01929	0.01309	0.01353	0.01767
HNR	21.033	19.085	20.651	20.644	19.649
status	1	1	1	1	1
RPDE	0.414783	0.458359	0.429895	0.434969	0.417356
DFA	0.815285	0.819521	0.825288	0.819235	0.823484
spread1	-4.81303	-4.07519	-4.44318	-4.1175	-3.74779
spread2	0.266482	0.33559	0.311173	0.334147	0.234513
D2	2.301442	2.486855	2.342259	2.405554	2.33218
PPE	0.284654	0.368674	0.332634	0.368975	0.410335

Table 2. Some statistics analysis of the Parkison disease dataset.

	count	mean	std	min	25%	50%	75%	max
MDVP:Fo(Hz)	195	154.2286	41.39007	88.333	117.572	148.79	182.769	260.105
MDVP:Fhi(Hz)	195	197.1049	91.49155	102.145	134.8625	175.829	224.2055	592.03
MDVP:Flo(Hz)	195	116.3246	43.52141	65.476	84.291	104.315	140.0185	239.17
MDVP:Jitter(%)	195	0.00622	0.004848	0.00168	0.00346	0.00494	0.007365	0.03316
MDVP:Jitter(Abs)	195	0.000044	0.000035	0.000007	0.00002	0.00003	0.00006	0.00026
MDVP:RAP	195	0.003306	0.002968	0.00068	0.00166	0.0025	0.003835	0.02144
MDVP:PPQ	195	0.003446	0.002759	0.00092	0.00186	0.00269	0.003955	0.01958
Jitter:DDP	195	0.00992	0.008903	0.00204	0.004985	0.00749	0.011505	0.06433
MDVP:Shimmer	195	0.029709	0.018857	0.00954	0.016505	0.02297	0.037885	0.11908
MDVP:Shimmer(dB)	195	0.282251	0.194877	0.085	0.1485	0.221	0.35	1.302
Shimmer:APQ3	195	0.015664	0.010153	0.00455	0.008245	0.01279	0.020265	0.05647
Shimmer:APQ5	195	0.017878	0.012024	0.0057	0.00958	0.01347	0.02238	0.0794
MDVP:APQ	195	0.024081	0.016947	0.00719	0.01308	0.01826	0.0294	0.13778
Shimmer:DDA	195	0.046993	0.030459	0.01364	0.024735	0.03836	0.060795	0.16942
NHR	195	0.024847	0.040418	0.00065	0.005925	0.01166	0.02564	0.31482
HNR	195	21.88597	4.425764	8.441	19.198	22.085	25.0755	33.047
status	195	0.753846	0.431878	0	1	1	1	1
RPDE	195	0.498536	0.103942	0.25657	0.421306	0.495954	0.587562	0.685151
DFA	195	0.718099	0.055336	0.574282	0.674758	0.722254	0.761881	0.825288
spread1	195	-5.6844	1.090208	-7.96498	-6.4501	-5.72087	-5.04619	-2.43403
spread2	195	0.22651	0.083406	0.006274	0.174351	0.218885	0.279234	0.450493
D2	195	2.381826	0.382799	1.423287	2.099125	2.361532	2.636456	3.671155
PPE	195	0.206552	0.090119	0.044539	0.137451	0.194052	0.25298	0.527367

Fig 1 shows the number of rows in each class of the dataset. The first class has 147 samples and the second class has 48 samples.

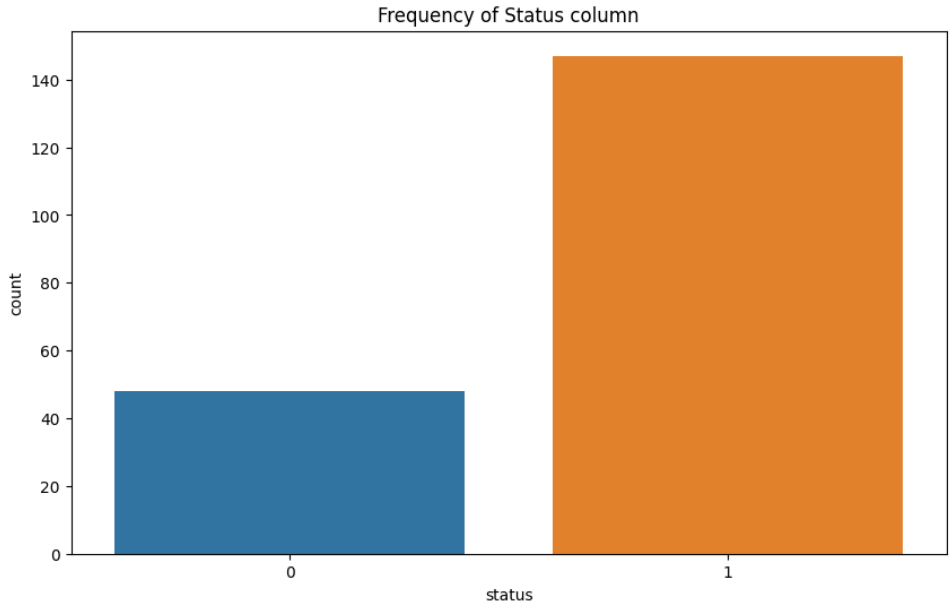

Fig 1. Samples of each class.

3.1 Metrics Measure

We can evaluate the ML models by different metrics evaluation such as accuracy, precision, recall, f1-score, and AUC score.

$$Accuracy = \frac{TP+TN}{TP+TN+FP+FN}$$

$$Precision = \frac{TP}{TP+FP}$$

$$Recall = \frac{TP}{TP+FN}$$

$$F1-Score = 2 * \frac{Precision*Recall}{Precision+Recall}$$

TP refers to true positive, TN refers to true negative, FP refers to false positive, and FN refers to false negative.

4. Analysis

In this section, we apply six ML models on the Parkinson's disease dataset such as Logistic Regression (LR), k-nearest Neighbor (KNN), Support Vector Machine (SVM), Naïve Bayes (NB), Decision Tree (DT), and Random Forest. Table 3 shows the comparison analysis of the six ML models.

Table 3. Comparative analysis of ML models.

	Accuracy	Precision	Recall	F1-Score	AUC Score
LR	0.820512821	0.9	0.870968	0.885246	0.747984
KNN	0.769230769	0.923077	0.774194	0.842105	0.762097

SVM	0.8974359	0.88571	1	0.93939	0.75
NB	0.61538462	1	0.51613	0.68085	0.75806
DT	0.74358974	0.95652	0.70968	0.81481	0.79234
RF	0.82051282	0.9	0.87097	0.88525	0.74798

To show the best ML models we use the MCDM methodology under the BNSs to deal with uncertainty information. We use the CRITIC methodology to compute the criteria weights and select the best criterion. Then we use the TOPSIS methodology to select the best ML models based on the criteria weights. Fig 2 shows the criteria and alternatives of this study.

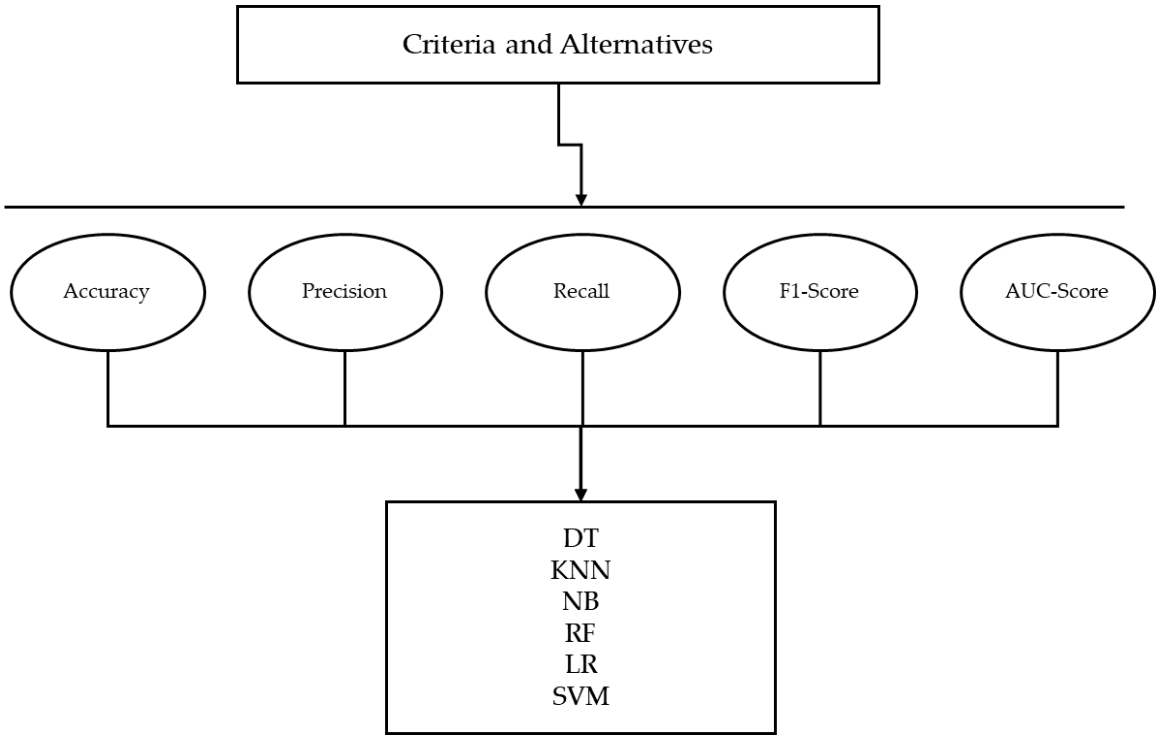

Fig 2. Set of criteria and alternatives.

Firstly, we apply the steps of the CRITIC methodology to compute the criteria weights, then we apply the steps of the TOPSIS methodology to show the best ML models.

Four experts and decision-makers are evaluating the criteria and alternatives to create the decision matrix using BNNs as shown in Table 4. Then we use Eq. (9) to obtain the crisp values. Then we combine the decision matrix.

Eq. (11) is used to normalize the decision matrix using equations. (11 and 12) as shown in table 5.

Eq. (13) is used to determine the correlation between the criteria F_{jk} as shown in table 6.

Then we determine the C index using eq. (13).

Then we compute the criteria weights using eq. (14) as shown in Fig 3.

Table 4. The decision matrix.

	BNC_1	BNC_2	BNC_3	BNC_4	BNC_5
BNA_1	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)
BNA_2	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)
BNA_3	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)
BNA_4	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)
BNA_5	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)
BNA_6	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)
	BNC_1	BNC_2	BNC_3	BNC_4	BNC_5
BNA_1	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)
BNA_2	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)
BNA_3	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)
BNA_4	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)
BNA_5	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)
BNA_6	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)
	BNC_1	BNC_2	BNC_3	BNC_4	BNC_5
BNA_1	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)
BNA_2	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)
BNA_3	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)
BNA_4	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)
BNA_5	(0.1,0.4,0.3,-0.1,-0.2,-0.3)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)
BNA_6	(0.1,0.4,0.4,-0.1,-0.3,-0.5)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.7,0.3,0.2,-0.4,-0.2,-0.1)
	BNC_1	BNC_2	BNC_3	BNC_4	BNC_5
BNA_1	(0.7,0.3,0.2,-0.4,-0.2,-0.1)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)
BNA_2	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.1,0.4,0.3,-0.1,-0.2,-0.3)
BNA_3	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)
BNA_4	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.6,0.4,0.4,-0.3,-0.2,-0.3)	(0.1,0.4,0.4,-0.1,-0.3,-0.5)
BNA_5	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)
BNA_6	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.4,0.1,0.4,-0.1,-0.2,-0.5)	(0.5,0.4,0.3,-0.4,-0.3,-0.3)	(0.4,0.3,0.3,-0.1,-0.2,-0.3)

Table 5. Normalized decision matrix.

	BNC_1	BNC_2	BNC_3	BNC_4	BNC_5
BNA_1	0.25	0.333333	0.125	0	0.444444
BNA_2	1	1	0.75	0.090909	0
BNA_3	0.5	0.166667	0	0.090909	0.444444
BNA_4	0	0.333333	0.375	0.454545	0.833333
BNA_5	0.416667	0	1	0.636364	0.833333
BNA_6	0.25	0.222222	0.375	1	1

Table 6. The correlation matrix.

	BNC_1	BNC_2	BNC_3	BNC_4	BNC_5
BNC_1	1	0.67387	0.3665	-0.39723	-0.80372
BNC_2	0.67387	1	0.1548	-0.44222	-0.78124
BNC_3	0.3665	0.1548	1	0.346198	0.030031
BNC_4	-0.39723	-0.44222	0.346198	1	0.836291
BNC_5	-0.80372	-0.78124	0.030031	0.836291	1

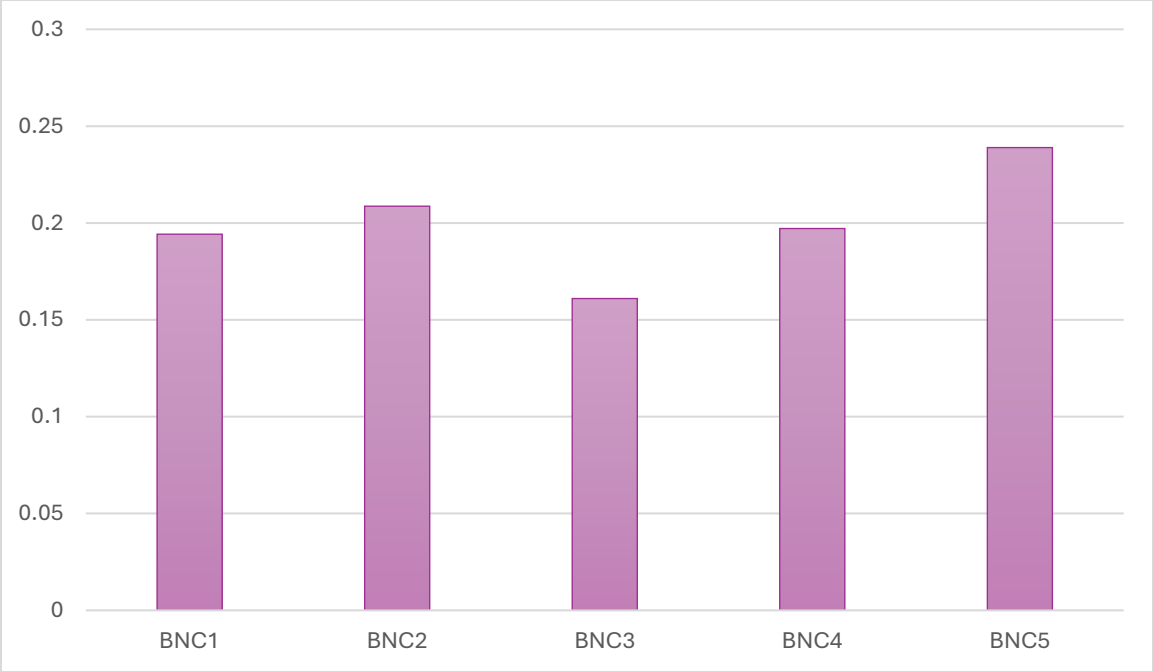

Fig 3. The importance of criteria.

Then we apply the steps of the TOPSIS methodology.

Eq. (15) is used for normalization decision matrix as shown in Table 7.

Eq. (16) is used to determine the weighted normalized matrix as shown in Table 8.

Then we determine the ideal and non-ideal solution

Then we determine the separation measures

Then we determine the relative closeness values using Eq. (21). Then we rank the alternatives as shown in Fig 4.

Table 7. Normalization matrix.

	BNC_1	BNC_2	BNC_3	BNC_4	BNC_5
BNA_1	0.052573651	0.053082501	0.053284	0.052243	0.053038
BNA_2	0.056420504	0.058178421	0.055504	0.052682	0.049502
BNA_3	0.053855935	0.051808521	0.05284	0.052682	0.053038
BNA_4	0.051291367	0.053082501	0.054172	0.054438	0.056131
BNA_5	0.053428507	0.050534541	0.056392	0.055316	0.056131
BNA_6	0.052573651	0.052233181	0.054172	0.057072	0.057457

Table 8. Weighted decision matrix.

	BNC_1	BNC_2	BNC_3	BNC_4	BNC_5

BNA_1	0.010210779	0.011077008	0.008579	0.0103	0.012673
BNA_2	0.010957909	0.012140401	0.008936	0.010386	0.011829
BNA_3	0.010459822	0.01081116	0.008508	0.010386	0.012673
BNA_4	0.009961735	0.011077008	0.008722	0.010732	0.013413
BNA_5	0.010376808	0.010545311	0.009079	0.010905	0.013413
BNA_6	0.010210779	0.010899776	0.008722	0.011252	0.01373

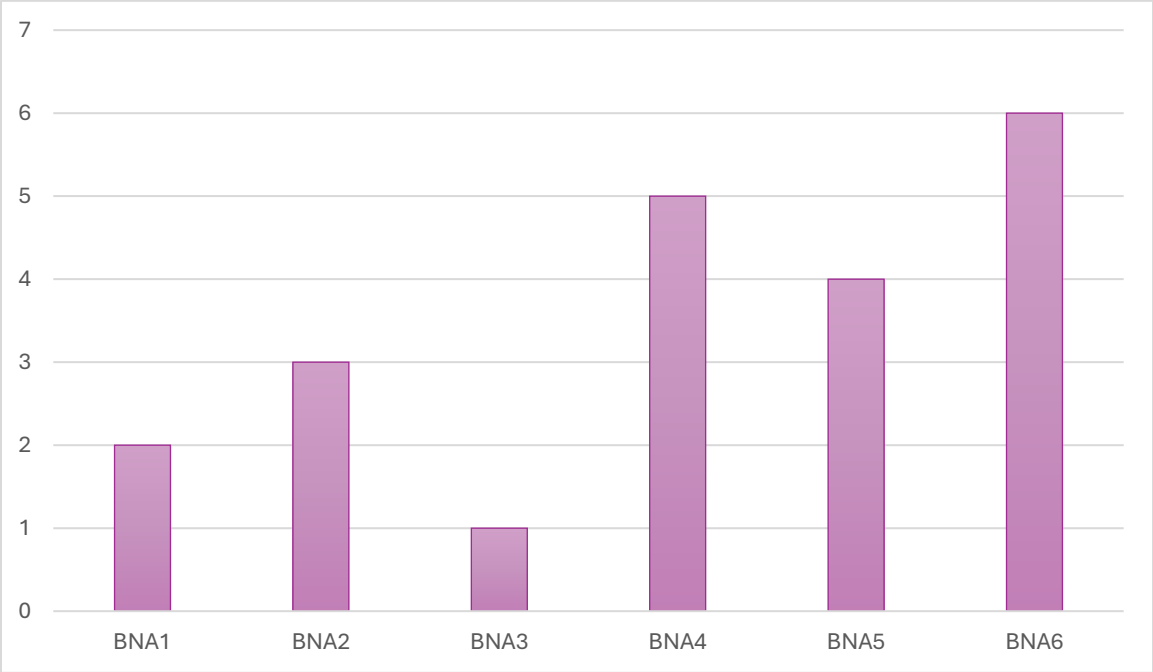

Fig 4. The rank of the alternatives.

5. Conclusions

This study proposed a ML model with the neutrosophic sets for PD prediction and classification. We used six ML models for the prediction of the PD such as LR, SVM, RF, DT, KNN, and NB. We applied these ML models to the PD dataset. The dataset has 24 features and 195 samples. These samples are divided into two classes. Then we used the MCDM methodology to select the best ML models based on the different metrics measures such as accuracy, precision, recall, f1-score, and AUC score. Two MCDM methods are used in this study, such as CRITIC method to compute the criteria weights and the TOPSIS method is used to rank the ML models. These methods are used under the bipolar neutrosophic sets to deal with uncertainty information. The results show the SVM is the best ML model in this study.

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SuperHyperSoft Set with MCDM Methods for Assessment of Ideological and Political Education in Colleges and Universities Under the Background of Big Data

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Abstract: The integration of big data into ideological and political education (IPE) in colleges and universities has reshaped traditional teaching methodologies, offering new ways to analyze student behavior, personalizing learning experiences, and enhance educational outcomes. As digital transformation accelerates, the need for an effective evaluation system to assess the impact of big data-driven IPE becomes essential. This paper uses the multicriteria decision making (MCDM) methodology for evaluating this problem. Two MCDM methods are used such as CRITIC methodology to compute the criteria weights and the MABAC methodology to rank the alternatives. We use the SuperHyperSoft set to deal with criteria and sub criteria. This study uses eight criteria and seven alternatives. We introduce the ranking of alternatives into four HyperSoft sets and in each set, we obtain the rank of alternatives.

Keywords: SuperHyperSoft Set; Ideological and Political Education; Big Data; Ideological and Political Education.

1. Introduction

In the digital age, the role of big data in higher education has become increasingly significant, revolutionizing the way knowledge is delivered and assessed. Among the various fields impacted, ideological and political education (IPE) has witnessed substantial transformation, shifting from conventional lecture-based instruction to data-driven, interactive, and adaptive learning models. Colleges and universities are now leveraging artificial intelligence, cloud computing, and real-time analytics to tailor educational content, monitor student engagement, and predict learning outcomes. However, despite these advancements, there remains a pressing need to establish an effective evaluation system that can measure the success of these new teaching methodologies and ensure that IPE continues to fulfill its mission of fostering critical thinking, civic responsibility, and ethical leadership among students[1], [2]. Evaluating the effectiveness of IPE under the background of big data requires a multi-dimensional approach.

Traditional evaluation methods, such as student surveys and academic assessments, no longer suffice in capturing the complexity of modern educational environments. Instead, a comprehensive framework that integrates quantitative and qualitative factors—such as curriculum relevance, teaching quality, student engagement, technological integration, and big data application—is necessary to provide a holistic view of IPE's impact[3], [4]. Through big data analytics, educators can track learning behaviors, measure knowledge retention, and refine teaching methods based on real-time feedback, thereby enhancing the precision and effectiveness of IPE delivery.

Despite its potential, the implementation of big data-driven IPE evaluation is not without challenges. Ethical concerns regarding student data privacy, digital literacy gaps among faculty, and the risk of excessive reliance on automated assessments pose significant obstacles. Additionally, the effectiveness of big data applications depends on the quality of the data collected, the analytical models used, and the extent to which educators and administrators are trained to interpret and utilize these insights[5], [6]. Therefore, a systematic and structured approach to evaluating IPE effectiveness must address both the technological advantages and the practical limitations associated with big data integration in education.

To bridge this gap, this study proposes an evaluation framework based on Multi-Criteria Decision-Making (MCDM) methodologies to systematically rank and assess various big data-driven IPE strategies. By applying these methods, institutions can identify the most effective educational approaches, optimize resource allocation, and refine their curricula to meet the evolving needs of students. Moreover, the results of this research can serve as a guide reference for educational policymakers, university administrators, and instructors in enhancing IPE quality through data-driven insights[7], [8].

Ultimately, as digital transformation continues to shape higher education, the need for a robust and adaptable evaluation mechanism becomes more crucial than ever. By integrating big data analytics with structured evaluation frameworks, universities can ensure that ideological and political education remains relevant, impactful, and aligned with the broader educational objectives of nurturing informed, responsible, and engaged citizens. This study aims to contribute to the ongoing discourse on IPE modernization by offering a strategic roadmap for optimizing its effectiveness in the era of big data[9], [10].

2. SuperHyperSoft-CRITIC-MABAC

We introduce the concept of SuperHyperSoft set with two MCDM methods such as CRITIC method to compute the criteria weights and ranking the alternatives.

SuperHyperSoft Set

Smarandache presents the SuperHyperSoft set which is an extension of HyperSoft set and has different HyperSoft sets. We use the SuperHyperSoft set to deal with different criteria values in the ranking the alternatives[11], [12].

Let U a universe discourse and K is a non-empty set; we can define the power set as P(K). let we have different criteria such as E_1, E_2, E_3 and their powerset can be defined as $P(E_1), P(E_2), P(E_3)$ [13], [14]

We can define the SuperHyperSoft set as:

$$F: P(E_1) \times P(E_2) \times P(E_3) \to P(K) \tag{1}$$

CRITIC Method

We use the CRITIC method in this study to compute the criteria weights[15], [16].

Create the decision matrix.

$$Y = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{m1} & \cdots & y_{mn} \end{pmatrix}_{m \times n} ; i = 1, \dots, m; j = 1, \dots, n \tag{2}$$

Normalize the decision matrix.

The decision matrix is normalized based on beneficial criteria and non-beneficial criteria such as:

$$x_{ij} = \frac{y_{ij} - y_i^-}{y_i^+ - y_i^-} \tag{3}$$

$$x_{ij} = \frac{y_{ij} - y_i^+}{y_i^- - y_i^+} \tag{4}$$

$$\begin{pmatrix} y_i^+ = \max(y_1, \dots . y_m) \\ y_i^- = \min(y_1, \dots . y_m) \end{pmatrix} \tag{5}$$

Determine the correlation between the criteria β_{jk} .

Compute the standard deviation for each criterion $\aleph_j$

Compute the C index

$$C_j = \aleph_j \sum_{k=1}^{n} (1 - \beta_{jk}) \tag{6}$$

Compute the criteria weights.

$$w_j = \frac{C_j}{\sum_{j=1}^{n} C_j} \tag{7}$$

MABAC Method

This study uses the MABAC method to rank the alternatives[17], [18].

Normalize the decision matrix

$$u_{ij} = \frac{y_{ij} - y_i^-}{y_i^+ - y_i^-} \tag{8}$$

$$u_{ij} = \frac{y_{ij} - y_i^+}{y_i^- - y_i^+} \quad (9)$$

Compute the weighted normalized decision matrix.

$$r_{ij} = w_j + w_j u_{ij} \quad (10)$$

Compute the border approximation area matrix.

$$t_j = \left(\prod_{i=1}^{m} r_{ij}\right)^{\frac{1}{m}} \quad (11)$$

Compute the distance from the t_j values.

$$d_j = r_{ij} - t_j \quad (12)$$

Compute the total distance

$$S_i = \sum_{j=1}^{n} d_{ij} \quad (13)$$

3. Case Study

This section shows the case study of this paper by implementing the proposed approach. This study uses eight criteria and seven alternatives such as: Curriculum Relevance(Highly Relevant, Weakly Relevant), Technology Integration (Advanced, Basic), Practical Impact on Students (Significant, Moderate, Insignificant), Student Engagement (High, Medium, Low), Big Data Application (Extensive, Adequate, Minimal), Evaluation and Feedback Mechanisms (Comprehensive, Standard, Limited), Teaching Quality (Excellent, Good, Average, Poor), Faculty Competency (Highly Competent, Competent, Needs Improvement). In each criterion, there are different values to present it.

We create the decision matrix using Eq. (2) between the criteria and alternatives. Three experts are invited to evaluate the criteria and alternatives. They used scale between 0.1 to 0.9.

We normalize the decision matrix using Eq. (3) as shown in Table 1.

We determine the correlation between the criteria β_{jk}.

We compute the standard deviation for each criterion $\aleph_j$

We compute the C index using Eq. (6).

We compute the criteria weights using Eq. (7) as shown in Table 2.

Table 1. Normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	-0.04375	-1	-0.39083	-1	-0.64242	-0.1369	0	-0.461
A_2	0	0	0	-0.65524	-1	-0.34395	-0.07253	-1
A_3	-1	-0.07018	-0.84779	-0.41379	-0.38054	-1	-1	-0.61517

A_4	-0.29852	-0.139	-1	-0.24391	-0.14279	0	-0.96583	0
A_5	-0.40403	-0.31994	-0.28209	-0.24391	-0.59522	-0.11408	-0.48502	-0.25586
A_6	-0.51059	-0.36155	-0.56519	0	0	-0.29802	-0.65586	0
A_7	-0.59549	-0.22259	-0.4344	0.343832	-0.57108	-0.29801	-0.10419	-0.61517

Table 2. Criteria weights.

Criteria	Weights
C_1	0.11176
C_2	0.1319272
C_3	0.12239
C_4	0.143527
C_5	0.122404
C_6	0.091427
C_7	0.158943
C_8	0.117621

We use the SuperHyperSoft set to divide the set of criteria values into different HyperSoft set such as: (Highly Relevant, Weakly Relevant), (Advanced, Basic), (Significant), (High), (Extensive), (Comprehensive), (Excellent), (Highly Competent).

First HyperSoft Set:

(Highly Relevant), (Advanced), (Significant), (High), (Extensive), (Comprehensive), (Excellent), (Highly Competent)

Second HyperSoft Set:

(Highly Relevant), (Basic), (Significant), (High), (Extensive), (Comprehensive), (Excellent), (Highly Competent)

Third HyperSoft Set:

(Weakly Relevant), Advanced), (Significant), (High), (Extensive), (Comprehensive), (Excellent), (Highly Competent)

Fourth HyperSoft Set:

(Weakly Relevant), (Basic), (Significant), (High), (Extensive), (Comprehensive), (Excellent), (Highly Competent)

Then we apply MABAC in each HyperSoft set.

First HyperSoft Set.

Eq. (8) is used to normalize the decision matrix as shown in Table 3.

Eq. (10) is used to compute the weighted normalized decision matrix as shown in Table 4.

Eq. (11) is used to compute the border approximation area matrix.

Eq. (12) is used to compute the distance from the t_j values as shown in Table 5.

Eq. (13) is used to compute the total distance.

Table 3. Normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.04375	1	0.390828	1	0.64242	0.136901	0	0.460998
A_2	0	0	0	0.743448	1	0.343948	0.072528	1
A_3	1	0.070183	0.847788	0.56378	0.380539	1	1	0.615171
A_4	0.29852	0.139002	1	0.437363	0.142785	0	0.965833	0
A_5	0.404034	0.319945	0.282087	0.437363	0.595215	0.114084	0.485021	0.25586
A_6	0.510595	0.361549	0.565192	0.25586	0	0.298019	0.655857	0
A_7	0.595486	0.222591	0.434397	0	0.571076	0.298014	0.104195	0.615171

Table 4. Weighted normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.11665	0.263854	0.170223	0.287054	0.201039	0.103944	0.158943	0.171844
A_2	0.11176	0.131927	0.12239	0.250232	0.244808	0.122874	0.170471	0.235242
A_3	0.223521	0.141186	0.226151	0.224444	0.168983	0.182855	0.317887	0.189978
A_4	0.145123	0.150265	0.24478	0.2063	0.139881	0.091427	0.312456	0.117621
A_5	0.156915	0.174137	0.156915	0.2063	0.19526	0.101858	0.236034	0.147715
A_6	0.168825	0.179625	0.191564	0.18025	0.122404	0.118675	0.263187	0.117621
A_7	0.178312	0.161293	0.175556	0.143527	0.192306	0.118674	0.175504	0.189978

Table 5. Distance values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	-0.03674	0.096182	-0.00962	0.077489	0.024487	-0.0134	-0.06622	0.009425
A_2	-0.04163	-0.03574	-0.05746	0.040667	0.068256	0.005533	-0.05469	0.072823
A_3	0.070134	-0.02649	0.046303	0.01488	-0.00757	0.065514	0.092723	0.027559
A_4	-0.00826	-0.01741	0.064932	-0.00326	-0.03667	-0.02591	0.087293	-0.0448
A_5	0.003529	0.006465	-0.02293	-0.00326	0.018708	-0.01548	0.010871	-0.0147
A_6	0.015438	0.011953	0.011716	-0.02931	-0.05415	0.001334	0.038024	-0.0448
A_7	0.024926	-0.00638	-0.00429	-0.06604	0.015754	0.001333	-0.04966	0.027559

Second HyperSoft Set.

Eq. (8) is used to normalize the decision matrix as shown in Table 6.

Eq. (10) is used to compute the weighted normalized decision matrix as shown in Table 7.

Eq. (11) is used to compute the border approximation area matrix.

Eq. (12) is used to compute the distance from the t_j values as shown in Table 8.

Eq. (13) is used to compute the total distance.

Table 6. Normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.04375	1	0.390828	1	0.64242	0.136901	0	0.460998
A_2	0	0.239888	0	0.743448	1	0.343948	0.072528	1
A_3	1	0	0.847788	0.56378	0.380539	1	1	0.615171
A_4	0.29852	0.520368	1	0.437363	0.142785	0	0.965833	0
A_5	0.404034	0.746629	0.282087	0.437363	0.595215	0.114084	0.485021	0.25586
A_6	0.510595	0.280161	0.565192	0.25586	0	0.298019	0.655857	0
A_7	0.595486	0.426875	0.434397	0	0.571076	0.298014	0.104195	0.615171

Table 7. Weighted normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.117518	0.282506	0.164584	0.288811	0.198659	0.107814	0.146301	0.177258
A_2	0.112592	0.175138	0.118336	0.251763	0.24191	0.127448	0.156912	0.242653
A_3	0.225184	0.141253	0.218659	0.225819	0.166983	0.189663	0.292602	0.195963
A_4	0.146203	0.214757	0.236671	0.207563	0.138225	0.094831	0.287604	0.121326
A_5	0.158083	0.246717	0.151717	0.207563	0.192949	0.10565	0.21726	0.152369
A_6	0.170081	0.180827	0.185218	0.181353	0.120955	0.123093	0.242254	0.121326
A_7	0.179639	0.201551	0.16974	0.144406	0.190029	0.123092	0.161545	0.195963

Table 8. Distance values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	-0.03701	0.080988	-0.00931	0.077964	0.024197	-0.0139	-0.06095	0.009722
A_2	-0.04194	-0.02638	-0.05555	0.040916	0.067448	0.005739	-0.05034	0.075118
A_3	0.070656	-0.06027	0.044769	0.014971	-0.00748	0.067953	0.085348	0.028428
A_4	-0.00832	0.013239	0.062781	-0.00328	-0.03624	-0.02688	0.08035	-0.04621
A_5	0.003555	0.045199	-0.02217	-0.00328	0.018487	-0.01606	0.010006	-0.01517
A_6	0.015553	-0.02069	0.011328	-0.02949	-0.05351	0.001383	0.035	-0.04621
A_7	0.025111	3.24E-05	-0.00415	-0.06644	0.015567	0.001383	-0.04571	0.028428

Third HyperSoft Set.

Eq. (8) is used to normalize the decision matrix as shown in Table 9.

Eq. (10) is used to compute the weighted normalized decision matrix as shown in Table 10.

Eq. (11) is used to compute the border approximation area matrix.

Eq. (12) is used to compute the distance from the t_j values as shown in Table 11.

Eq. (13) is used to compute the total distance.

Table 9. Normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.210384	1	0.390828	1	0.64242	0.136901	0	0.460998
A_2	0.105192	0	0	0.743448	1	0.343948	0.072528	1
A_3	0	0.070183	0.847788	0.56378	0.380539	1	1	0.615171
A_4	0.351175	0.139002	1	0.437363	0.142785	0	0.965833	0
A_5	1	0.319945	0.282087	0.437363	0.595215	0.114084	0.485021	0.25586
A_6	0.666688	0.361549	0.565192	0.25586	0	0.298019	0.655857	0
A_7	0.438504	0.222591	0.434397	0	0.571076	0.298014	0.104195	0.615171

Table 10. Weighted normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.169237	0.231134	0.169447	0.267068	0.187566	0.11723	0.149977	0.178176
A_2	0.154529	0.115567	0.121832	0.232809	0.228402	0.138579	0.160854	0.24391
A_3	0.139821	0.123678	0.225119	0.208818	0.157659	0.206227	0.299953	0.196978
A_4	0.188923	0.131631	0.243663	0.191937	0.130507	0.103114	0.294829	0.121955
A_5	0.279642	0.152542	0.156199	0.191937	0.182175	0.114877	0.222718	0.153158
A_6	0.233038	0.15735	0.19069	0.1677	0.114201	0.133843	0.24834	0.121955
A_7	0.201133	0.141291	0.174755	0.133534	0.179419	0.133843	0.165603	0.196978

Table 11. Distance values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	-0.02111	0.084255	-0.00958	0.072094	0.022846	-0.01511	-0.06248	0.009773
A_2	-0.03582	-0.03131	-0.0572	0.037836	0.063682	0.00624	-0.05161	0.075507
A_3	-0.05053	-0.0232	0.046092	0.013844	-0.00706	0.073888	0.087493	0.028575
A_4	-0.00143	-0.01525	0.064636	-0.00304	-0.03421	-0.02923	0.082368	-0.04645
A_5	0.089292	0.005663	-0.02283	-0.00304	0.017455	-0.01746	0.010258	-0.01524
A_6	0.042688	0.010471	0.011663	-0.02727	-0.05052	0.001504	0.035879	-0.04645
A_7	0.010783	-0.00559	-0.00427	-0.06144	0.014698	0.001503	-0.04686	0.028575

Fourth HyperSoft Set.

Eq. (8) is used to normalize the decision matrix as shown in Table 12.

Eq. (10) is used to compute the weighted normalized decision matrix as shown in Table 13.

Eq. (11) is used to compute the border approximation area matrix.

Eq. (12) is used to compute the distance from the t_j values as shown in Table 14.

Eq. (13) is used to compute the total distance.

Table 12. Normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.363133	0.294418	0.390828	1	0.64242	0.136901	0	0.460998
A_2	1	0.404824	0	0.743448	1	0.343948	0.072528	1
A_3	0.423855	0	0.847788	0.56378	0.380539	1	1	0.615171
A_4	0.787711	0.33122	1	0.437363	0.142785	0	0.965833	0
A_5	0.696866	0.220813	0.282087	0.437363	0.595215	0.114084	0.485021	0.25586
A_6	0.060722	0.297094	0.565192	0.25586	0	0.298019	0.655857	0
A_7	0	1	0.434397	0	0.571076	0.298014	0.104195	0.615171

Table 13. Weighted normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.141784	0.346854	0.153904	0.238062	0.145678	0.098066	0.133978	0.130621
A_2	0.208026	0.376439	0.110656	0.207524	0.177394	0.115926	0.143695	0.178811
A_3	0.1481	0.267962	0.204469	0.186138	0.12245	0.172515	0.267956	0.144405
A_4	0.185946	0.356716	0.221313	0.17109	0.101362	0.086258	0.263379	0.089405
A_5	0.176496	0.327131	0.141871	0.17109	0.141491	0.096098	0.19896	0.112281
A_6	0.110329	0.347571	0.173198	0.149486	0.088697	0.111964	0.221849	0.089405
A_7	0.104013	0.535923	0.158725	0.119031	0.13935	0.111964	0.147938	0.144405

Table 14. Distance values.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	-0.00741	-0.01161	-0.0087	0.064264	0.017744	-0.01264	-0.05582	0.007164
A_2	0.058835	0.017977	-0.05195	0.033726	0.04946	0.00522	-0.0461	0.055354
A_3	-0.00109	-0.0905	0.041864	0.012341	-0.00548	0.061809	0.078159	0.020948
A_4	0.036754	-0.00175	0.058707	-0.00271	-0.02657	-0.02445	0.073582	-0.03405
A_5	0.027305	-0.03133	-0.02073	-0.00271	0.013557	-0.01461	0.009163	-0.01118
A_6	-0.03886	-0.01089	0.010593	-0.02431	-0.03924	0.001258	0.032052	-0.03405
A_7	-0.04518	0.177461	-0.00388	-0.05477	0.011416	0.001258	-0.04186	0.020948

Then we obtain the rank of the alternatives under each HyperSoft set as shown in Table 15.

Table 15. Rank of alternatives.

	First HyperSoft	Second HyperSoft	Third HyperSoft	Fourth HyperSoft	Final
A_1	6	6	6	3	6
A_2	4	3	3	7	3
A_3	7	7	7	6	7
A_4	5	5	4	5	5

A_5	3	4	5	2	4
A_6	2	1	2	1	1
A_7	1	2	1	4	2

4. Conclusions

The evaluation of ideological and political education in colleges and universities under the background of big data presents both opportunities and challenges. While data-driven insights can significantly enhance personalized learning experiences and teaching efficiency, their effectiveness depends on a well-structured evaluation system that considers key factors such as curriculum relevance, student engagement, and technological integration. This study underscores the necessity of Multi-Criteria Decision-Making (MCDM) methodologies to systematically assess and rank educational strategies, ensuring that institutions adopt the most effective approaches.

We used the SuperHyperSoft set to deal with criteria and sub criteria. This study used eight criteria and seven alternatives to be evaluated. We divided the ranking of the alternatives into four HyperSoft sets. In each HyperSoft set we obtain the ranking of the alternatives. Then we obtain the final ranks of the alternatives. Two MCDM methods are used such as CRITIC method to compute the criteria weights and the MABAC method to rank the alternatives.

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Multi-Criteria Decision-Making Algorithm based on HyperSoft Set for Civil Litigation Efficiency Evaluation in the Context of Artificial Intelligence

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Abstract: Artificial Intelligence (AI) is increasingly being integrated into civil litigation to enhance efficiency, reduce costs, and improve decision-making accuracy. This study evaluates the effectiveness of AI-driven tools in civil litigation efficiency using a multi-criteria decision-making (MCDM) framework. Key evaluation criteria include case processing speed, judicial decision accuracy, cost reduction, transparency, accessibility, and AI adoption rates. Two MCDM methods are used in this study such as LMAW to compute the criteria weights and the CoCoSo method to rank the alternatives. We use the concept of the HyperSoft set to deal with various criteria and sub criteria in the evaluation problem. We conducted a case study with eight criteria and ten alternatives to show the validation of the proposed approach. The sensitivity analysis is conducted to show the stability of the proposed approach.

Keywords: HyperSoft Set; CoCoSo Method; Artificial Intelligence; Civil Litigation.

1. Introduction

The rapid advancements in Artificial Intelligence (AI) have significantly transformed various industries, including the legal sector. In the realm of civil litigation, AI-powered tools are being integrated to improve efficiency, streamline legal processes, and reduce the burden on courts. Traditional litigation is often characterized by lengthy case proceedings, high costs, and complex legal research, all of which can hinder access to justice. With AI-driven innovations such as predictive analytics, natural language processing (NLP), and automated document review systems, the landscape of civil litigation is evolving toward greater efficiency and transparency[1], [2]. Evaluating the effectiveness of AI in civil litigation efficiency is crucial to understanding its potential, limitations, and impact on legal systems. One of the key benefits of AI in civil litigation is its ability to accelerate case processing times. AI-driven case management systems and automated legal research tools can reduce the time required for attorneys and judges to analyze documents, identify relevant precedents, and make informed decisions[3], [4].

Moreover, AI-powered predictive analytics can assess case outcomes based on historical data, enabling litigants to make better strategic decisions before proceeding to court. However, while these advancements contribute to efficiency, concerns about accuracy, bias, and ethical considerations remain significant challenges that require continuous evaluation and refinement.

Cost reduction is another major advantage associated with AI-driven litigation systems. Automated document review, AI-assisted contract analysis, and virtual legal assistants can help reduce the workload of legal professionals, thereby lowering litigation costs for both clients and the judiciary. Small and medium-sized law firms, which often struggle with resource constraints, can particularly benefit from these technologies. However, the adoption of AI tools varies across legal jurisdictions, with some courts embracing AI-driven solutions while others remain skeptical about their reliability and fairness[5], [6]. This disparity in implementation necessitates a comprehensive evaluation of AI's impact on litigation efficiency across different legal systems.

The integration of AI in civil litigation also raises important questions regarding transparency and fairness. While AI systems can analyze large volumes of legal data and identifying patterns, they may inadvertently reinforce existing biases if trained on flawed datasets. Ensuring that AI-driven legal tools align with judicial ethics, procedural fairness, and data privacy regulations is essential for maintaining public trust in the legal system. Moreover, the need for human oversight in AI-assisted legal decision-making remains crucial to mitigate risks associated with algorithmic errors and bias.

Given these evolving dynamics, conducting a systematic evaluation of AI's role in civil litigation efficiency is imperative. This study employs a multi-criteria decision-making (MCDM) framework to assess AI-driven litigation tools based on factors such as processing speed, cost-effectiveness, accuracy, transparency, accessibility, and user adoption[7], [8]. By examining these criteria, this research aims to provide insights into how AI can be optimally integrated into legal proceedings, ensuring a balanced approach that enhances efficiency while upholding justice and fairness.

Several groups of academics from a variety of fields, including supply chain and manufacturing, business and management, energy, and construction, have expanded their study on MCDM. To handle a decision-making problem, MCDM offers a variety of techniques with various algorithms and structures. This type of decision-making issue involves several variables (criteria) and alternatives (choices, solutions). The function of each MCDM approach is to generate the weights of the criteria and the alternatives (score or ranking). Depending on their needs and the intricacy of the challenge, experts and researchers from various fields would use such techniques. This area of business analytics has made significant contributions to the solution of issues including ranking, sorting, classification, and assessment.

To identify the best answer to multi-criteria decision-making problems, a variety of decision-making techniques have been created during the past three decades. The MCDM literature

demonstrates the rapidly expanding use of CoCoSo. Experts from a variety of fields attempt to use CoCoSo because of its simple algorithm and ease of use[9], [10].

The main contributions of this study are organized as follows:

We propose an MCDM method to compute the criteria weights and ranking the alternatives.

We use The LMAW method to compute the criteria weights and the CoCoSo method to rank the alternatives.

We use the HyperSoft Set to deal with various values of alternatives.

Sensitivity analysis is conducted to show the stability of ranks of the alternatives under different cases and criteria.

2. LMAW-CoCoSo Model

This section shows the steps of the LMAW with the CoCoSo methodology. We used the LMAW methodology to compute the criteria weights and the CoCoSo methodology to rank the alternatives.

Steps of the LMAW methodology are organized as follows[11], [12]:

Prioritize the criteria based on the opinions of experts and decision makers.

Define the absolute anti-ideal point x_{AIP}

$$x_{AIP} = \frac{x_{min}}{S} \tag{1}$$

S is a number greater than the base of logarithm (A).

Determine the relation between the elements of priority vector and x_{AIP}.

$$r = \frac{x_{cn}}{x_{AIP}} \tag{2}$$

Where x_{cn} refers to the value from the relation vector.

Compute the criteria weights.

$$w_j = \frac{\log_A r}{\log_A b} \tag{3}$$

$$b = \prod_{j=1}^{n} r \tag{4}$$

Steps of the CoCoSo method.

By merging concepts from compromised solutions, such as power weight aggregation and mean evaluation weighting, CoCoSo begins to identify the best option. Here is an interpretation of the CoCoSo step-by-step solution:

This method's first step is to formulate the decision issue, which includes identifying the criteria, alternatives, weights for each criterion, and the direction of optimization. After that, we ought to create a preliminary choice matrix. Accessing a data collection or decision-making preference (using linguistic values) should be used to carry out this activity[13], [14].

Normalizing the matrix is the second phase, which is accomplished using two methodologies for the benefit and non-benefit categories of criteria, such as product pricing or energy use, among others.

CoCoSo calculates two methods to aggregate the weights of the criteria in the decision-making process. The first is to add up the normalized matrix's multiplication by the weight values (S), and the second is to add up the normalized matrix's power weight (P).

We must combine the S and P variables in this phase. We compute a total of relative WSM and WPM scores in relation to the best, then we compute the arithmetic mean of the sums of WSM and WPM scores. The balanced compromise of the WSM and WPM model scores is released.

The U values are used to establish the alternatives' final ranking.

To help decision experts confirm the results, CoCoSo's applicability is compared to other comparable tools or sensitivity analysis tests. The CoCoSo implementation's mathematical formulation is shown here.

Determine the decision matrix

$$x_{ij} = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}; i = 1, \dots, m; j = 1, \dots, n \tag{5}$$

Normalize the decision matrix

The decision matrix is normalized for beneficial and non-beneficial criteria such as:

$$q_{ij} = \frac{x_{ij} - \min_i x_{ij}}{\max_i x_{ij} - \min_i x_{ij}} \tag{6}$$

$$q_{ij} = \frac{\max_i x_{ij} - x_{ij}}{\max_i x_{ij} - \min_i x_{ij}} \tag{7}$$

Compute the power weight of decision matrix and total weighted decision matrix

$$S_i = \sum_{j=1}^{n} w_j q_{ij} \tag{8}$$

$$P_i = \sum_{j=1}^{n} \left(q_{ij}\right)^{w_j} \tag{9}$$

Compute the relative weights of the alternatives based on a set of strategies.

$$U_{ia} = \frac{P_i + S_i}{\sum_{i=1}^{m} (P_i + S_i)} \tag{10}$$

$$U_{ib} = \frac{S_i}{\min\limits_i S_i} + \frac{P_i}{\min\limits_i P_i} \tag{11}$$

$$U_{ic} = \frac{\varphi(S_i)+(1-\varphi)(P_i)}{\left(\varphi \max\limits_i S_i+(1-\varphi) \max\limits_i P_i\right)} \; 0 \le \varphi \le 1 \tag{12}$$

Rank the alternatives based on

$$U_i = (U_{ia}U_{ib}U_{ic})^{\frac{1}{3}} + \frac{1}{3}(U_{ia}+U_{ib}+U_{ic}) \tag{13}$$

HyperSoft Set

Smarandache extends the soft set by introducing the HyperSoft Set to deal with various criteria and sub criteria. HyperSoft set can transform the single-criterion function into a multi-criteria function[15], [16].

Let U and L be a universe of discourse and non-empty set in U. The powerset of U is a P(L).

Let $d_1, d_2, \dots, d_n \; where \; n \ge 1$ be a distinct criteria and their values can be represented as $D_1, \dots D_n \; with \; D_i \cap D_j = \emptyset \; for \; i \ne j \; and \; i, j \in 1,2, \dots, n$

The pair $(F, D_1 \times D_2 \times D_3 \times \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \times D_n)$ where $D_1 \times D_2 \times D_3 \times \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \times D_n$ presents the cartesian product with $F: D_1 \times D_2 \times D_3 \times \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \times D_n \to P(L)$ is called a HyperSoft Set.

3. Results and Discussion

We present an application to show the criteria weights and ranking the alternatives. The CoCoSo method is used to rank alternatives. Several variants of the CoCoSo approach have been adopted and used in several case studies since its inception. From the original or classical CoCoSo to interval or gray models and a few fuzzy expansions. Researchers exploited CoCoSo intuitive and relaxed algorithm to concentrate on its usefulness in ambiguous situations. To the best of our knowledge, after reviewing every resource that applied CoCoSo, none of them attempted to utilize an enhanced or modified version of the program. We saw several mistakes when using the CoCoSo technique on exceptional occasions, even though this is a thorough complete ranking approach, and the structure has been tested extensively. This scenario will be presented and discussed below.

We employed several techniques, including CoCoSo, in our study on the application of MCDM techniques for Civil Litigation Efficiency Evaluation in the Context of Artificial Intelligence. There are eight criteria that allow you to choose them based on the accommodation that is available. The criteria of this study are organized as follows:

User Satisfaction and Adoption Rate (High, Moderate, Low), Accessibility to Justice (Broad, Standard, Limited), AI-Assisted Legal Research Efficiency (Efficient, Standard, Inefficient), Cost Reduction in Litigation (Significant, Moderate, Minimal), Judicial Decision Accuracy (High,

Moderate, Low), Data Security and Privacy Protection (Strong, Moderate, Weak), Case Processing Speed (Fast, Moderate, Slow), Transparency and Fairness (High, Moderate, Low). The alternatives of this study are:

Online Dispute Resolution System, AI-Enabled Court Transcription System, Predictive Analytics for Case Outcomes, AI-Powered Sentencing and Judicial Decision Support, AI-Assisted Legal Research Platform, AI-Based Legal Chatbots, Blockchain-Based Legal Data Security System, Automated Document Review System, Natural Language Processing (NLP) for Legal Texts, AI-Powered Case Management System

The results of the LMAW methodology are presented as:

We prioritize the criteria by the opinions of the experts.

We define the absolute anti-ideal point x_{AIP} using Eq. (1).

Eq. (2) is used to determine the relation between the elements of priority vector and x_{AIP}.

Eq. (3) is used to compute the criteria weights as shown in Fig 1.

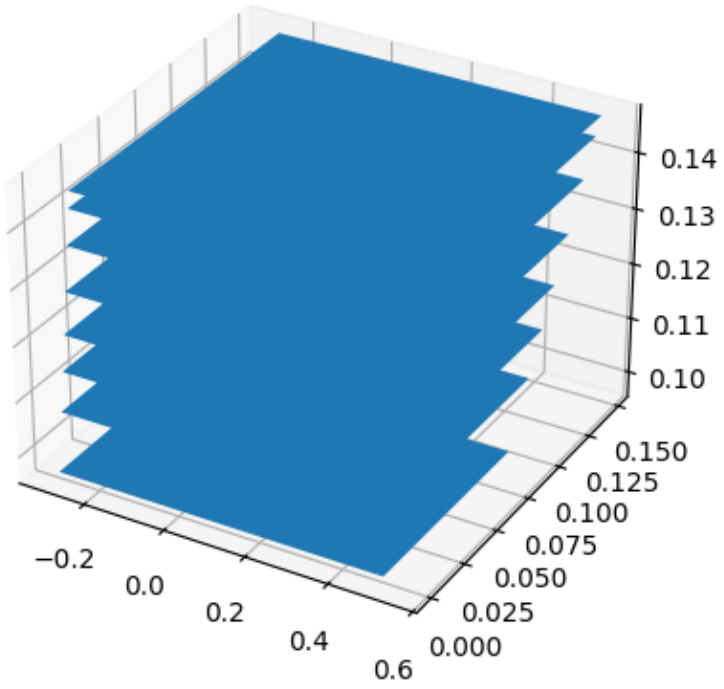

Fig 1. The criteria weights.

We are ranking the alternatives we use the concept of the HyperSoft set to deal with various criteria and values. This study selects the best values such as:

{High}, {Broad}, {Efficient}, {Moderate}, {High}, {Strong}, {Fast}, {Moderate}

Results of the CoCoSo method.

We determine the decision matrix using Eq. (5). We use scale between 0.1 to 0.99 to evaluate the criteria and alternatives based on three experts and decision makers. Then we combine the decision matrix into a single matrix as shown in Fig 2.

We normalize the decision matrix using Eq. (6) as shown in Fig 3.

Then we compute the power weight of the decision matrix and total weighted decision matrix using Eq. (8) and Eq. (9) as shown in Fig 4 and Fig 5.

Then we compute the relative weights of the alternatives based on a set of strategies using Eqs. (10-12).

Then we rank the alternatives based on U_i using Eq. (13). Fig 6 shows the rank of the alternatives.

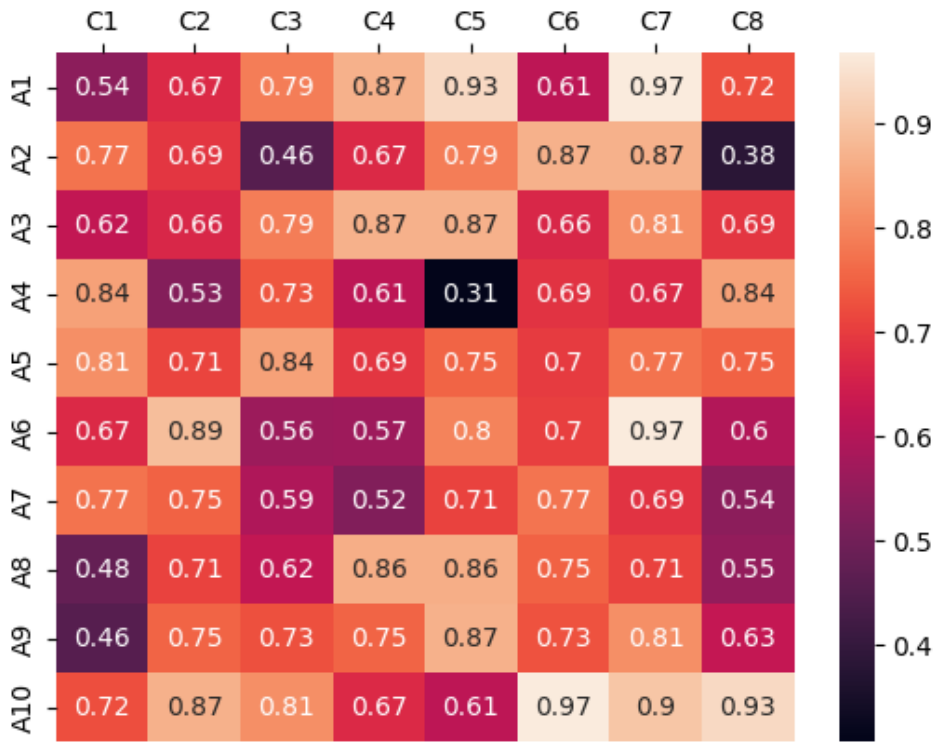

Fig 2. The combined decision matrix.

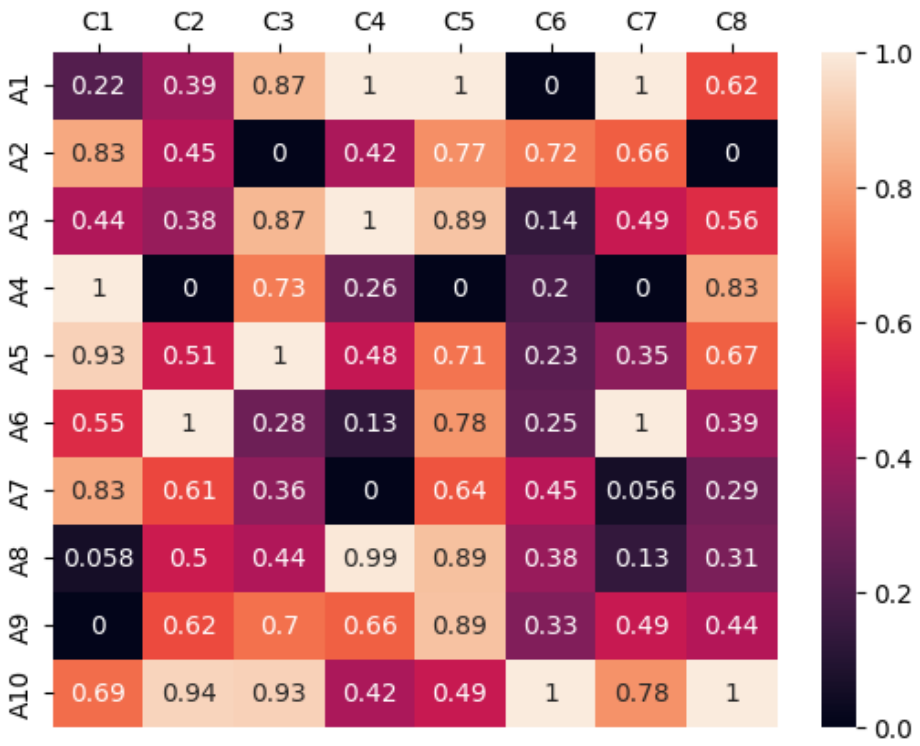

Fig 3. The normalized decision matrix.

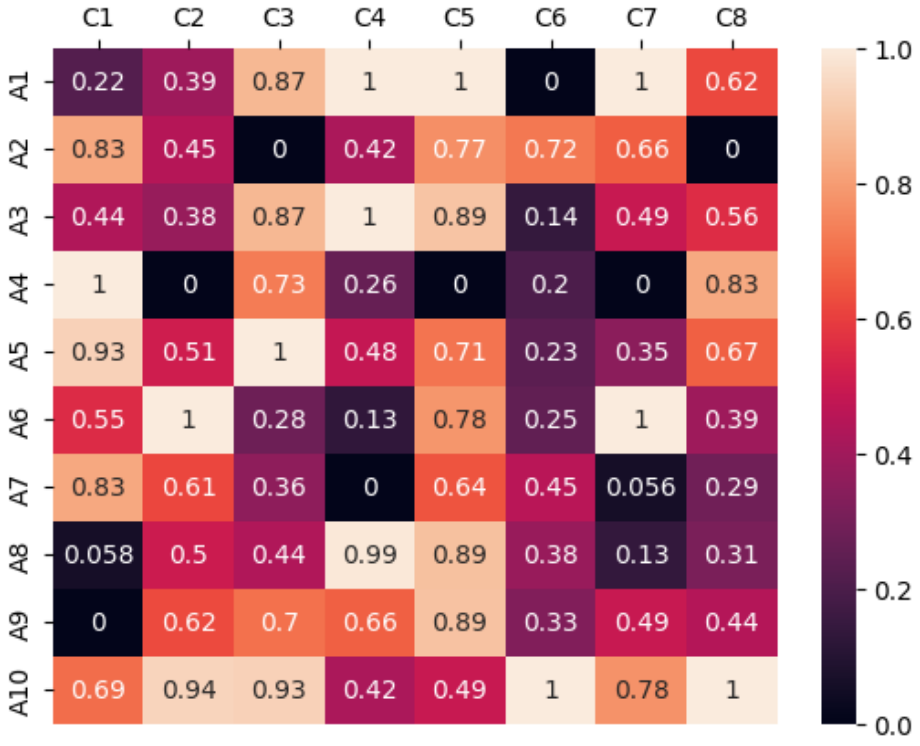

Fig 4. The weighted decision matrix.

Fig 5. The power weight decision matrix.

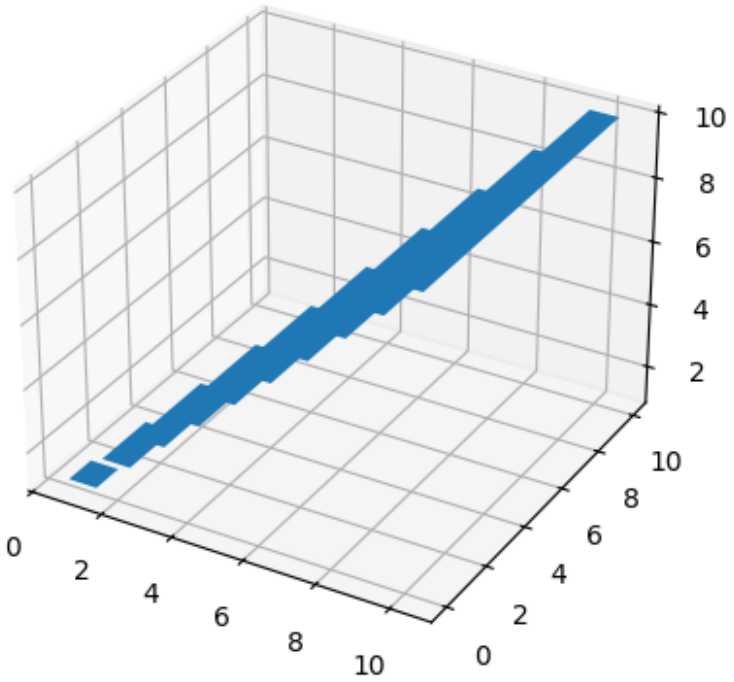

Fig 6. The rank of alternatives.

4. Analysis

Sensitivity analysis is a powerful methodological tool used to assess how variations in input parameters influence the outcomes of a given model or decision-making process. It is widely applied across disciplines, including engineering, economics, environmental sciences, and MCDM. By systematically adjusting key factors within a model, sensitivity analysis helps to identify which variables have the most significant impact on results, ensuring robust decision-making even in uncertain conditions. This approach is particularly valuable when dealing with complex systems where small changes in input can lead to substantial shifts in outputs, highlighting the stability and reliability of analytical frameworks.

This study conducted the sensitivity analysis to show the validation of the proposed approach and stability of the ranks. We change the values of φ between 0 and 1 to show different ranks of the proposed approach. Fig 7 shows the different values of U_i. Then we rank the alternatives based on these cases. Fig 8 shows the different ranks of the proposed approach. We show alternative 10 is the best and alternative 4 is the worst. We show the ranks are stable under different cases.

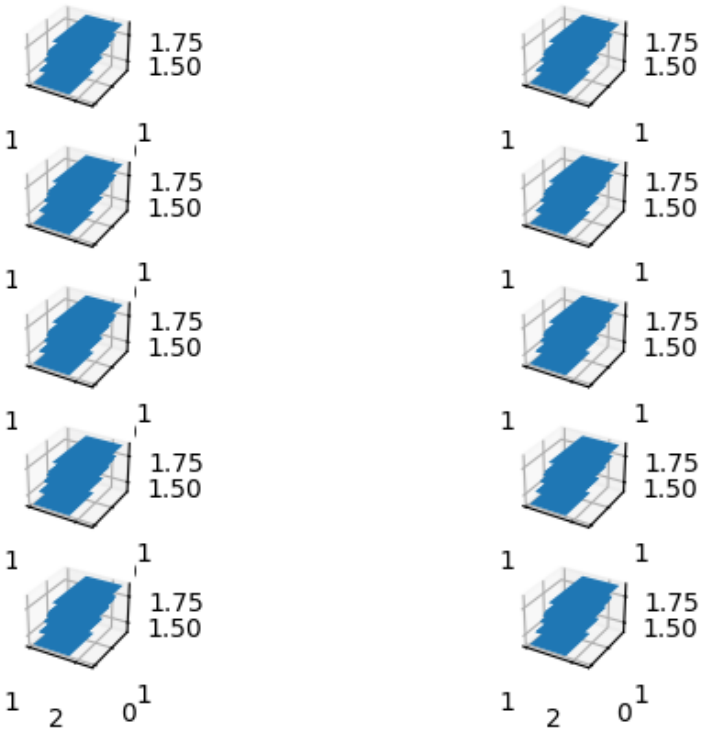

Fig 7. Different values of U_i.

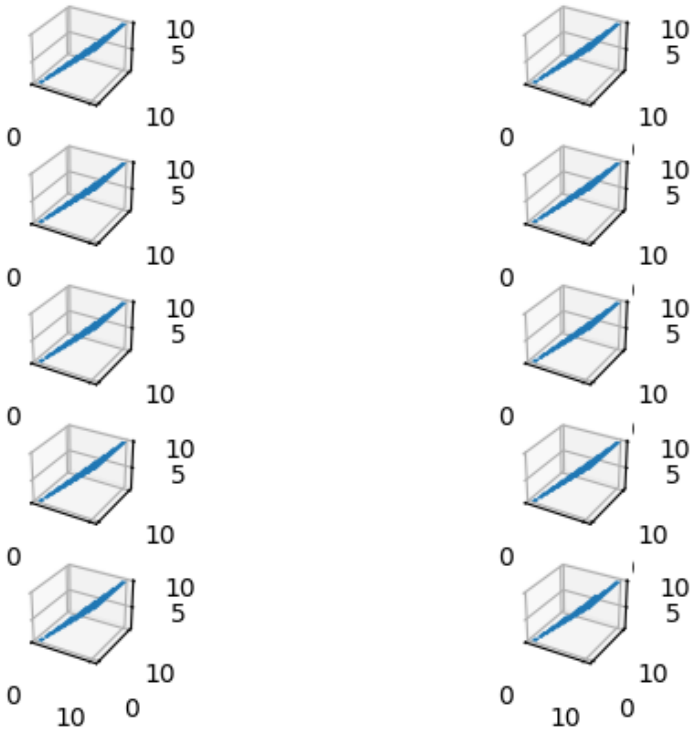

Fig 8. Different ranks of alternatives.

In the context of decision-making, sensitivity analysis enhances transparency and confidence in model-driven conclusions. It enables researchers and policymakers to prioritize influential criteria, optimize resource allocation, and mitigate risks associated with uncertainties in data. Sensitivity analysis provides different perspectives on how model parameters interact. By incorporating sensitivity analysis into evaluation processes, organizations can make more data-driven, resilient, and adaptable decisions, ensuring that strategies remain effective under varying real-world conditions.

5. Conclusions

The adoption of AI in civil litigation has the potential to revolutionize legal proceedings by reducing case processing times, lowering litigation costs, and improving judicial decision-making accuracy. The study's findings suggest that AI-powered tools such as automated legal research, predictive analytics, and document review systems contribute significantly to enhancing litigation efficiency. However, while AI enhances speed and accessibility, concerns regarding data security, algorithmic bias, and ethical transparency must be addressed to maintain trust in the legal system. A balanced approach combining AI-driven automation with human oversight is crucial to ensuring both efficiency and fairness in civil litigation.

This study proposed an MCDM method to compute the criteria weights by the LMAW method and ranking the alternative by the CoCoSo method. We used the HyperSoft set to deal with criteria and sub criteria. Eight criteria and ten alternatives are used in this study. The results show alternative 10 is the best and alternative 4 is the worst. We conducted sensitivity analysis to show the stability of the ranks. The results show the results of the proposed approach are stable in different cases.

Despite the advantages AI brings to civil litigation, the unequal adoption of AI technologies across legal jurisdictions remains a major challenge. Some legal systems have embraced AI-driven case management, while others remain cautious due to concerns about reliability, bias, and regulatory uncertainties. A structured approach to AI implementation, supported by regulatory frameworks, ethical guidelines, and continuous monitoring mechanisms, is necessary to standardize AI's role in litigation.

Another critical finding of this research is the need for data integrity and security measures in AI-driven legal systems. Given the sensitive nature of legal data, AI-powered litigation tools must comply with data privacy regulations, encryption standards, and cybersecurity best practices to ensure confidentiality and prevent misuse. Additionally, efforts to enhance AI transparency and explainability can help build confidence in AI-assisted judicial decision-making.

AI presents a transformative opportunity for improving civil litigation efficiency, but its implementation must be carefully managed to ensure fairness, accuracy, and ethical compliance. The integration of AI in the legal domain should be guided by strategic policy frameworks, interdisciplinary collaboration, and ongoing assessments to mitigate risks while maximizing benefits. By leveraging AI responsibly, the legal sector can achieve a more efficient, accessible, and transparent judicial system, ultimately enhancing the delivery of justice in an increasingly digital world.

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Neutrosophic λ-Closed Set and Related Mappings in Neutrosophic Topological Spaces

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Abstract: Studying open and closed set variations is crucial in neutrosophic topology, given the growing significance of neutrosophic sets in various applications. This study introduces neutrosophic λ-closed sets, generalizing neutrosophic closed and pre-closed sets within neutrosophic topological spaces. Their fundamental properties are examined alongside comparisons with various existing neutrosophic sets. To illustrate their significance, we define neutrosophic λ-continuous mapping, neutrosophic λ-closed mapping, and neutrosophic λ-$T_{1/2}$ spaces, exploring their connections with established mappings. Additionally, new concepts such as neutrosophic-w-continuous, neutrosophic-w-closed, neutrosophic-Gα-closed, and neutrosophic-αG-closed mappings are proposed to further enrich neutrosophic topological spaces. Several theorems and counterexamples validate these findings and highlight their theoretical significance.

Keywords: neutrosophic λ-closed set; neutrosophic λ-$T_{1/2}$ space; neutrosophic λ-continuous mapping; neutrosophic-w-continuous mapping; neutrosophic λ-closed mapping; neutrosophic-w-closed mapping; neutrosophic-αG-closed mapping; neutrosophic-Gα-closed mapping

1. Introduction

In 1965, Zadeh [24] introduced the concept of fuzzy sets to handle vague, imprecise, and uncertain data. Later, in 1986, Atanassov [6] expanded this concept by proposing intuitionistic fuzzy sets, which subsequently gained widespread recognition as a fundamental tool in mathematics, engineering [13], and medicine [8]. In 1997, Coker [7] further advanced this field by developing the notion of intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets. Building on this, Rajarajeswari and Bagyalakshmi [15] introduced the concept of intuitionistic fuzzy λ (lambda)-closed sets in 2011. In our study, we explore this concept within the framework of neutrosophic (or $\mathcal{N}_{eu}$) topological spaces. Smarandache [22] introduced the concept of $\mathcal{N}_{eu}$-sets, characterized by truth, indeterminacy, and falsity values, to represent complex real-world information. Over time, researchers have contributed significantly to $\mathcal{N}_{eu}$-theory, leading to the development of $\mathcal{N}_{eu}$-vector spaces [4], $\mathcal{N}_{eu}$-group theory [21], $\mathcal{N}_{eu}$-ring theory [3], and Dombi $\mathcal{N}_{eu}$-graphs [12]. The study of $\mathcal{N}_{eu}$-sets continues to evolve, with academicians actively exploring their diverse applications. AboElHamd et al. [1] contributed to the development of $\mathcal{N}_{eu}$-logic and its applications. In 2012, Salama and Alblowi [18] established $\mathcal{N}_{eu}$-topological spaces using $\mathcal{N}_{eu}$-sets, introducing the concepts of $\mathcal{N}_{eu}$-closed and $\mathcal{N}_{eu}$-open sets. Subsequently, in 2014, Salma et al. [19] examined continuous and closed

mappings in $\mathcal{N}_{eu}$-topological spaces. Santhi and Udhayarani [20] later introduced $\mathcal{N}_{eu}$-w-closed sets in 2016, proposing various properties of these sets. In 2017, Arokiarani et al. [5] introduced the concept of $\mathcal{N}_{eu}$-semi-open sets in $\mathcal{N}_{eu}$-topological spaces. Following this, several researchers have proposed and explored various types of open and closed sets within $\mathcal{N}_{eu}$-topological spaces. The study of weaker and stronger forms of open and closed sets plays a crucial role in $\mathcal{N}_{eu}$-topology, as the importance of $\mathcal{N}_{eu}$-set is rapidly growing across numerous applications. Consequently, it remains a prominent research focus for researchers worldwide. Additionally, Arokiarani [5] introduced $\mathcal{N}_{eu}$-pre-open sets as a weaker variant of α-open sets in $\mathcal{N}_{eu}$-topological spaces. In 2018, Dhavaseelan and Jafari [10] proposed $\mathcal{N}_{eu}$-g-closed sets and examined their properties. The same year, Jayanthi [11] presented $\mathcal{N}_{eu}$-αg-closed sets and analysed their characteristics. Further developments in the field include the introduction of $\mathcal{N}_{eu}$-gα-closed sets by Sreeja and Sarankumar [23] and the exploration of $\mathcal{N}_{eu}$-g-closed mappings by Ramesh [17] in 2020. In 2021, Rajeshwaran and Chandramathi [16] proposed $\mathcal{N}_{eu}$-g-semi-pre-closed sets, later extending their work in 2024 by introducing $\mathcal{N}_{eu}$-g-semi-pre-continuous mappings. That same year, Debnath and Mukherjee [9] examined $\mathcal{N}_{eu}$-$p\delta s$-irresolute and $\mathcal{N}_{eu}$-$\alpha\delta s$-irresolute mappings in $\mathcal{N}_{eu}$-topological spaces. Mohammed et al. [14] also contributed in 2024 by introducing Q^*-closed sets within the framework of fuzzy $\mathcal{N}_{eu}$-topology, along with relevant definitions and theorems. In this paper, we introduce λ-closed sets in $\mathcal{N}_{eu}$-topological spaces, a novel concept weaker than $\mathcal{N}_{eu}$-closed and pre-closed sets but independent of $\mathcal{N}_{eu}$-g-closed and $\mathcal{N}_{eu}$-semi-closed sets. We apply this new concept to define the notions of $\mathcal{N}_{eu}$-λ-continuous mapping, $\mathcal{N}_{eu}$-λ-$T_{1/2}$ space, and $\mathcal{N}_{eu}$-λ-closed mapping, comparing them with existing $\mathcal{N}_{eu}$-mappings. Additionally, we presented the notions of $\mathcal{N}_{eu}$-w-continuous mapping, $\mathcal{N}_{eu}$-w-closed mapping, $\mathcal{N}_{eu}$-Gα-closed mapping, and $\mathcal{N}_{eu}$-αG-closed mapping, thereby contributing to the theoretical foundation of $\mathcal{N}_{eu}$-topological spaces.

For simplicity, we use the symbol '$\mathcal{N}_{eu}$' to represent the notion of 'neutrosophic'.

2. Preliminaries

Definition 2.1. [22] Consider a fixed set X ($\neq \phi$). Then the $\mathcal{N}_{eu}$-set N on X is defined as N $=\{< x, \eta_{\mathrm{N}}(x), \theta_{\mathrm{N}}(x), \xi_{\mathrm{N}}(x) >: x \in \mathrm{X}\}$, where, $\eta_{\mathrm{N}}(x), \theta_{\mathrm{N}}(x)$, and $\xi_{\mathrm{N}}(x)$ are values in the range $[0, 1]$, representing the degree of membership, indeterminacy, and non-membership functions respectively for each element $x \in$ X. These values satisfy the condition, $0 \leq \eta_{\mathrm{N}}(x) + \theta_{\mathrm{N}}(x) + \xi_{\mathrm{N}}(x) \leq 3$.

Definition 2.2 [18] Consider two $\mathcal{N}_{eu}$-sets N and Q, where N $=\{<x, \eta_{\mathrm{N}}(x), \theta_{\mathrm{N}}(x), \xi_{\mathrm{N}}(x)>: x \in \mathrm{X}\}$, and Q $=\{< x, \eta_{\mathrm{Q}}(x), \theta_{\mathrm{Q}}(x), \xi_{\mathrm{Q}}(x)>: x \in \mathrm{X}\}$, then

(i) $\mathrm{N} \subseteq \mathrm{Q} \Leftrightarrow \eta_{\mathrm{N}}(x) \leq \eta_{\mathrm{Q}}(x)$; $\theta_{\mathrm{N}}(x) \leq \theta_{\mathrm{Q}}(x)$, and $\xi_{\mathrm{N}}(x) \geq \xi_{\mathrm{Q}}(x)$
(ii) $\mathrm{N} = \mathrm{Q} \Leftrightarrow \eta_{\mathrm{N}}(x) = \eta_{\mathrm{Q}}(x)$; $\theta_{\mathrm{N}}(x) = \theta_{\mathrm{Q}}(x)$ and $\xi_{\mathrm{N}}(x) = \xi_{\mathrm{Q}}(x)$ $\forall$x ∈ X
(iii) $\mathrm{N}' = \{< x, \xi_{\mathrm{N}}(x), \theta_{\mathrm{N}}(x), \eta_{\mathrm{N}}(x) >: x \in \mathrm{X}\}$
(iv) $\mathrm{N} \cap \mathrm{Q} = \{< x, \min\{\eta_{\mathrm{N}}(x), \eta_{\mathrm{Q}}(x)\}, \min\{\theta_{\mathrm{N}}(x), \theta_{\mathrm{Q}}(x)\}, \max\{\xi_{\mathrm{N}}(x), \xi_{\mathrm{Q}}(x)\} >\}$
(v) $\mathrm{N} \cup \mathrm{Q} = \{< x, \max\{\eta_{\mathrm{N}}(x), \eta_{\mathrm{Q}}(x)\}, \max\{\theta_{\mathrm{N}}(x), \theta_{\mathrm{Q}}(x)\}, \min\{\xi_{\mathrm{N}}(x), \xi_{\mathrm{Q}}(x)\} >\}$
(vi) $\tilde{0} = \{< x, 0, 0, 1 >: x \in \mathrm{X}\}$, and $\tilde{1} = \{< x, 1, 1, 0 >: x \in \mathrm{X}\}$ are respectively the empty set, and the whole set of X.

Definition 2.3 [18] Let X be any non-empty set, and ζ be the collection of $\mathcal{N}_{eu}$-subsets of X that satisfy the following axioms:

(i) $\tilde{0}, \tilde{1} \in \zeta$

(ii) $H_1 \cap H_2 \in \zeta$ for any $H_1, H_2 \in \zeta$
(iii) $\cup H_i \in \zeta, \forall \{H_i : i \in \Delta\} \subseteq \zeta$.

Then the pair (X, ζ) is referred to as $\mathcal{N}_{eu}$-topological space, and any $\mathcal{N}_{eu}$-set in ζ is called $\mathcal{N}_{eu}$-open set in X. The components of ζ are known as $\mathcal{N}_{eu}$-open sets, and the supplement of $\mathcal{N}_{eu}$-open set is called $\mathcal{N}_{eu}$-closed set. We will write X instead of (X, ζ) to represent $\mathcal{N}_{eu}$-topological space.

Definition 2.4 [18] Let N be any $\mathcal{N}_{eu}$-set in a $\mathcal{N}_{eu}$-topological space X where, N ={< x, $\eta_N(x), \theta_N(x), \xi_N(x)$>: $x \in X$}. Then $\mathcal{N}_{eu}$-closure of N is defined as $\mathcal{N}_{eu}$-cl(N) = $\bigcap${O: O is $\mathcal{N}_{eu}$-closed set in X, and N $\subseteq$ O}, and $\mathcal{N}_{eu}$-interior of N is defined as $\mathcal{N}_{eu}$-int(N) = $\bigcup${K: K is $\mathcal{N}_{eu}$-open set in X, and K $\subseteq$ N}.

Definition 2.5 [5] Consider a fixed set X ($\neq \phi$). A $\mathcal{N}_{eu}$-set $p_{m,i,n}$ is a $\mathcal{N}_{eu}$- point in X, and it is given by

$$p_{m,i,n}(p_j) = \begin{cases} (m,i,n), if\ p = p_j \\ (0,0,1), if\ p \neq p_j \end{cases}$$

Where m, i, $n \in$ [0, 1] respectively represent the degree of membership, indeterminacy, and non-membership values of ($p_{m,i,n}$) with $0 < m + i + n \leq 3$. The point $p_j \in$ X is the support of $p_{m,i,n}$.

Definition 2.6 Any $\mathcal{N}_{eu}$-subset N of a $\mathcal{N}_{eu}$-topological space X is called

(i) $\mathcal{N}_{eu}$-semi-closed (respectively, $\mathcal{N}_{eu}$-semi-open) set [5] if $\mathcal{N}_{eu}$-int($\mathcal{N}_{eu}$-cl(N)) $\subseteq$ N (respectively, N $\subseteq$ $\mathcal{N}_{eu}$-cl($\mathcal{N}_{eu}$-int(N))).
(ii) $\mathcal{N}_{eu}$-pre-closed (respectively, $\mathcal{N}_{eu}$-pre-open) [5] if $\mathcal{N}_{eu}$-cl($\mathcal{N}_{eu}$-int(N)) $\subseteq$ N (respectively, N $\subseteq$ $\mathcal{N}_{eu}$-int($\mathcal{N}_{eu}$-cl(N))).
(iii) $\mathcal{N}_{eu}$-g-closed set [10] if $\mathcal{N}_{eu}$-cl(N) $\subseteq$ G, whenever N $\subseteq$ G and G is $\mathcal{N}_{eu}$-open set. The supplement of $\mathcal{N}_{eu}$-g-closed set is $\mathcal{N}_{eu}$-g-open set.
(iv) $\mathcal{N}_{eu}$-αg-closed set [11] if $\mathcal{N}_{eu}$-αcl(N) $\subseteq$ G, whenever N $\subseteq$ G and G is $\mathcal{N}_{eu}$-open set in X. The supplement of $\mathcal{N}_{eu}$-α generalized closed set is $\mathcal{N}_{eu}$-αg-open set in X.
(v) $\mathcal{N}_{eu}$-gα-closed set [23] if $\mathcal{N}_{eu}$-αcl(N) $\subseteq$ G, whenever N $\subseteq$ G and G is $\mathcal{N}_{eu}$-α open set in X.
(vi) $\mathcal{N}_{eu}$-g-semi-pre-closed set [16] if $\mathcal{N}_{eu}$-spcl(N) $\subseteq$ G, whenever N $\subseteq$ G and G is $\mathcal{N}_{eu}$- open set. The supplement of $\mathcal{N}_{eu}$-gsp-closed set is called the $\mathcal{N}_{eu}$-g-semi-pre-open set.
(vii) $\mathcal{N}_{eu}$-w-closed set [20] if $\mathcal{N}_{eu}$-cl(N) $\subseteq$ G whenever N $\subseteq$ G and G is $\mathcal{N}_{eu}$-semi-open set.

Definition 2.7 [2] Suppose f is a mapping from X to Y.

(i) If N ={< $x, \eta_N(x), \theta_N(x), \xi_N(x)$>: $x \in$ X} be an arbitrary $\mathcal{N}_{eu}$-subset in X, then the image of N under f, written as f(N) is the $\mathcal{N}_{eu}$-subset in Y, and it is defined as:

$$\eta_{f(N)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\eta_N(x)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{if } f^{-1}(y) = \phi \end{cases}$$

$$\theta_{f(N)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\theta_N(x)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{if } f^{-1}(y) = \phi \end{cases}$$

$$\xi_{f(\mathrm{N})}(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \{\xi_{\mathrm{N}}(x)\}, & \text{if } f^{-1}(\mathrm{y}) \neq \phi \\ 0, & \text{if } f^{-1}(\mathrm{y}) = \phi \end{cases}$$

for each $y \in Y$, where $f^{-1}(\mathrm{y}) = \{x: f(x) = y\}$.

(ii) If Q ={< $y, \eta_{\mathrm{Q}}(y), \theta_{\mathrm{Q}}(y), \xi_{\mathrm{Q}}(y)$>: $y \in$ Y} be an arbitrary $\mathcal{N}_{eu}$-subset in Y, then the pre-image of Q under f, written as $f^{-1}(\mathrm{Q})$ is the $\mathcal{N}_{eu}$-subset in X, and it is described as:

$$\eta_{f^{-1}(\mathrm{Q})}(x) = \eta_Q\big(f(x)\big), \forall\, x \in \mathrm{X}$$

$$\theta_{f^{-1}(\mathrm{Q})}(x) = \theta_Q\big(f(x)\big), \forall\, x \in \mathrm{X}$$

$$\xi_{f^{-1}(\mathrm{Q})}(x) = \xi_Q\big(f(x)\big), \forall\, x \in \mathrm{X}$$

Definition 2.8 Consider a mapping f: X ⟶ Y. Then f is called

(i) $\mathcal{N}_{eu}$-continuous [19] if f^{-1}(N) is $\mathcal{N}_{eu}$-closed set in X for each $\mathcal{N}_{eu}$-closed set N in Y.
(ii) $\mathcal{N}_{eu}$-g-continuous [10] if f^{-1}(N) is $\mathcal{N}_{eu}$-g-closed set in X for each $\mathcal{N}_{eu}$-closed set N in Y
(iii) $\mathcal{N}_{eu}$-semi-continuous [5] if f^{-1}(N) is $\mathcal{N}_{eu}$-semi closed set in X for each $\mathcal{N}_{eu}$-closed set N in Y
(iv) $\mathcal{N}_{eu}$-g-semi-pre-continuous [16] if f^{-1} (N) is $\mathcal{N}_{eu}$-g-semi-pre-closed set in X for each $\mathcal{N}_{eu}$-closed set N in Y.

Definition 2.9 Consider a mapping f: X ⟶ Y. Then f is called

(i) $\mathcal{N}_{eu}$-closed (respectively$\mathcal{N}_{eu}$-open) mapping [5] if f(N) is $\mathcal{N}_{eu}$-closed (respectively, $\mathcal{N}_{eu}$-open) set in Y for each $\mathcal{N}_{eu}$-closed (respectively, $\mathcal{N}_{eu}$-open) set N in X.
(ii) $\mathcal{N}_{eu}$-g-closed mapping [17] if f(N) is $\mathcal{N}_{eu}$-g-closed set in Y for each $\mathcal{N}_{eu}$-closed set N in X.
(iii) $\mathcal{N}_{eu}$-semi-closed mapping [5] if f(N) is $\mathcal{N}_{eu}$-semi-closed set in Y for each $\mathcal{N}_{eu}$-closed set N in X.

3. Neutrosophic λ-Closed Sets

In this section, we introduce $\mathcal{N}_{eu}$-λ-closed (respectively, $\mathcal{N}_{eu}$-λ-open) sets, and explore various characteristics of these sets.

Definition 3.1 Any $\mathcal{N}_{eu}$-subset N of a $\mathcal{N}_{eu}$- topological space X is called $\mathcal{N}_{eu}$-λ-closed set if $\mathcal{N}_{eu}$-cl(O) ⊆ N for each $\mathcal{N}_{eu}$-open set O in X, satisfying O ⊆ N. The supplement of $\mathcal{N}_{eu}$-λ-closed set is called $\mathcal{N}_{eu}$-λ-open set in X. The collection of all $\mathcal{N}_{eu}$-λ-closed sets is represented by the symbol $\mathcal{N}_{eu}$-$\lambda CS(X)$.

Theorem 3.2 Every $\mathcal{N}_{eu}$-closed set (respectively, $\mathcal{N}_{eu}$-open set) is $\mathcal{N}_{eu}$-λ-closed set (respectively, $\mathcal{N}_{eu}$-λ-open set) in X.

Proof: Let N be an arbitrary $\mathcal{N}_{eu}$-closed set in $\mathcal{N}_{eu}$-topological space X, and let O be any $\mathcal{N}_{eu}$-open subset in X, provided O ⊆ N, then $\mathcal{N}_{eu}$-cl(O) ⊆ $\mathcal{N}_{eu}$-cl(N). As N is $\mathcal{N}_{eu}$- closed set, $\mathcal{N}_{eu}$-cl(N) = N, and $\mathcal{N}_{eu}$-cl(O) ⊆ N. Hence, N is $\mathcal{N}_{eu}$-λ-closed set.

Remark 3.3 The reverse of the above theorem need not be true, as seen in the subsequent example.

Example 3.4 Let X = $\{q, t\}$, and $\zeta = \{\tilde{0}, \tilde{1}, M\}$ be the $\mathcal{N}_{eu}$-topology defined on X, where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.30, 0.50, 0.60 >\}$. Then $\mathcal{N}_{eu}$-set N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.80, 0.50, 0.20 >\}$ is $\mathcal{N}_{eu}$-λ-closet set, but it is not $\mathcal{N}_{eu}$-closed set in X.

Theorem 3.5 If M and N are any two $\mathcal{N}_{eu}$-λ-closed sets in a $\mathcal{N}_{eu}$-topological space X, then M⋂N is $\mathcal{N}_{eu}$-λ-closed set in X.

Proof: Let O be any $\mathcal{N}_{eu}$-open set in X such that O ⊆ M ⋂ N. Since M, and N are $\mathcal{N}_{eu}$-closed sets, $\mathcal{N}_{eu}$-cl(O) ⊆ M, and $\mathcal{N}_{eu}$-cl(O) ⊆ N. Therefore, $\mathcal{N}_{eu}$-cl(O) ⊆ M ⋂ N. Hence, M ⋂ N is $\mathcal{N}_{eu}$-λ-closed set.

Remark 3.6 The union of two $\mathcal{N}_{eu}$-λ-closed sets need not be $\mathcal{N}_{eu}$-λ-closed, as seen in the subsequent example.

Example 3.7 Let X = $\{q, t\}$, and let $\zeta = \{\tilde{0}, \tilde{1}, M\}$ be an $\mathcal{N}_{eu}$-topology defined on X where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.60, 0.50, 0.30 >\}$. Then $\mathcal{N}_{eu}$-sets N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.30 >\}$, and Q = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.60, 0.50, 0.40 >\}$ are $\mathcal{N}_{eu}$-λ-closed sets in X, but N ⋃ Q is not $\mathcal{N}_{eu}$-λ-closed set.

Note: The concept of $\mathcal{N}_{eu}$-λ-closed set is independent of the concepts of $\mathcal{N}_{eu}$-g-closed sets, $\mathcal{N}_{eu}$-semi-closed sets, and $\mathcal{N}_{eu}$-w-closed sets, as seen in the subsequent example.

Example 3.8

(i) Let X = $\{q, t\}$, and $\zeta = \{\tilde{0}, \tilde{1}, M\}$ be the $\mathcal{N}_{eu}$-topology defined on X, where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.20 >\}$. Then N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.40 >\}$ is $\mathcal{N}_{eu}$-λ-closed set but it is not $\mathcal{N}_{eu}$-g-closed set.

(ii) Let X = $\{q, t\}$, and $\zeta = \{\tilde{0}, \tilde{1}, M\}$ be the $\mathcal{N}_{eu}$-topology defined on X, where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.60, 0.50, 0.30 >\}$. Then N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.60, 0.50, 0.20 >\}$ is $\mathcal{N}_{eu}$-g-closed set but it is not $\mathcal{N}_{eu}$-λ-closed set.

(iii) Let X = $\{q, t\}$, and $\zeta = \{\tilde{0}, \tilde{1}, M\}$ be the $\mathcal{N}_{eu}$-topology defined on X, where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.20 >\}$. Then N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.40 >\}$ is $\mathcal{N}_{eu}$-λ-closed set, but it is not $\mathcal{N}_{eu}$-semi-closed set.

(iv) Let X = $\{q, t\}$, and $\zeta = \{\tilde{0}, \tilde{1}, M, S\}$ be the $\mathcal{N}_{eu}$-topology on X, where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.30, 0.50, 0.50 >\}$, and S = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.50 >\}$. Then N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.30, 0.50, 0.50 >\}$ is not $\mathcal{N}_{eu}$-λ-closed set, but it is $\mathcal{N}_{eu}$-semi-closed set.

(v) Let X = $\{q, t\}$, and $\zeta = \{\tilde{0}, \tilde{1}, M\}$ be the $\mathcal{N}_{eu}$-topology defined on X, where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.60 >\}$. Then N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.50 >\}$ is $\mathcal{N}_{eu}$-w-closed set but not $\mathcal{N}_{eu}$-λ-closed set.

(vi) Let X = $\{q, t\}$, and $\zeta = \{\tilde{0}, \tilde{1}, M\}$ be the $\mathcal{N}_{eu}$-topology defined on X, where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.20 >\}$. Then N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.50 >\}$, is $\mathcal{N}_{eu}$-λ-closed set but not $\mathcal{N}_{eu}$-w-closed set.

Remark 3.9 Every $\mathcal{N}_{eu}$-pre-closed set is $\mathcal{N}_{eu}$-λ-closed set, but the reverse is not true, as seen in the subsequent example.

Example 3.10 Let X= $\{q, t\}$, and $\zeta = \{\tilde{0}, \tilde{1}, M\}$ be the $\mathcal{N}_{eu}$-topology defined on X, where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.20 >\}$. Then, $\mathcal{N}_{eu}$-set N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.50 >\}$ is $\mathcal{N}_{eu}$-λ-closed set but not $\mathcal{N}_{eu}$-pre-closed set.

Here, we have the following representation among different kinds of $\mathcal{N}_{eu}$-closed sets.

$\mathcal{N}_{eu}$-closed set $\longrightarrow$ $\mathcal{N}_{eu}\lambda$-closed set $\longleftarrow$ $\mathcal{N}_{eu}$-pre-closed set

4. Neutrosophic Continuous Mappings

In this section, we introduce $\mathcal{N}_{eu}$-continuous mappings and establish various characteristics and counter examples of these mappings.

Definition 4.1 Consider a mapping f: X $\longrightarrow$ Y. Then f is called

(i) $\mathcal{N}_{eu}$-λ-continuous mapping if f^{-1}(N) is $\mathcal{N}_{eu}$-λ-closed set in X for each $\mathcal{N}_{eu}$-closed set N in Y.
(ii) $\mathcal{N}_{eu}$-w-continuous mapping if f^{-1}(N) is $\mathcal{N}_{eu}$-w-closed set in X of each $\mathcal{N}_{eu}$-closed set N in Y.

Remark 4.2 Every $\mathcal{N}_{eu}$-continuous mapping is $\mathcal{N}_{eu}$-λ-continuous, but the reverse is not true, as seen in the subsequent example.

Example 4.3 Let X = $\{q, t\}$ and Y =$\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, N\}$ on X, and Y resp., where M =$\{< q, 0.50, 0.50, 0.50 >, < t, 0.60, 0.50, 0.30 >\}$, and N = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.20, 0.50, 0.80 >\}$. Define f: X $\longrightarrow$ Y such that $f(q) = w$, and $f(t) = z$. Then f is $\mathcal{N}_{eu}$-λ-continuous but not $\mathcal{N}_{eu}$-continuous.

Remark 4.4 Each $\mathcal{N}_{eu}$-pre-continuous mapping f: X $\longrightarrow$ Y is $\mathcal{N}_{eu}$-λ-continuous, but the reverse is not true, as seen in the subsequent example.

Example 4.5 Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, N\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.20 >\}$, and N = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.50, 0.50, 0.40 >\}$. Define f: X $\longrightarrow$ Y such that $f(q) = w$, and $f(t) = z$. Consider the $\mathcal{N}_{eu}$-set N = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.40, 0.50, 0.50 >\}$, then f^{-1}(N) = $f^{-1}(\{< w, 0.50, 0.50, 0.50 >, < z, 0.40, 0.50, 0.50 >\}) = \{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.50 >\}$ is $\mathcal{N}_{eu}$-λ-closed set in X but not $\mathcal{N}_{eu}$-pre-closed set in X. Hence, f is $\mathcal{N}_{eu}$-λ-continuous mapping but not $\mathcal{N}_{eu}$-pre-continuous mapping.

Note: The concept of $\mathcal{N}_{eu}$-λ-continuous mapping is independent of the concepts of $\mathcal{N}_{eu}$-g-continuous mapping, $\mathcal{N}_{eu}$-semi continuous mapping, $\mathcal{N}_{eu}$-generalized semi-pre-continuous mapping, and $\mathcal{N}_{eu}$-w-continuous mapping, as seen in the subsequent example.

Example 4.6

(i) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.30, 0.50, 0.60 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.20, 0.50, 0.60 >\}$. Define f: X $\longrightarrow$ Y such that $f(q) = w$, and $f(t) = z$. Then f is $\mathcal{N}_{eu}$-g-continuous but not $\mathcal{N}_{eu}$-λ-continuous.
(ii) Let X = {q, t}, and Y = {w, z}. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.20, 0.50, 0.50 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.40, 0.50, 0.50 >\}$. Define f: X $\longrightarrow$ Y such that $f(q) = w$, and $f(t) = z$. Then f is $\mathcal{N}_{eu}$-λ-continuous, but it is neither $\mathcal{N}_{eu}$-g- continuous nor $\mathcal{N}_{eu}$-semi continuous.
(iii) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M =$\{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.60 >\}$, and P = $\{< w, 0.20, 0.50, 0.80 >, < z, 0.10, 0.50, 0.90 >\}$. Define f: X $\longrightarrow$ Y such that $f(q) = w$, and $f(t) = z$. Then f is $\mathcal{N}_{eu}$-semi continuous but not $\mathcal{N}_{eu}$-λ-continuous.

(iv) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M =$\{< q, 0.50, 0.50, 0.50 >, < t, 0.60, 0.50, 0.30 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.20, 0.50, 0.80 >\}$. Define f: X ⟶ Y such that $f(q)$ = w, and $f(t) = z$. Then f is $\mathcal{N}_{eu}$-λ-continuous but not $\mathcal{N}_{eu}$-continuous.

(v) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.60, 0.50, 0.30 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.20, 0.50, 0.80 >\}$. Define f: X ⟶ Y such that $f(q)$ = w, and $f(t) = z$. Then f is $\mathcal{N}_{eu}$-λ-continuous but not $\mathcal{N}_{eu}$-generalized semi-pre-continuous.

(vi) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.30 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.50, 0.50, 0.40 >\}$. Define f: X ⟶ Y such that $f(q)$ = w, and $f(t) = z$. Then f is $\mathcal{N}_{eu}$-generalized semi-pre-continuous but not $\mathcal{N}_{eu}$-λ-continuous.

(vii) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.20 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.50, 0.50, 0.40 >\}$. Define f: X ⟶ Y such that $f(q)$ = w, and $f(t) = z$. Let N = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.40, 0.50, 0.50 >\}$. Then f is $\mathcal{N}_{eu}$-λ-continuous but not $\mathcal{N}_{eu}$-w-continuous.

(viii) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.60 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.50, 0.50, 0.50 >\}$. Define f: X ⟶ Y such that $f(q)$ = w, and $f(t) = z$. Let N = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.50, 0.50, 0.50 >\}$, then f^{-1}(N) is not $\mathcal{N}_{eu}$-λ-closed set, but it is $\mathcal{N}_{eu}$-w-closed set. Hence, f is $\mathcal{N}_{eu}$-w-continuous but not $\mathcal{N}_{eu}$-λ-continuous.

Here we have the following representation among different kinds of $\mathcal{N}_{eu}$-continuous mappings.

$\mathcal{N}_{eu}$-continuous mapping ⟶ $\mathcal{N}_{eu}\lambda$-continuous mapping ⟵ $\mathcal{N}_{eu}$-pre-continuous mapping

Theorem 4.7 Let f: X ⟶ Y be any $\mathcal{N}_{eu}$-λ-continuous mapping.

(i) For any$\mathcal{N}_{eu}$-point $p_{m,i,n} \in$ X, and any $\mathcal{N}_{eu}$-neighbourhood N of $f(p_{m,i,n})$, there exists a $\mathcal{N}_{eu}$-λ-open set O in X such that $p_{m,i,n} \in$ O ⊆ f^{-1}(N).

(ii) For any$\mathcal{N}_{eu}$-point $p_{m,i,n} \in$ X, and any $\mathcal{N}_{eu}$-neighbourhood N of $f(p_{m,i,n})$, there exists a $\mathcal{N}_{eu}$-λ-open set O in X such that $p_{m,i,n} \in$ O, and f(O) ⊆ N.

Proof:

(i) Let $p_{m,i,n} \in$ X, and let N be arbitrary $\mathcal{N}_{eu}$-neighbourhood of $f(p_{m,i,n})$. Then there exists $\mathcal{N}_{eu}$-open set O in Y, provided that $f(p_{m,i,n}) \in$ O ⊆ N. As f is $\mathcal{N}_{eu}$-λ-continuous, f^{-1}(O) is $\mathcal{N}_{eu}$-λ-open set in X, and $p_{m,i,n} \in f^{-1}(f(p_{m,i,n})) \subseteq f^{-1}(O) \subseteq f^{-1}(N)$.

(ii) Let $p_{m,i,n} \in$ X, and let N be arbitrary $\mathcal{N}_{eu}$-neighbourhood of $f(p_{m,i,n})$. From (i), there exists $\mathcal{N}_{eu}$-λ-open set O in X, provided that $p_{m,i,n} \in$ O ⊆ f^{-1}(N). Thus, $p_{m,i,n} \in$ O, and $f(O) \subseteq f(f^{-1}(N) \subseteq$ N. That is, $p_{m,i,n} \in$ O, and f(O) ⊆ N.

Theorem 4.8 Consider mappings f: X ⟶ Y, and g: Y ⟶ Z in such a way that the mapping g is $\mathcal{N}_{eu}$-continuous, and the mapping f is $\mathcal{N}_{eu}$-λ-continuous then composition mapping g ∘ f is $\mathcal{N}_{eu}$-λ-continuous.

Proof: Let N be any $\mathcal{N}_{eu}$-closed set in Z. As g is $\mathcal{N}_{eu}$-continuous, $g^{-1}(N)$ is $\mathcal{N}_{eu}$-closed set in Y, and as f is $\mathcal{N}_{eu}$-λ-continuous, $f^{-1}(g^{-1}(N))$ is $\mathcal{N}_{eu}$-λ-closed set in X, and we know that $f^{-1}(g^{-1}(N)) = (g \circ f)^{-1}(N)$, consequently, $g \circ f$ is $\mathcal{N}_{eu}$-λ-continuous mapping.

Definition 4.9 Any $\mathcal{N}_{eu}$-topological space X is called $\mathcal{N}_{eu}$-λ-$T_{1/2}$ space if every $\mathcal{N}_{eu}$-λ-closed set is $\mathcal{N}_{eu}$-closed set.

Theorem 4.10 If a mapping f: X ⟶ Y is $\mathcal{N}_{eu}$-λ-continuous, and the space X is $\mathcal{N}_{eu}$-λ-$T_{1/2}$ space then the mapping f is $\mathcal{N}_{eu}$-continuous.

Proof: Let N be any $\mathcal{N}_{eu}$-closed set in Y. As the mapping f is $\mathcal{N}_{eu}$-λ-continuous, $f^{-1}(N)$ is $\mathcal{N}_{eu}$-λ-closed set in X. By given hypothesis X is $\mathcal{N}_{eu}$-λ-$T_{1/2}$ space, therefore, $f^{-1}(N)$ is $\mathcal{N}_{eu}$-closed set in X. Hence, the mapping f is $\mathcal{N}_{eu}$-continuous mapping as pre-image of every $\mathcal{N}_{eu}$-closed set in Y is $\mathcal{N}_{eu}$-closed in X.

Theorem 4.11 Any $\mathcal{N}_{eu}$-topological space X is $\mathcal{N}_{eu}$-λ-$T_{\frac{1}{2}}$ space if and only if $\mathcal{N}_{eu}$-λ-$OS(X)$ = $\mathcal{N}_{eu}$-$OS(X)$.

Proof: Let $N \in \mathcal{N}_{eu}$-λ-$OS(X)$, then the supplement of N (or N') is $\mathcal{N}_{eu}$-λ-closed set in X. As X is $\mathcal{N}_{eu}$-λ-$T_{1/2}$ space, N' is $\mathcal{N}_{eu}$-closed set in X, and hence, N is $\mathcal{N}_{eu}$-open set in X. That is $N \in \mathcal{N}_{eu}$-$OS(X)$ Thus, $\mathcal{N}_{eu}$-λ-$OS(X) \subseteq \mathcal{N}_{eu}$-$OS(X)$. Now, using Theorem 4.2, we have $\mathcal{N}_{eu}$-$OS(X) \subseteq \mathcal{N}_{eu}$-$\lambda$-$OS(X)$. Hence, $\mathcal{N}_{eu}$-λ-$OS(X)$=$\mathcal{N}_{eu}$-$OS(X)$. Conversely, as N is an arbitrary $\mathcal{N}_{eu}$-λ-closed set in X, N' is $\mathcal{N}_{eu}$-λ- open set in X, and so $N' \in \mathcal{N}_{eu}$-$OS(X)$, as $\mathcal{N}_{eu}$-λ-$OS(X)$ = $\mathcal{N}_{eu}$-$OS(X)$, therefore, N is $\mathcal{N}_{eu}$-closed set. Hence, X is $\mathcal{N}_{eu}$-λ-$T_{\frac{1}{2}}$ space.

5. Neutrosophic-Closed Mappings

In this section, we introduce and study $\mathcal{N}_{eu}$-closed mappings with their characteristics.

Definition 5.1 A mapping f: X ⟶ Y is called

(i) $\mathcal{N}_{eu}$-λ-closed mapping if f(N) is $\mathcal{N}_{eu}$-λ-closed set in Y for each $\mathcal{N}_{eu}$-closed set N in X.
(ii) $\mathcal{N}_{eu}$-w-closed mapping if f(N) is $\mathcal{N}_{eu}$-w-closed set in Y for each $\mathcal{N}_{eu}$-closed set N in X.
(iii) $\mathcal{N}_{eu}$-Gα–closed mapping if f(N) is $\mathcal{N}_{eu}$-generalized α–closed set in Y for each $\mathcal{N}_{eu}$-closed set N in X.
(iv) $\mathcal{N}_{eu}$-αG-closed mapping if f(N) is $\mathcal{N}_{eu}$-α-generalized closed set in Y for each $\mathcal{N}_{eu}$-closed set N in X.

Theorem 5.2 Each $\mathcal{N}_{eu}$-closed mapping f: X ⟶ Y is $\mathcal{N}_{eu}$-λ-closed mapping.

Proof: Consider a mapping f: X ⟶ Y, and consider any $\mathcal{N}_{eu}$-closed set N in X. Then f(N) is $\mathcal{N}_{eu}$-closed set in Y. Since, each $\mathcal{N}_{eu}$-closed set in X is $\mathcal{N}_{eu}$-λ-closed set, f(N) is $\mathcal{N}_{eu}$-λ-closed set in Y. Hence, f is $\mathcal{N}_{eu}$-λ-closed mapping.

In general, the reverse of the above theorem is not true, as seen in the subsequent example.

Example 5.3 Let X = $\{q, t\}$ and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.20, 0.50, 0.80 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.30, 0.50, 0.60 >\}$. Define f: X ⟶ Y such that $f(q) = w$, and $f(t) = z$. Let N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.80, 0.50, 0.20 >\}$ be the $\mathcal{N}_{eu}$-closed set in X, then the image of N, given by f(N) = $f(\{< q, 0.50, 0.50, 0.50 >, < t, 0.80, 0.50, 0.20 >\}) = \{< w, 0.50, 0.50, 0.50 >, < z, 0.80, 0.50, 0.20 >\}$ is $\mathcal{N}_{eu}$-λ-closed set in Y as $\mathcal{N}_{eu}$-cl($\tilde{0}$) = $\tilde{0} \subseteq$ A, and $\mathcal{N}_{eu}$-cl(P) = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.60, 0.50, 0.30 >\} \subseteq$ N, whenever $\tilde{0}$

$\subseteq$ N, and P $\subseteq$ N, respectively (using Definition 3.1). But N is not $\mathcal{N}_{eu}$-closed in Y as $\mathcal{N}_{eu}$-cl(N) $=\tilde{1} \neq$ N. Hence, f is $\mathcal{N}_{eu}$-λ-closed mapping but not $\mathcal{N}_{eu}$-closed mapping.

Theorem 5.4 If f: X $\longrightarrow$ Y is $\mathcal{N}_{eu}$-λ-closed mapping and Y is $\mathcal{N}_{eu}$-λ-$T_{\frac{1}{2}}$ space, then f is $\mathcal{N}_{eu}$-closed mapping.

Proof: Let N be an arbitrary $\mathcal{N}_{eu}$-closed set in X. As f is $\mathcal{N}_{eu}$-λ-closed mapping, f(N) is $\mathcal{N}_{eu}$-λ-closed set in Y. As Y is $\mathcal{N}_{eu}$-λ-$T_{\frac{1}{2}}$ space, f(N) is $\mathcal{N}_{eu}$-closed set in Y. Hence, f is $\mathcal{N}_{eu}$-closed mapping.

Note: The concept of $\mathcal{N}_{eu}$-λ-closed mapping is independent of the concepts of $\mathcal{N}_{eu}$-w-closed mappings $\mathcal{N}_{eu}$-g–closed mapping, $\mathcal{N}_{eu}$-semi closed mapping, $\mathcal{N}_{eu}$-Gα–closed mapping, and $\mathcal{N}_{eu}$-αG-closed mapping, as seen in the subsequent example.

Example 5.5

(i) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.40 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.50, 0.50, 0.20 >\}$. Define f: X $\longrightarrow$ Y such that $f(q) = w$, and $f(t) = z$. Let N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.50 >\}$, then f(N) = $f(\{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.50 >\}) = \{< w, 0.50, 0.50, 0.50 >, < z, 0.40, 0.50, 0.50 >\}$ is $\mathcal{N}_{eu}$-λ-closed set in Y but not $\mathcal{N}_{eu}$-w-closed set. Hence, f is $\mathcal{N}_{eu}$-λ-closed mapping but not $\mathcal{N}_{eu}$-w-closed mapping.

(ii) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.50 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.40, 0.50, 0.60 >\}$. Define f: X $\longrightarrow$ Y such that $f(q) = w$, and $f(t) = z$. Let N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.50 >\}$, then f(N) = $f(\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.50 >\}) = \{< w, 0.50, 0.50, 0.50 >, < z, 0.50, 0.50, 0.50 >\}$ is not $\mathcal{N}_{eu}$-λ-closed set, but it is $\mathcal{N}_{eu}$-w-closed set. Hence, f is $\mathcal{N}_{eu}$-w-closed mapping but not $\mathcal{N}_{eu}$-λ-closed mapping.

(iii) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.60, 0.50, 0.30 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.20, 0.50, 0.60 >\}$. Define f: X $\longrightarrow$ Y such that $f(q) = w$, and $f(t) = z$. Let N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.30, 0.50, 0.60 >\}$, then f(N) = $f(\{< q, 0.50, 0.50, 0.50 >, < t, 0.30, 0.50, 0.60 >\}) = \{< w, 0.50, 0.50, 0.50 >, < z, 0.30, 0.50, 0.60 >\}$ is not $\mathcal{N}_{eu}$-λ-closed set, but it is $\mathcal{N}_{eu}$-g-closed set. Hence, f is $\mathcal{N}_{eu}$-g-closed mapping but not $\mathcal{N}_{eu}$-λ-closed mapping.

(iv) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M =$\{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.50 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.50, 0.50, 0.20 >\}$. Define f: X $\longrightarrow$ Y such that $f(q) = w$, and $f(t) = z$. Let N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.40 >\}$, then f(N) = $f(\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.40 >\}) = \{< w, 0.50, 0.50, 0.50 >, < z, 0.50, 0.50, 0.40 >\}$ is $\mathcal{N}_{eu}$-λ-closed set but not $\mathcal{N}_{eu}$-g-closed set. Hence, f is $\mathcal{N}_{eu}$-λ-closed mapping but not $\mathcal{N}_{eu}$-g-closed mapping.

(v) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.50 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.10, 0.50, 0.90 >\}$. Define f: X $\longrightarrow$ Y such that $f(q) = w$, and $f(t) = z$. Let N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.40 >\}$, then f(N) = $f(\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.40 >\}) = \{< w, 0.50, 0.50, 0.50 >, <$

$z, 0.50, 0.50, 0.40 >\}$ is not $\mathcal{N}_{eu}$-λ-closed set, but it is $\mathcal{N}_{eu}$-semi-closed set. Hence, f is $\mathcal{N}_{eu}$-semi-closed mapping but not $\mathcal{N}_{eu}$-λ-closed mapping.

(vi) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.20 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.50, 0.50, 0.40 >\}$. Define f: X ⟶ Y such that $f(q)$ = w, and $f(t) = z$. Let N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.20, 0.50, 0.50 >\}$, then f(N) = $f(\{< q, 0.50, 0.50, 0.50 >, < t, 0.20, 0.50, 0.50 >\}) = \{< w, 0.50, 0.50, 0.50 >, < z, 0.20, 0.50, 0.50 >\}$ is $\mathcal{N}_{eu}$-λ-closed set but not $\mathcal{N}_{eu}$-semi-closed set. Hence, f is $\mathcal{N}_{eu}$-λ-closed mapping but not $\mathcal{N}_{eu}$-semi-closed mapping.

(vii) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.30 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.50, 0.50, 0.30 >\}$. Define f: X ⟶ Y such that $f(q)$ = w, and $f(t) = z$. Let N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.30 >\}$, then f(N) = $f(\{< q, 0.50, 0.50, 0.50 >, < t, 0.30, 0.50, 0.50 >\}) = \{< w, 0.50, 0.50, 0.50 >, < z, 0.30, 0.50, 0.50 >\}$ is $\mathcal{N}_{eu}$-λ-closed set but not $\mathcal{N}_{eu}$-semi-pre-closed set. Hence, f is $\mathcal{N}_{eu}$-λ-closed mapping but not $\mathcal{N}_{eu}$-semi-pre-closed mapping.

(viii) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M =$\{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.60 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.10, 0.50, 0.90 >\}$. Define f: X ⟶ Y such that $f(q)$ = w, and $f(t) = z$. Let N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.60, 0.50, 0.40 >\}$, then f(N) = $f(\{< q, 0.50, 0.50, 0.50 >, < t, 0.60, 0.50, 0.40 >\}) = \{< w, 0.50, 0.50, 0.50 >, < z, 0.60, 0.50, 0.40 >\}$ is $\mathcal{N}_{eu}$-semi-pre-closed set but not $\mathcal{N}_{eu}$-λ-closed set. Hence, f is $\mathcal{N}_{eu}$-semi-pre-closed mapping but not $\mathcal{N}_{eu}$-λ-closed mapping.

(ix) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M = $\{< q, 0.40, 0.50, 0.60 >, < t, 0.30, 0.50, 0.70 >\}$, and P = $\{< w, 0.20, 0.50, 0.80 >, < z, 0.30, 0.50, 0.70 >\}$. Define f: X ⟶ Y such that $f(q)$ = w, and $f(t) = z$. Let N = $\{< q, 0.60, 0.50, 0.40 >, < t, 0.70, 0.50, 0.30 >\}$, then f(N) = $f(\{< q, 0.60, 0.50, 0.40 >, < t, 0.70, 0.50, 0.30 >\}) = \{< w, 0.60, 0.50, 0.40 >, < z, 0.70, 0.50, 0.30 >\}$ is $\mathcal{N}_{eu}$-Gα–closed set but not $\mathcal{N}_{eu}$-λ-closed set. Hence, f is $\mathcal{N}_{eu}$-Gα-closed mapping but not $\mathcal{N}_{eu}$-λ-closed mapping.

(x) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M =$\{< q, 0.10, 0.50, 0.90 >, < t, 0.30, 0.50, 0.70 >\}$, and P = $\{< w, 0.80, 0.50, 0.20 >, < z, 0.80, 0.50, 0.10 >\}$. Define f: X ⟶ Y such that $f(q)$ = w, and $f(t) = z$. Let N = $\{< q, 0.90, 0.50, 0.10 >, < t, 0.70, 0.50, 0.30 >\}$, then f(N) = $f(\{< q, 0.90, 0.50, 0.10 >, < t, 0.70, 0.50, 0.30 >\}) = \{< w, 0.90, 0.50, 0.10 >, < z, 0.70, 0.50, 0.30 >\}$ is not $\mathcal{N}_{eu}$-Gα–closed set but it is $\mathcal{N}_{eu}$-λ-closed set. Hence, f is not $\mathcal{N}_{eu}$-Gα-closed mapping but it is $\mathcal{N}_{eu}$-λ-closed mapping.

(xi) Let X = $\{q, t\}$, and Y = $\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M =$\{< q, 0.60, 0.50, 0.40 >, < t, 0.70, 0.50, 0.20 >\}$, and P = $\{< w, 0.20, 0.50, 0.60 >, < z, 0.20, 0.50, 0.70 >\}$. Define f: X ⟶ Y such that $f(q)$ = w, and $f(t) = z$. Let N = $\{< q, 0.40, 0.50, 0.60 >, < t, 0.20, 0.50, 0.70 >\}$, then f(N) = $f(\{< q, 0.40, 0.50, 0.60 >, < t, 0.20, 0.50, 0.70 >\}) = \{< w, 0.40, 0.50, 0.60 >, < z, 0.20, 0.50, 0.70 >\}$ is $\mathcal{N}_{eu}$-αG-closed set in Y but not $\mathcal{N}_{eu}$-λ-closed set in Y. Hence, f is $\mathcal{N}_{eu}$-αG-closed mapping but not $\mathcal{N}_{eu}$-λ-closed mapping.

(xii) Let X = $\{q, t\}$, and Y =$\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies $\zeta = \{\tilde{0}, \tilde{1}, M\}$, and $\eta = \{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M =$\{< q, 0.10, 0.50, 0.50 >, < t, 0.20, 0.50, 0.60 >\}$, and P = $\{< w, 0.20, 0.50, 0.40 >, < z, 0.30, 0.50, 0.50 >\}$. Define f: X ⟶ Y such that $f(q)$ = w, and $f(t) = z$. Let N = $\{< q, 0.50, 0.50, 0.10 >, < t, 0.60, 0.50, 0.20 >\}$, then f(N) = $f(\{< q, 0.50, 0.50, 0.10 >, < t, 0.60, 0.50, 0.20 >\}) = \{< w, 0.50, 0.50, 0.10 >, <$

$z, 0.60, 0.50, 0.20 >\}$ is $\mathcal{N}_{eu}$-λ-closed set in Y but not $\mathcal{N}_{eu}$-αG-closed set in Y. Hence, f is $\mathcal{N}_{eu}$-λ-closed mapping but not $\mathcal{N}_{eu}$-αG-closed mapping.

Remark 5.6 Each $\mathcal{N}_{eu}$-pre-closed mapping f: X $\longrightarrow$ Y is $\mathcal{N}_{eu}$-λ-closed mapping, but the reverse is not true as seen in the subsequent example.

Example 5.7 Let X = $\{q, t\}$ and Y=$\{w, z\}$. Define the $\mathcal{N}_{eu}$-topologies ζ =$\{\tilde{0}, \tilde{1}, M\}$, and η = $\{\tilde{0}, \tilde{1}, P\}$ on X, and Y resp., where M =$\{< q, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.40 >\}$, and P = $\{< w, 0.50, 0.50, 0.50 >, < z, 0.50, 0.50, 0.20 >\}$. Define f: X $\longrightarrow$ Y such that $f(q) = w$, and $f(t) = z$. Let N = $\{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.50 >\}$ be the $\mathcal{N}_{eu}$-set in X, then f(N) = $f(\{< q, 0.50, 0.50, 0.50 >, < t, 0.40, 0.50, 0.50 >\}) = \{< w, 0.50, 0.50, 0.50 >, < z, 0.40, 0.50, 0.50 >\}$ is $\mathcal{N}_{eu}$-λ-closed set in Y, but it is not $\mathcal{N}_{eu}$-pre-closed set in Y. Hence, f is $\mathcal{N}_{eu}$-λ-closed mapping, but not $\mathcal{N}_{eu}$-pre-closed mapping.

Here, we have the following representation among different kinds of $\mathcal{N}_{eu}$-closed mappings.

$\mathcal{N}_{eu}$-closed mapping $\longrightarrow$ $\mathcal{N}_{eu}\lambda$-closed mapping $\longleftarrow$ $\mathcal{N}_{eu}$-pre-closed mapping

Theorem 5.8 Let f be any $\mathcal{N}_{eu}$-λ-closed mapping. Then for each $\mathcal{N}_{eu}$- set N in X, f($\mathcal{N}_{eu}$-cl(N)) is $\mathcal{N}_{eu}$-λ-closed set in Y.

Proof: Let N be any $\mathcal{N}_{eu}$-set in X. Then $\mathcal{N}_{eu}$-cl(N) is $\mathcal{N}_{eu}$-closed set in X. As f is $\mathcal{N}_{eu}$-λ-closed mapping, f($\mathcal{N}_{eu}$-cl(N)) is $\mathcal{N}_{eu}$-λ-closed set in Y.

Theorem 5.9 If f: X $\longrightarrow$ Y is $\mathcal{N}_{eu}$-closed mapping and g: Y $\rightarrow$ Z is $\mathcal{N}_{eu}$-λ-closed mapping, then the composition mapping g $\circ$ f is $\mathcal{N}_{eu}$-λ-closed mapping.

Proof. As f is $\mathcal{N}_{eu}$-closed mapping, for any $\mathcal{N}_{eu}$-closed set N in X, f(N) is $\mathcal{N}_{eu}$-closed set in Y, and since g is $\mathcal{N}_{eu}$-λ-closed mapping, g(f(N)) is $\mathcal{N}_{eu}$-λ-closed set in Z. Hence, g $\circ$ f is $\mathcal{N}_{eu}$-λ-closed mapping.

Remark 5.10 The composition of two $\mathcal{N}_{eu}$-λ-closed mappings is not $\mathcal{N}_{eu}$-λ-closed mapping, as seen in the subsequent example.

Example 5.11 Let X = $\{k, p\}$, Y = $\{w, t\}$ and Z = $\{d, j\}$. Let M = $\{< k, 0.50, 0.50, 0.50 >, < p, 0.50, 0.50, 0.20 >\}$, Q = $\{< w, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.40 >\}$, and P = $\{< d, 0.50, 0.50, 0.50 >, < j, 0.60, 0.50, 0.40 >\}$ be $\mathcal{N}_{eu}$-sets in X, Y, and Z resp., and let ζ = $\{\tilde{0}, \tilde{1}, M\}$, η = $\{\tilde{0}, \tilde{1}, Q\}$, and σ = $\{\tilde{0}, \tilde{1}, P\}$, be $\mathcal{N}_{eu}$-topologies on X, Y, and Z resp. Define a mapping f: X $\longrightarrow$ Y such that $f(k) = w$, and $f(p) = t$, then $f(\{< k, 0.50, 0.50, 0.50 >, < p, 0.20, 0.50, 0.50 >\}) = \{< w, 0.50, 0.50, 0.50 >, < t, 0.20, 0.50, 0.50 >\}$ is $\mathcal{N}_{eu}$- λ-closed set in Y. Thus, f is $\mathcal{N}_{eu}$-λ-closed mapping. Now, consider g: Y $\rightarrow$ Z such that g(w) = d, and g(t) = j, then g($\{< w, 0.50, 0.50, 0.50 >, < t, 0.50, 0.50, 0.40 >\}$)=$\{< d, 0.50, 0.50, 0.50 >, j, 0.50, 0.50, 0.40 >\}$ is $\mathcal{N}_{eu}$-λ-closed set in Z. Hence, g is $\mathcal{N}_{eu}$-λ-closed mapping. But the mapping g $\circ$ f: X $\rightarrow$ Z is not $\mathcal{N}_{eu}$-λ-closed mapping, as g(f(N)) = g($f(\{< k, 0.50, 0.50, 0.50 >, < p, 0.20, 0.50, 0.50 >\})$) = g($\{< w, 0.50, 0.50, 0.50 >, < t, 0.20, 0.50, 0.50 >\}$) = $\{< d, 0.50, 0.50, 0.50 >, < j, 0.20, 0.50, 0.50 >\}$ is not $\mathcal{N}_{eu}$-λ-closed set in Z. Hence, g(f(A)) is not $\mathcal{N}_{eu}$-λ-closed mapping. That is, g $\circ$ f is not $\mathcal{N}_{eu}$-λ-closed mapping.

6. Conclusion and Future Work

In this paper, we introduced the concepts of $\mathcal{N}_{eu}$-λ-closed and $\mathcal{N}_{eu}$-λ-open sets in $\mathcal{N}_{eu}$-topological spaces and explored their fundamental properties. Through a comparative analysis, we established that $\mathcal{N}_{eu}$-λ-closed sets are weaker than $\mathcal{N}_{eu}$-closed and $\mathcal{N}_{eu}$-pre-closed sets, but independent of the concepts of $\mathcal{N}_{eu}$-g-closed, $\mathcal{N}_{eu}$-semi-closed, and $\mathcal{N}_{eu}$-w-closed sets. As an

application, we proposed and examined $\mathcal{N}_{eu}$-λ-continuous mapping, $\mathcal{N}_{eu}$-λ-closed mapping, and $\mathcal{N}_{eu}$-λ-$T_{1/2}$ space. We have studied various characteristics of these mappings using $\mathcal{N}_{eu}$-λ-$T_{1/2}$ space. We also compare these newly defined mappings with various existing mappings, and it is found that $\mathcal{N}_{eu}$-λ-continuous mapping is weaker than $\mathcal{N}_{eu}$-continuous and $\mathcal{N}_{eu}$-pre-continuous mappings, but independent of the concepts of $\mathcal{N}_{eu}$-g-continuous, $\mathcal{N}_{eu}$-semi continuous, and $\mathcal{N}_{eu}$-generalized semi pre-continuous mappings. Similarly, we established that $\mathcal{N}_{eu}$-λ-closed mapping is weaker than $\mathcal{N}_{eu}$-closed and $\mathcal{N}_{eu}$-pre-closed mappings, but independent of the concepts of $\mathcal{N}_{eu}$-g-closed, and $\mathcal{N}_{eu}$-semi closed mappings. Some new notions like $\mathcal{N}_{eu}$-w-continuous mapping, $\mathcal{N}_{eu}$-w-closed mapping, $\mathcal{N}_{eu}$-Gα−closed mapping, and $\mathcal{N}_{eu}$-αG-closed mapping are discussed, and it is found that these notions are independent of $\mathcal{N}_{eu}$-λ-continuous mapping, and $\mathcal{N}_{eu}$-λ-closed mappings, respectively. These findings contribute to the ongoing development of topology, and open avenues for future research. Future work can extend these concepts in the study of homeomorphisms, connectedness, compactness, and so on.

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Advanced deep learning models based on neutrosophic logic for the analysis of brain tumor medical images

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Abstract

The categorization of medical photographs poses considerable difficulties owing to noise, uncertainty, and ambiguous information. Conventional deep learning models frequently encounter difficulties in addressing this issue, resulting in diminished diagnostic precision, particularly in the context of low-quality or ambiguous situations. This work presents a hybrid methodology that combines Neutrosophic Set (NS) theory with deep learning models to improve magnetic resonance imaging (MRI) picture classification in uncertain settings. NS theory delineates three domains: True (T), Indeterminate (I) and False (F) to address picture uncertainty and noise, hence enhancing deep learning models' capacity to analyze complex, ambiguous visual data. To assess the methodology, four advanced deep learning models MobileNet, VGG16, DenseNet121 and InceptionV3 were employed, and their efficacy was analyzed on brain tumor medical image datasets. The findings demonstrate that models trained on NS-transformed data, especially DenseNet and inception, produce enhanced results relative to those trained on the original data, attaining notably higher accuracy, precision, and recall. This illustrates that integrating NS theory into deep learning models markedly improves their capacity to categorize uncertain and noisy MRI pictures, offering a reliable method for enhancing diagnostic accuracy in medical imaging.

Keywords: Deep Learning, Neutrosophic Set, Convolutional Neural Networks, MRI, Tumor Detection.

1. Introduction

With 9.6 million deaths in 2018, cancer ranks as the second most common cause of death globally. Comprising billions of cells, the sophisticated organ, known as the brain is responsible for the grouping of aberrant cells leading to brain tumors. These tumors could be benign or malignant; malignant tumors spread quickly to other parts of the brain. Various classifications of brain cancers exist, such as glioma, meningioma, and pituitary tumors [1]. MRI is a commonly employed method for visualizing the internal structures of the body through cross-sectional images. It offers comprehensive information about various tissues with excellent resolution and contrast, rendering it extensively utilized in the anatomical adjunctive

assessment of brain tissue. MRI scans yield images of tissues exhibiting varying contrasts, facilitating precise diagnosis [2].

Computer-Aided Diagnosis (CAD) technologies are employed to expedite and enhance the accuracy of tumor detection for early illness identification. CAD systems can identify the tumor's location, kind, and severity, if applicable. Nevertheless, physicians' diagnoses are prone to inaccuracies, oversights, and delays owing to the vast volume of data. Because of this, the steps needed to classify, segment, localize, and find brain tumors have become the hardest parts of disease identification [3].

The term neutrosophy denotes the understanding of neutral thought, which constitutes the primary distinction between fuzzy logic and intuitionistic fuzzy logic and sets .The neutrosophic set possesses the necessary capabilities to serve as a comprehensive framework for uncertainty analysis in data sets, particularly in the realm of artificial intelligence and deep learning involving images [4]. In 1995, Smarandache came up with the idea of neutrosophic reasoning. After that, in 1999, its founder unified and made it more general. Neutrosophic logic has been used in many areas of computer science since then, such as pattern recognition, picture segmentation, and processing. It helps with a lot of study and real-world problems in many areas, like healthcare, economics, space satellites, and farming. There are a lot of new mathematical ideas that come from neutrosophy, including both classical and fuzzy ones.

The Neutrosophic Set (NS) theory provides a workable way to handle uncertainty in picture classification. In addition to T, F and I, it adds three more domains to normal binary logic. These groups show information that isn't complete, isn't regular, or isn't clear, which is common in medical MRI. NS theory emphasizes how reliable certain visual elements are while reducing the impact of areas that are unclear or noisy. It works by giving each item in a set one of three membership numbers (T, I, F), which show how much the item fits into the truth, indeterminacy, or falsehood domains [5]. When there isn't a lot of information or information that doesn't make sense, like in medical imaging, this three-part model is very helpful. The zones of uncertainty are clearly modeled in NS, which makes it a more advanced way to evaluate medical imaging. Still, not much study has been done on adding uncertainty management strategies like NS theory to the preprocessing or training pipeline of deep learning models. This means that uncertainty-aware methods for medical MRI interpretation are not widely used. This method uses the NS framework for MRI images, which makes classification easier by better representing image features and making the method more resilient in unclear situations [6]. The concepts of Artificial Intelligence (AI), Machine Learning (ML), and Deep Learning (DL) are individually distinct from one another

Generally, DL is a part of AI. ML is a methodology for data modeling and AI is a technology that is characterized by cognitive skills. Different types of ML approaches include reinforcement learning, supervised learning, and unsupervised learning. An increase in proficiency in speech recognition, picture analysis, object detection and disease diagnosis can be achieved through the use of DL [3]. It encompasses numerous categories, including Recurrent Neural Networks (RNN) and Convolutional Neural Networks (CNN). Convolution is implemented in CNNs, which are deep neural networks, as opposed to matrix multiplication. It gained recognition as a result of AlexNet, a 2012 model that exhibited exceptional

performance in the ILSVRC. In the fields of computer vision and natural language processing, Convolutional Neural Networks (CNNs) are employed. Their architecture comprises concealed layers for feature extraction, with specific layers designated for feature extraction during training and others for classification. The overall efficacy and feature extraction are improved by the augmented hidden layer depth [7]. Four advanced deep learning models were employed for evaluation, including MobileNet, VGG16, DenseNet121, and InceptionV3. The findings from experiments indicate that models that were trained on NS-transformed data outperformed.

The subsequent sections of the paper are organized as follows. Section 2 delineates the literature review on the application of neural networks using deep learning techniques in medical domains, while section 3 clarifies the dataset employed and the deep learning methodologies implemented in this study. Section 4 outlines the experimental setting, presents the experimental data, and discusses the suggested models alongside other deep learning models for the detection. The endpoint of this work is situated in section 5.

2. Literature review

This part summarizes relevant studies on the use of NS functions with DL methods in image categorization. By means of the study of past studies, we want to identify shortcomings and constraints while stressing important findings, methods, and achievements in the integration of NS with DL models spanning several domains. Unsupervised approaches are becoming more and more important to researchers since they are so remarkably effective and able to independently generate features, hence reducing error rates. Medical image analysis requires deep learning models for segmentation, reconstruction, and classification as well as for other purposes.

In [8], the authors suggested an automatic segmentation approach for the identification of brain tumors utilizing MRI brain images. Convolutional Neural Networks (CNNs), machine learning algorithms inspired by biological processes, are under investigation for their application in brain tumor classification and radionics analysis. In [9], the researchers developed a useful brain tumor segmentation system including benign and malignant classification of brain tumors. The NS-EMFSE method helped to define brain tumors. CNN architectures—more especially, AlexNet—derived the features of the segmented images; later, SVM and KNN classifiers were used to identify them. Comprising feed-forward layers, CNN is a deep learning method. The authors Created an automatic detection method employing NS domain transformation, then implementing a process similar to that of a traditional CAD system within the NS domain [10]. A DCNN1 classifier was explicitly trained to recognize individual MCs and used as the phase for reducing false positive MCs.

In order to convert medical images from the grayscale spatial domain into the neutrosophic domain, the research makes use of the neutrosophic set theory [11]. True (T) images, indeterminacy (I) images, and falsity (F) images are the three types of images that can be found in the neutrosophic realm. Several different sources were utilized in the collection of the dataset that was utilized in this research. The dataset is divided into four distinct categories, which are as follows: COVID-19, normal, bacterial pneumonia, and viral pneumonia. An investigation into the influence that neutrosophic sets have on deep transfer learning models is the purpose

of this study. Specifically, the deep learning models, such as AlexNet, GoogleNet, and ResNet18 are utilized in this investigation.

The study [12] made use of both picture and text data, and took inspiration from earlier techniques for embedding text onto photographs, with the intention of identifying the images through the application of neutrosophic classification algorithms. Neutrosophic convolutional neural networks, also known as NCNNs, are utilized for the purpose of acquiring feature representations of generated images for the purpose of classification objectives. It is necessary to demonstrate the utilization of a pipeline that makes use of NCNN in order to obtain representations of the unique fusion technique. Convolutional neural networks that are conventional are susceptible to unanticipated noisy scenarios during the testing process, which results in a decline in their performance when it comes to classifying noisy data. Positive results are obtained when our strategy is evaluated against a variety of sources using a multi-modal classification dataset.

A computer-aided diagnosis approach [13] was offered for categorizing malignant lesions whereby the acquired image undergoes first pre-processing using creative methods. Digital artifacts—including blood vessels and hair follicles—are deleted; then, using a novel histogram equalizing method, image improvement follows. The pre-processed image then moves through the segmentation procedure, during which the likely lesion is defined with the Neutrosophic technique.

Prior research has enhanced the application of NS in Deep Learning methodologies involving uncertainty and noise. Nonetheless, a requirement for robustness persists, and several of these investigations exhibit insufficient precision. Several more investigations were undertaken with a limited dataset, rendering the long-term effects uncertain.

3. Materials and Methods

This research utilizes NS domain photos to develop a transfer deep learning model aimed at classifying various types of brain tumors. Following this, the models are subjected to rigors testing, thorough evaluation, and detailed comparative analysis. This study will be carried out in a series of methodical steps, as depicted in figure 1, to analyze and evaluate the performance of various DL models on NS.

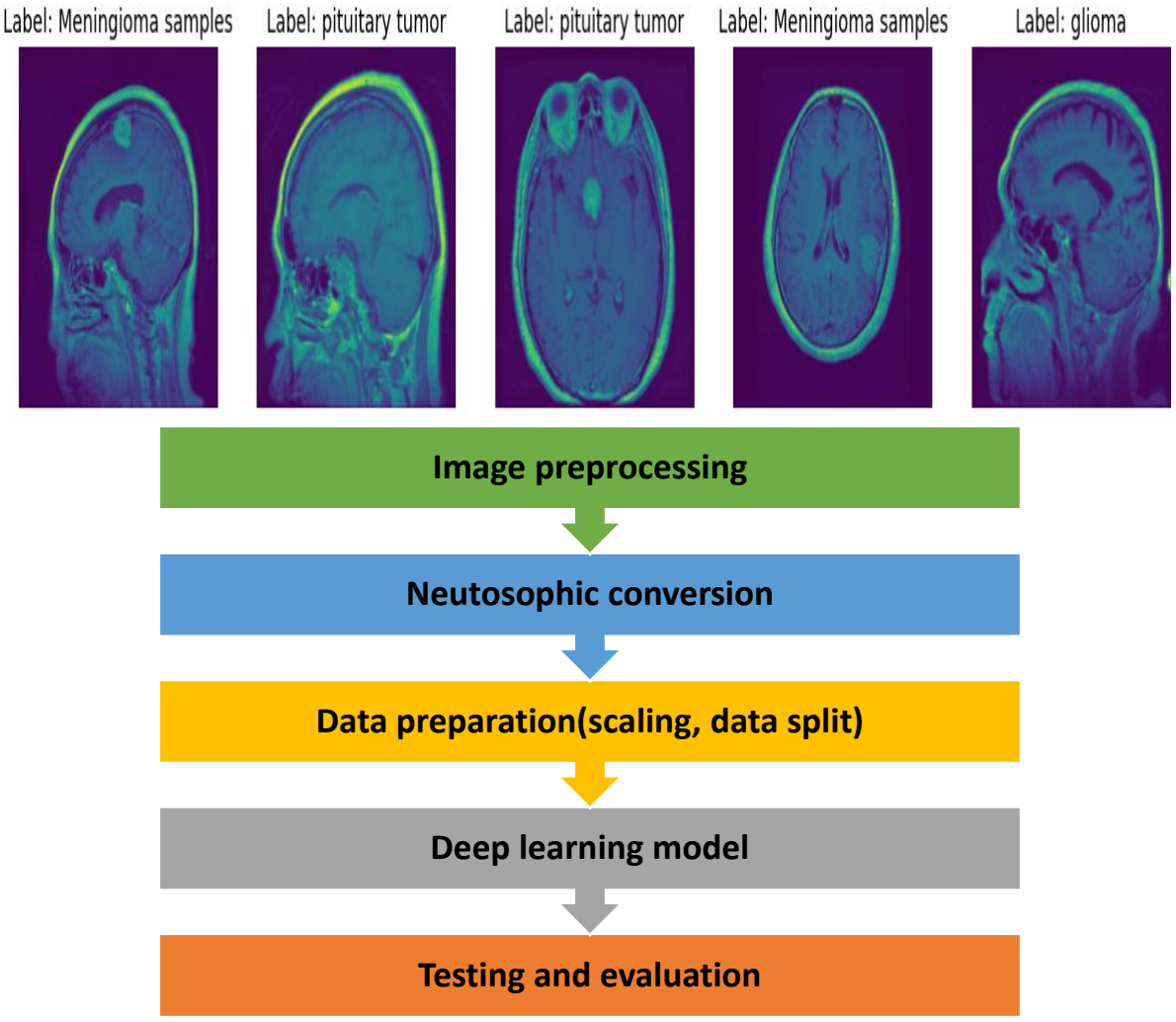

Figure 1 Methodical steps of our study for various DL on NS

3.1 Dataset description

The MRI dataset of the brain was collected from Nanfang Hospital in Guangzhou and General Hospital at Tianjin Medical University in China, spanning the years 2005 to 2010. The dataset consists of 3064 slices derived from 233 individuals, which includes 708 meningiomas, 1426 gliomas and 930 pituitary tumors. The images feature an in-plane resolution of 512×512, accompanied by a pixel size of 0.49×0.49 mm^2. The measurement of slice thickness is 6 mm,

and the gap between slices is 1 mm. The tumor margin was meticulously delineated by three experienced radiologists [14]. The sample of dataset is displayed in figure 2.

Figure 2 Dataset sample

3.2 Neuotrosphic set

Neutrosophy (NS) is a sophisticated theoretical framework established by Florentin Smarandache [15]. NS represents a significant and advantageous concept within the context of computing fuzzy scenarios. As t changes within the T subsets, events in NS theory are classified into one of three sets: true (T) significance, where the status indicates the percentage of truth; indeterminacy (I) significance, where the status indicates the percentage of indeterminacy; and falsity (F) significance, where the status indicates the percentage of falsehood. In image processing, particularly in object and edge identification, all pixels of the image are classified into three subsets: T, I and F. The edge detection and object recognition processes of the image are then performed through necessary operations on these subsets. The input image undergoes transformation into the neutrosophic domain, as illustrated in Eqs. (1–5). The pixel P (q, r) in the picture domain is transformed into the NS p_{NS} (q, r).

$$p_{NS}(\mathrm{q},\mathrm{r}) = \{\mathrm{T}_{q,r}, \mathrm{I}_{q,r}, \mathrm{F}_{q,r}\} \tag{1}$$

$$T(q,r) = \frac{\bar{N}(\mathrm{q},\mathrm{r}) - \bar{N}_{\min}}{\bar{N}_{\max} - \bar{N}_{\min}} \tag{2}$$

$$I(q,r) = 1 - \frac{\bar{H}(\mathrm{q},\mathrm{r}) - \bar{H}_{\min}}{\bar{H}_{\max} - \bar{H}_{\min}} \tag{3}$$

$$H(\mathrm{q},\mathrm{r}) = \mathrm{abs}(\mathrm{N}(\mathrm{q},\mathrm{r}) - \mathrm{N}(\bar{\mathrm{q}},\mathrm{r}) \tag{4}$$

$$F(\mathrm{q},\mathrm{r}) = 1 - \mathrm{T}(\mathrm{q},\mathrm{r}) \tag{5}$$

Where $N(q,r)$ denotes the gray value of the corresponding pixel, $\bar{N}(q,r)$ signifies the regional average value of $N(q,r)$ and $H(\mathrm{i},\mathrm{j})$ represents the homogeneity value, defined as the absolute difference between the intensity of $N(q,r)$ and its local mean value $\bar{N}(q,r)$. Upon converting the image to the NS domain, the brain tumor image should be retained in the (T) domain, while the edges are to be positioned in the (I) domain and the background layer should be located in the (F) domain. Figure 3 demonstrates the difference between the original image and the images following conversion into NS image domains.

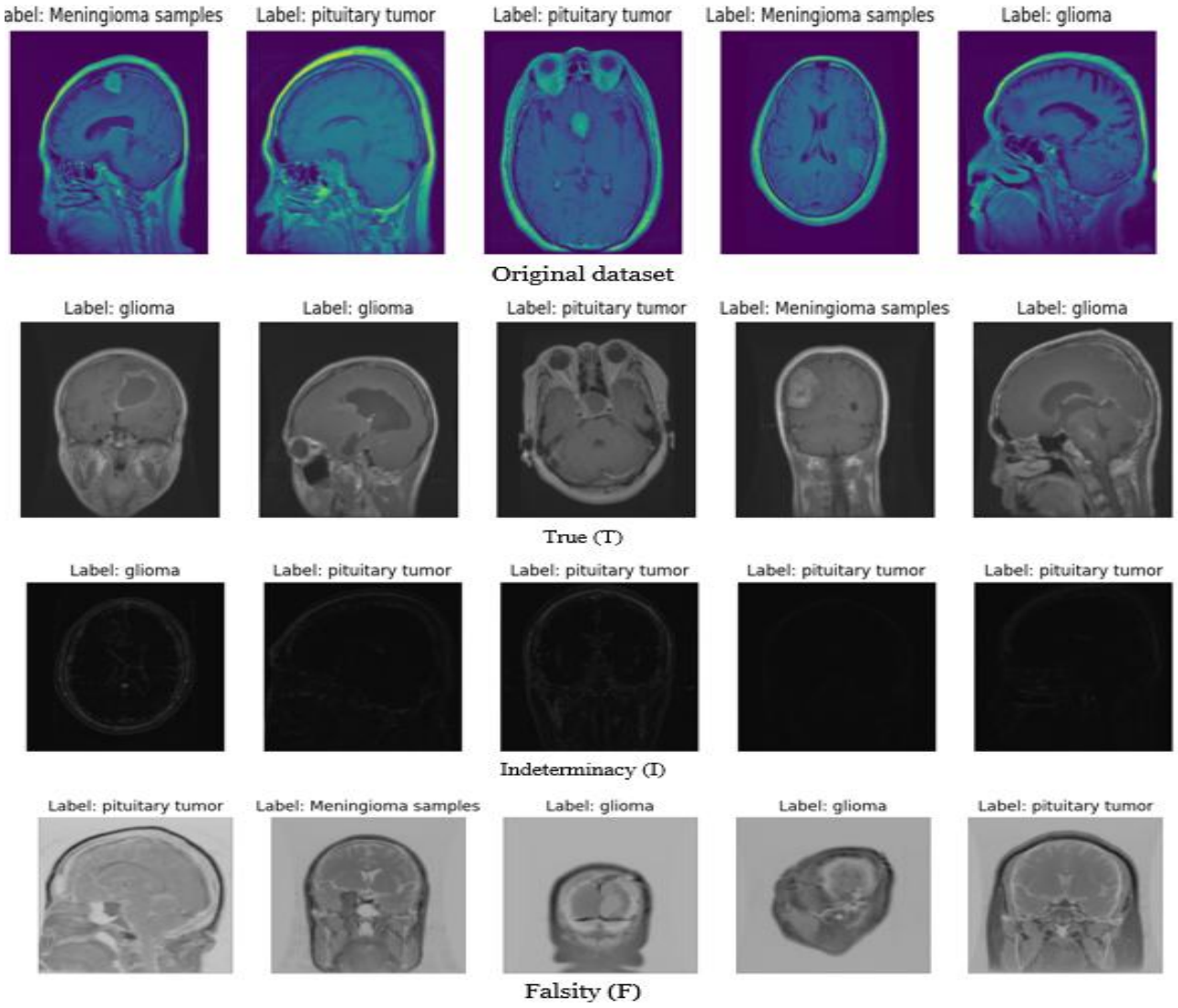

Figure 3 Different NS Images Domains Conversion Original images, True domain, Indeterminacy domain, and Falsity domain image

3.3 Deep learning models

Deep learning has emerged as a strong method for addressing complicated challenges across several domains, including medical picture processing. Convolutional Neural Networks (CNNs) are pivotal in deep learning, revolutionizing image classification by extracting hierarchical feature representations from data. They examine grid-structured data, such as images, to find prominent characteristics including edges, textures, and patterns. In medical imaging, CNNs can discern intricate patterns that elude human detection, rendering them essential for illness diagnosis and fracture recognition. Transfer learning is employed to modify models for new domains, hence minimizing the requirement for substantial training data [16]. This study involves training deep learning models to operate within the neutrosophic domain, wherein medical images are converted to represent true, false, and indeterminate information based on neutrosophic sets.

By identifying these properties, the models are prepared to manage the basic uncertainty inherent in medical imaging data. Transfer learning methodologies are utilized to augment performance by fine-tuning pertained CNN models for novel medical imaging domains. Fine-tuning pertained models on medical MRI datasets enables the extraction of more relevant information related to fractures and pulmonary illnesses [17]. Numerous existing deep learning models, noted for their computational efficiency and feature extraction skills, were utilized to classify medical MRI images. These models have exhibited exceptional performance in diverse medical imaging applications and have been chosen for their demonstrated capacity to learn complex characteristics from data and their promise to tackle the complexities of classification jobs using MRI images.

Many DL models are utilized, including: MobileNetV2, VGG16, NN, Inception-V3 and DenseNet121. The fundamental architecture of MobileNetV2 is derived from its predecessor, MobileNetV1. MobileNetV2 employs Depth wise Separable Convolutions (DSC) to enhance portability, addressing the issue of information degradation in non-linear layers within convolutional blocks through Linear Bottlenecks, and introduces a novel architecture called Inverted Residuals to maintain information integrity [18]. Another DL model is VGG16 which is a neural network architecture that garnered attention during the 2014 ILSVRC (ImageNet) challenge [19].

Also, NN is classified among the premier designs for classification of images models. The model is significant for utilizing a 3x3 filter convolution while maintaining the same max pooling and padding layer as a 2x2 filter, instead of an excessive number of hyper-parameters. The organization of convolutional and max pooling layers is consistent throughout the system. Ultimately, the output comprises two completely connected softmax layers. DenseNet121 is generally used because it can improve feature reuse, solve the vanishing gradient problem, and minimize parameter use—all of which are benefits for deep learning models. Moreover, DenseNet-121 has shown performance in disease diagnosis using image databases [20].

The designation 16 in VGG16 signifies that the architecture comprises 16 layers with diverse weights, totaling around 138 million parameters [21]. Moreover, the Inception-V3 model is a development of the V1 model. Several techniques are used in the Inception-V3 model to maximize network optimization for higher model adaption. Its network is more complete than those of the V1 and V2 models. It is designed especially on a low-specification machine, the V3 model is a neural network. Training takes a lot of time and could last several days, hence it is really demanding. Transfer learning keeps the last layer of the model for new categories, therefore addressing this problem. The parameters of the previous levels remain the same; the V3 model is deconstructed by removing the last layer using the transfer learning approach [22].

4. Experiments and results

In this section, we will investigate the performance of several DL models, such as VGG16, DenseNet121, InceptionV3 and MobileNetV2. All models were trained using images generated through NS into T, I and F domain images, and FL into membership and non-membership

images. The models employ the Adam optimizer with a learning rate of 0.001, a total of 100 epochs and a batch size of 32.

4.1 Experiment setup

Using a MATLAB application, photographs were transformed to NS images; the testing with DL models was place on the Kaggle platform with GPU P100. With sixteen GB of RAM, the proposed model was built and trained using Python 3.10, Keras 3.5 and TensorFlow 2.15.

4.2 Evaluation metrics

Using accuracy, precision, recall, and F1-score—mathematically defined in Eqs. (7–10)—a set of tests is conducted to evaluate the proposed model and compare it with existing state-of- the-art models.

- In machine learning, accuracy is a measure of the frequency with which a model correctly forecasts results. It is computed by dividing the number of correct forecasts by the total number of all forecasts. The correct forecasts are composed of both True Positives (TP) and True Negatives (TN) forecasts. All predictions are composed of Positive (P) and Negative (N) forecasts, where P consists of TP and False Positive (FP), and N consists of TN and False Negative (FN). It can be expressed as follows:

$$Acc = \frac{TP + TN}{TP + FP + TN + FN} \tag{6}$$

- One measure of a machine learning model's accuracy in predicting the positive class is precision. Dividing the total number of cases expected as positive by the number of genuine positives yields precision—both true and false positives included. It is mathematically formulated as:

$$\Pr ecision = \frac{TP}{TP + FP} \tag{7}$$

- Recall is a metric measuring, among all the actual positive samples in the dataset, the frequency with which a machine learning model correctly finds positive examples true positives. Dividing the total number of positive cases by the amount of true positives can help one to get recall. The latter consists in false negatives (missed cases) and true positives—accurately identified cases. The recall metric can be calculated as:

$$\mathrm{Re}\, call = \frac{TP}{TP + FN} \tag{8}$$

- We want both precision and recall to equal one; so, false positives and false negatives equal zero as well. We thus need a statistic that takes recall into account as well as precision. A measure combining recall and accuracy is the F1-score. It can be computed as:

$$F1\_score = 2 \times \frac{\mathrm{Re}\, call \times \Pr ecision}{\mathrm{Re}\, call + \Pr ecision} \tag{9}$$

4.3 *Statistical Analysis of different deep learning models*

This part analyses the performance of the transfer learning model across accuracy, precision, recall, and F1-score measurements by means of the employed picture data. Table 1 presents how well the model performed using original data. Table 2 illustrates how well the model performed on true. Table 3 presents model performance on indeterminacy. Table 4 presents how well the model performed on falsehood.

The findings of the trial showed that NS domain images obtained better test accuracy than original images. Over all experimental trials, the True (T) NS domain achieved the highest testing accuracy when compared among the T, I, and F NS domain images. Among Transfer learning models, the DenseNest121 model has shown exceptional performance in all picture categories. Table 2 shows the ideal accuracy of a model for a certain image type; DenseNet121 obtained an accuracy of 90.87%, with precision at 90.71% and recall at 90.87% for NS Domain images.

Table 1 original dataset

Model	Accuracy	Precision	Recall	F1-score
MobileNet	0.8783	0.8751	0.8783	0.8750
VGG16	0.8848	0.8824	0.8848	0.8820
InceptionV3	0.8674	0.8651	0.8674	0.8660
DenseNet121	0.8783	0.8753	0.8783	0.8748

Table 2 True NS

Model	Accuracy	Precision	Recall	F1-score
MobileNet	0.8935	0.8978	0.8935	0.8942
VGG16	0.8891	0.8898	0.8891	0.8891
InceptionV3	0.9065	0.9056	0.9065	0.9056
DenseNet121	0.9087	0.9071	0.9087	0.9073

Table 3 Indeterminacy NS

Model	Accuracy	Precision	Recall	F1-score
MobileNet	0.7935	0.7935	0.7717	0.7740
VGG16	0.7435	0.7465	0.7435	0.7240
InceptionV3	0.7696	0.7675	0.7696	0.7677
DenseNet121	0.8130	0.8147	0.8130	0.8135

Table 4 Falsity NS

Model	Accuracy	Precision	Recall	F1-score
MobileNet	0.8717	0.8679	0.8717	0.8656
VGG16	0.8391	0.8329	0.8391	0.8344
InceptionV3	0.8761	0.8749	0.8761	0.8745
DenseNet121	0.8957	0.8931	0.8957	0.8933

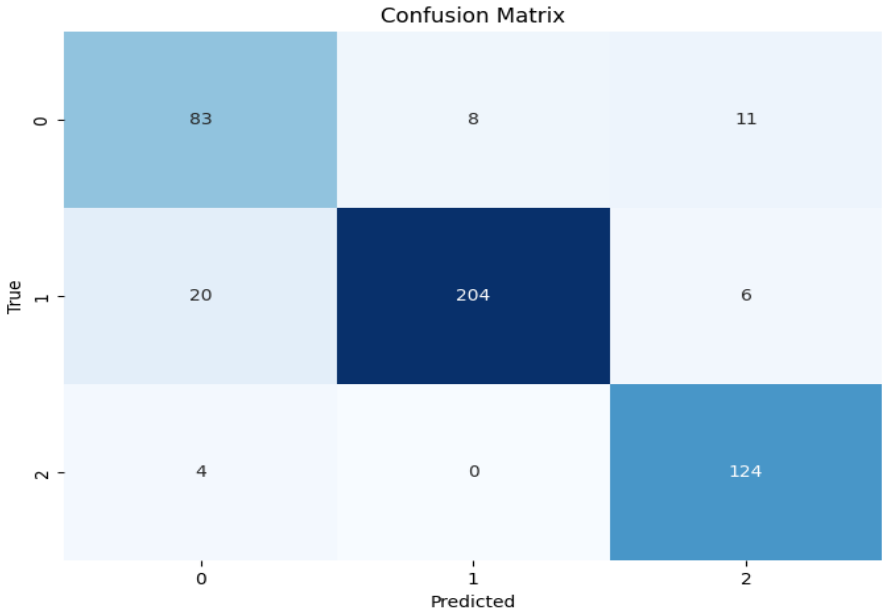

Figure 4 The confusion matrix MobileNet model on True NS

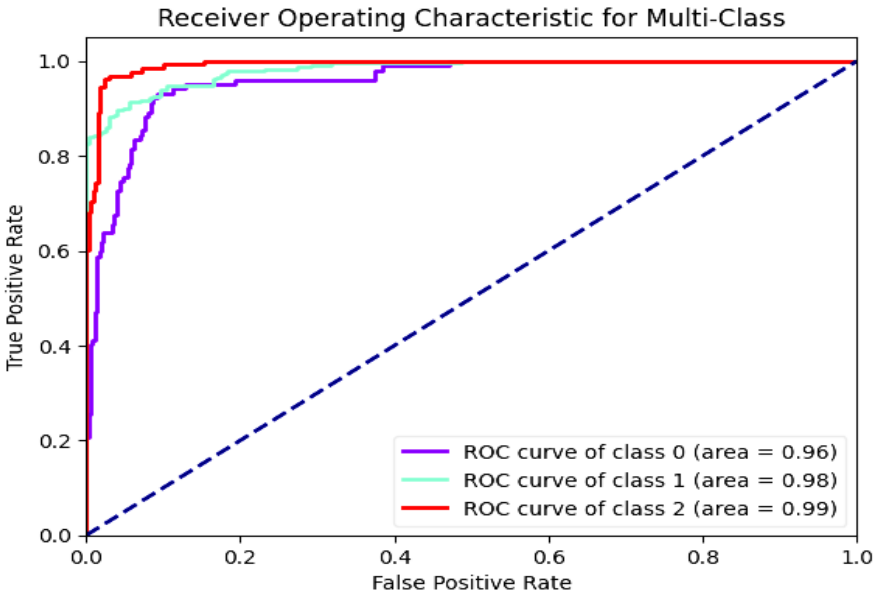
Receiver Operating Characteristic for Multi-Class
True Positive Rate
False Positive Rate
ROC curve of class 0 (area = 0.96)
ROC curve of class 1 (area = 0.98)
ROC curve of class 2 (area = 0.99)

Figure 5 The ROC MobileNet model on True NS

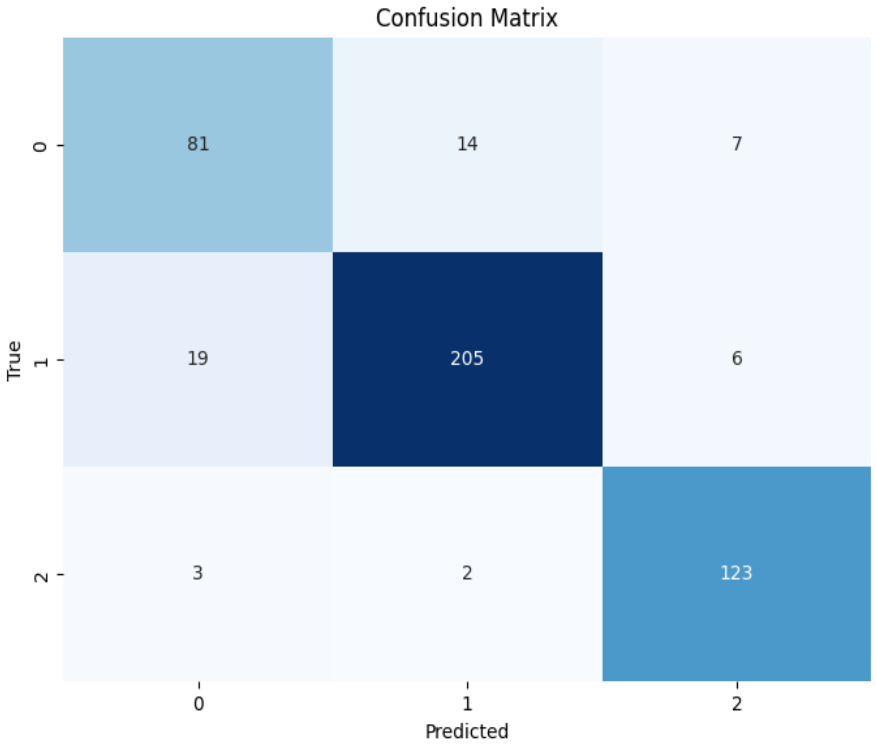

Figure 6 The confusion matrix VGG16 model on True NS

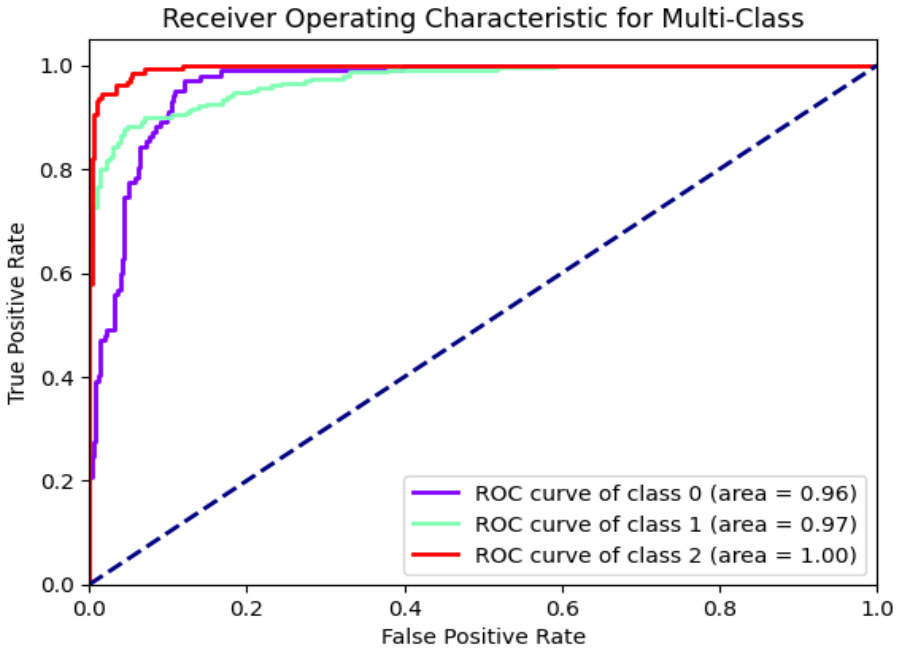
Receiver Operating Characteristic for Multi-Class
True Positive Rate
False Positive Rate
ROC curve of class 0 (area = 0.96)
ROC curve of class 1 (area = 0.97)
ROC curve of class 2 (area = 1.00)

Figure 7 The ROC VGG16 model on True NS

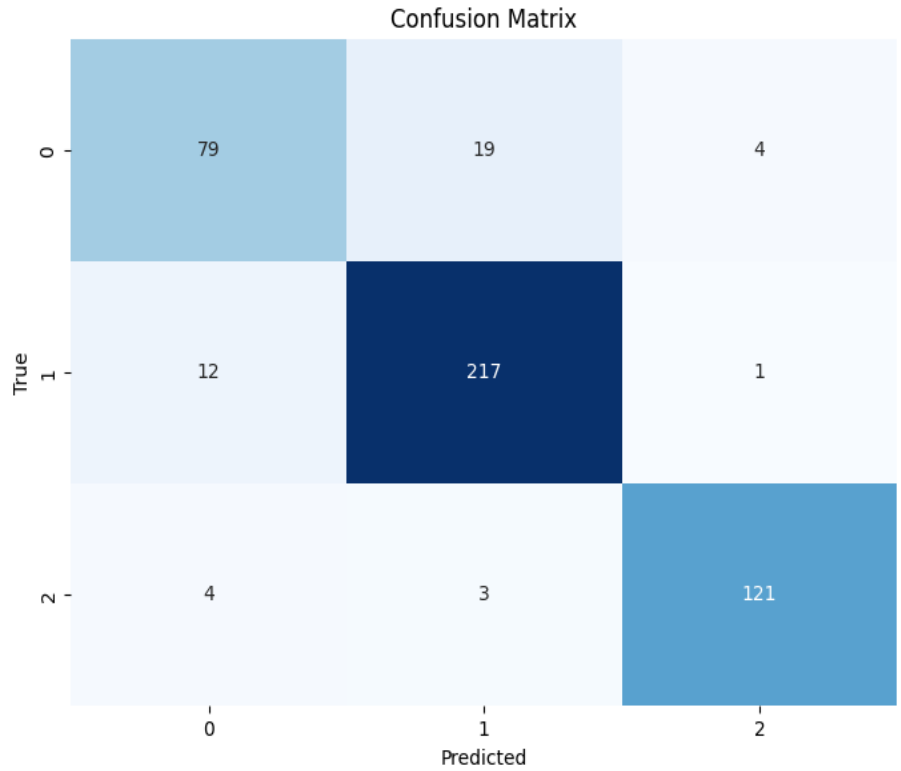

Figure 8 The confusion matrix Inception v3 model on True NS

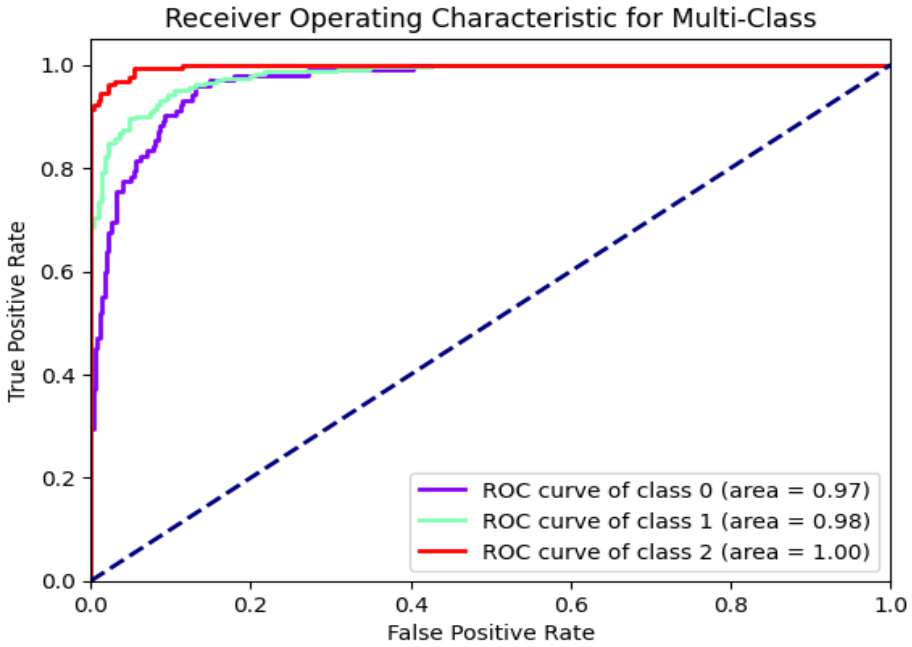
Receiver Operating Characteristic for Multi-Class
True Positive Rate
False Positive Rate
ROC curve of class 0 (area = 0.97)
ROC curve of class 1 (area = 0.98)
ROC curve of class 2 (area = 1.00)

Figure 9 The ROC Inception V3 model on True NS

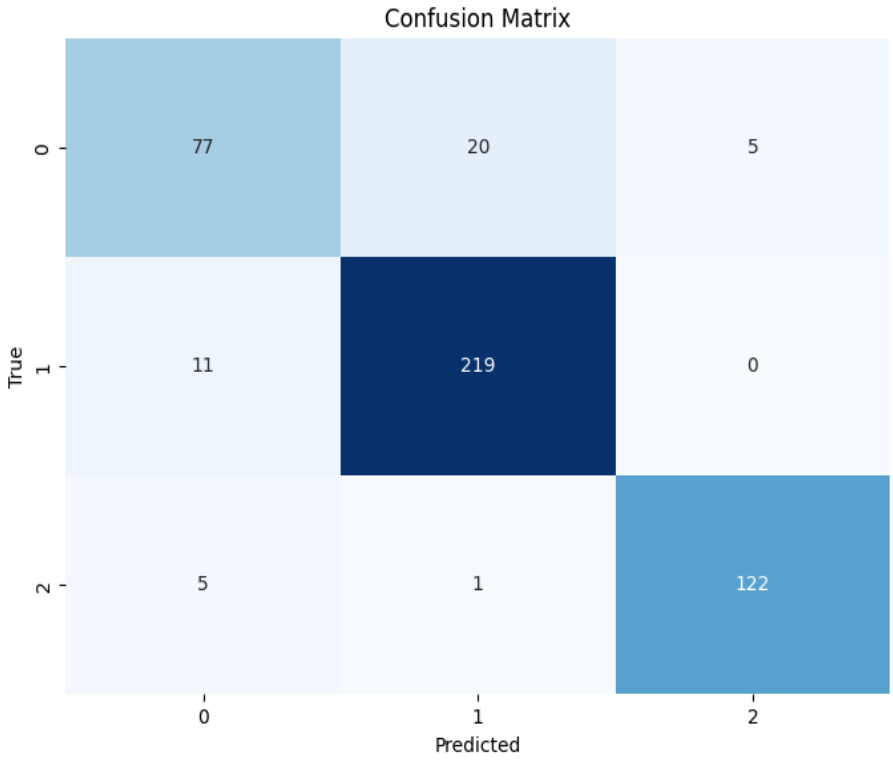

Figure 10 The confusion matrix DenseNet121 model on True NS

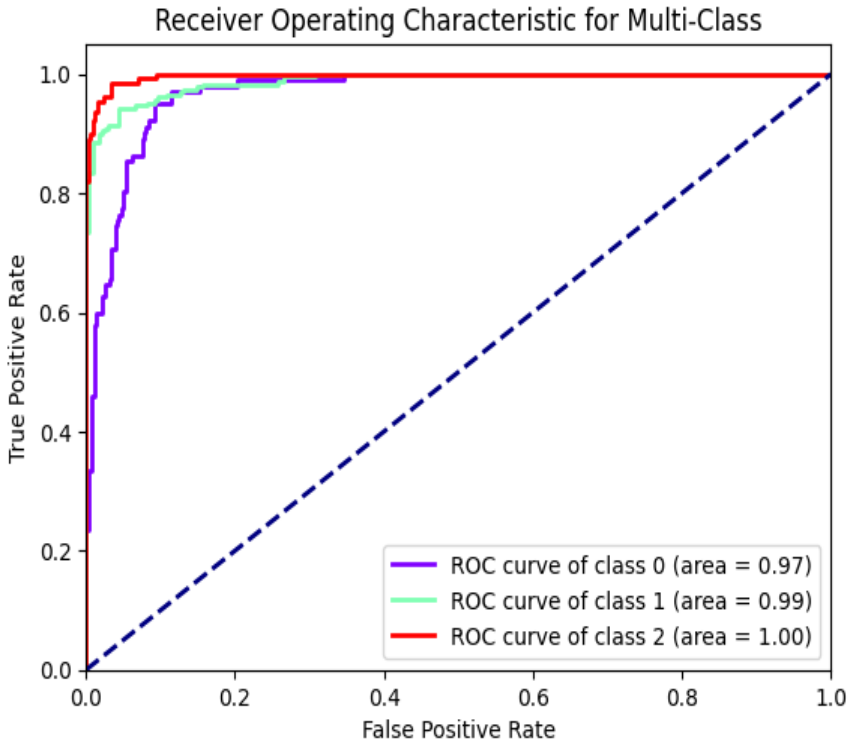
Receiver Operating Characteristic for Multi-Class
True Positive Rate
False Positive Rate
ROC curve of class 0 (area = 0.97)
ROC curve of class 1 (area = 0.99)
ROC curve of class 2 (area = 1.00)

Figure 11 The ROC DenseNet121 model on True Ns

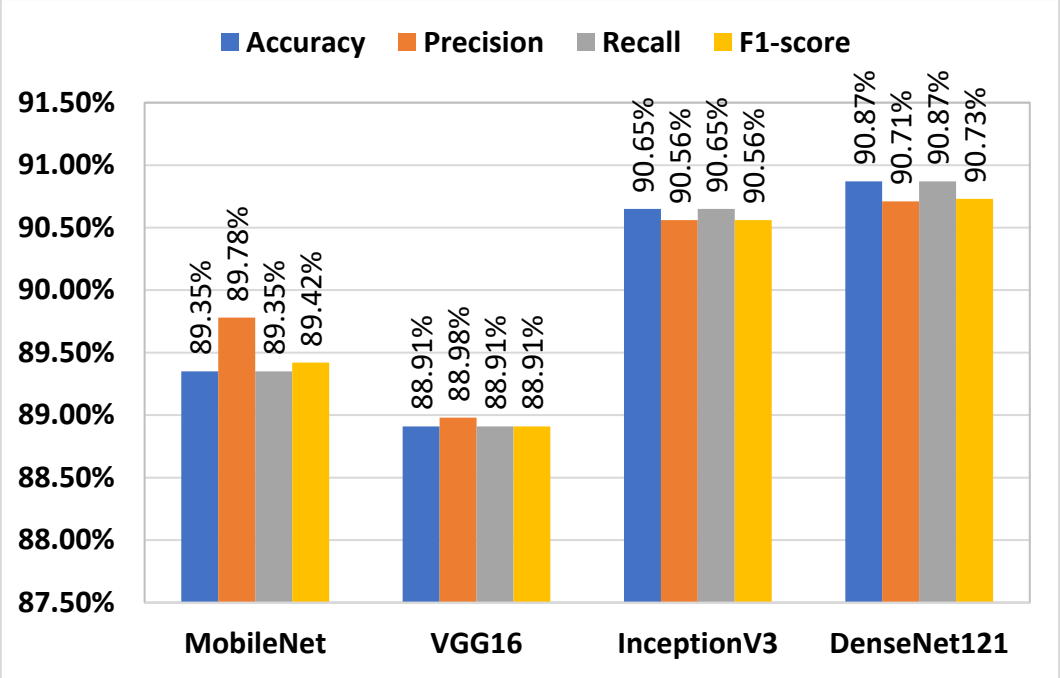

Figure 12 Performance comparisons among four models on Ns domain (True)

The application of the neutrosophic domain markedly impacts the classification accuracy and robustness of the model. Upon analyzing the confusion matrix in figure 4, figure 6, figure 8 and figure 10 of each model utilizing the neutrosophic True (T). The performance disparity demonstrates that the neutrosophic domain model attains superior classification accuracy and has a reduced incidence of misclassification. Figure 12 performs a graphical comparison among the four DL models in terms of accuracy, precision, recall and f1-score metrics. As can be shown from figure 12, DenseNet121 achieves the best results with values of 90.87%, 90.71%, 90.87% and 90.73% for accuracy, precision, recall and f1-score, respectively. On the other hand, VGG16 shows the worst performance for accuracy metric with a value of 88.91%.

5. Conclusion

This paper presents an exciting strategy for enhancing classification accuracy in brain tumors through the merging of NS and DL techniques. Four DL models have been created to assess their performance inside the NS domain. The NS is capable of managing uncertainty and ambiguity in higher dimensions. DL is an improved tool for categorization compared to conventional ML models. The assessment of our work has been conducted on 3,064 pictures to categorize brain tumor histology. Ultimately, the NS demonstrates higher performance regarding accuracy, precision, recall, and F1-score.

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